MTH 42 LECTURE NOTES (Ojakian)

Topic 9: Inverses

OUTLINE
(References: 3.3)

1. Invertible Linear Transformations
2. Invertible Matrices

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1. Function Inverses

PROBLEM 1. *For the following two functions, first observe what the domain and codomain are. Then convince yourself that neither is linear.*

After seeing the below definitions, determine if each is invertible or not. If it is invertible, find its inverse and see how it is one-to-one and onto. If it is not invertible, see how being one-to-one and/or onto fails.

(a) $f(x, y) = (x^2, y^2)$

(b) $h(x, y, z) = (x + 1, z, y)$

Definition 1. *A function $f : A \to B$ is invertible if there exists a function $g : B \to A$, such that the following holds:*

$$f(a) = b \text{ if and only if } g(b) = a$$

*If the function g exists it is called the **inverse** of f and denoted f^{-1}.*

Note: It can be proved that a function either does **not** have an inverse or has **exactly one** inverse.

Theorem 1. *A function $f : A \to B$ is invertible if and only if there exists a function $g : B \to A$ such that the following two conditions hold:*

- $f(g(b)) = b$
- $g(f(a)) = a$

Such a function g is the inverse of f.

Theorem 2. *A function $f : A \to B$ is invertible if and only if f is one-to-one and onto.*

2. Linear Transformations and Inverses

   (a) Read Theorem 3.19

   PROBLEM 2. *For the following functions, answer this: 1) Is it linear? And if so, find its corresponding matrix, 2) Is it invertible? And if so, find its corresponding matrix.*

   i. $T(x_1, x_2, x_3) = (x_1 + x_2, 0)$

   ii. $T(x_1, x_2) = (x_1 + x_2, x_2)$

   PROBLEM 3. *Prove part (b) of Theorem 3.19*

3. Matrices and Inverses

   PROBLEM 4. *Recall the invertible linear transformation from Problem 2. Find the product of the two matrices.*

   (a) Definition 3.20

   (b) Theorem 3.21

   (c) Definition 3.22

   (d)

   Theorem 3. *Suppose T is a linear transformation represented by matrix A. Then*

   i. T is invertible if and only if A is invertible.

   ii. T^{-1} is represented by A^{-1}.

4. Calculting the Matrix Inverse

   PROBLEM 5. *Use the standard procedure to complete the inverse of one of the matrices from Problem 4.*

   PROBLEM 6. *From Section 3.3 (page 124): Do exercise 24 and thn exercise 11.*

   PROBLEM 7. *Check the last two problems using Anaconda.*

5. The power of the inverse

   (a) Theorem 3.23, i.e. the inverse and "cancellation" basically works as we are used to.

   (b) Theorem 3.24, i.e. solving systems by "cancellation"

   (c) Theorem 3.25 (the big theorem again)

   PROBLEM 8. *Use Anaconda. Create a complicated matrix equation based on an invertible matrix. Then do the following:*

   i. Solve the system using the inverse matrix and check your answer.

   ii. Solve the homogeneous system to see that it really has only the trivial solution.

6. Application: Encryption

   (a) Usual Caesar Shift (with 27 for space) or even more complicated single letter replacement.

   (b) Problem with above: Frequency Analysis

   (c) A better option: Use blocks of letters

   PROBLEM 9. *Use Anaconda to encode and decode messages.*

   (d) Serious Encryption: The difficulty of factoring, i.e. inverting multiplication.