# MTH 42 LECTURE NOTES (Ojakian)

## Topic 7: Linear Transformations and Matrices

---

**OUTLINE**
(References: 3.1)

1. Linear Transformations
2. Connection between linear transformations and matrices

---

1. Functions
   (a) Definition of a function. Example: $f(x, y) = x^2 + y^2 + 1$
   (b) Domain, codomain, and range.
   **PROBLEM 1.** *For the above function, choose a reasonable domain and codomain. Then find the range.*
2. Linear Transformations
   (a) See Definition 3.1
   **PROBLEM 2.** *Consider the function in Example 2. Evaluate it for some example inputs.*
   **PROBLEM 3.** *Do Example 2. Then modify it to make it linear (and show it is linear).*
   *Note: Intuition - make each output entry a linear expression.*
   (b) Equivalent definition (section 3.1, exercise 58).
   **PROBLEM 4.** *Consider the theorem of section 3.1 - exercise 58.*
      i. *We will make up a linear function and then use this fact to prove it is linear.*
      ii. *Prove one direction of the fact - Just 58a.*
   **PROBLEM 5.** *Observe that $T(0) = 0$ for the above linear examples, then do exercise 3.1: 55.*

3. Relation to Matrix

   (a) Useful facts to note: Theorem 2.16 (page 73).

   (b) Defining a function using a matrix

   **PROBLEM 6.** *From Section 3.1 (page 92): Do exercise 3.*
   *Then prove that this function is linear.*

   (c) Equivalent definition (Theorem 3.8):
   $T$ is defined by from a matrix if and only if $T$ is a linear transformation.

      i. Example of forward direction : see last problem.

      ii. Example of backwards direction.

      **PROBLEM 7.** *See Example 7 (page 89). Find its corresponding matrix in a way that mimicks the proof of Theorem 3.8, giving a "proof by example" of the theorem.*

4. Recall: Injective and Surjective Funcions

   (a) One-to-one (or injective)

   **PROBLEM 8.** *Consider the two functions* $f(x) = x^2 - 3$ *and* $g(x) = x^3 + 1$. *One is injective and one is not. Determine which is which, and justify!*

   (b) Onto (or surjective)

   **PROBLEM 9.** *For the last two functions are they onto or not?*

   (c) One-to-one correspondance (or bijective)

   **PROBLEM 10.** *Which of the above functions are bijective?*

   **PROBLEM 11.**

      *i. Find a function with domain and codomain R which is onto, but* **not** *injective.*

      *ii. Find a function with domain and codomain R which is injective, but* **not** *onto.*

5. Injective and Surjective Linear Transformations

   (a)

   **Theorem 1.** *("One-to-one theorem" - Theorem 3.6) Suppose A is the matrix defining a linear transformation.*
   *The linear transformation is one-to-one* $\iff$ *The columns of A are linearly independent*

   **Theorem 2.** *("Onto theorem" - Theorem 3.7) Suppose A is the matrix defining a linear transformation.*
   *The linear transformation is onto* $\iff$ *The columns of A span the codomain.*

   **PROBLEM 12.** *Do section 3.1 (page 92): Exercises 23, 25*

   (b) Briefly, why on above theorems:

      i. One-to-One: If vectors are linearly independent, then two linear combinations yielding the same vector would create the contradiction of a linear dependence.

      ii. Onto: If vectors span, then you get all possibilities.

   (c) NOTE: The Big Theorem (Theorem 3.9) on page 89.

6. Geometric Aspects

**Theorem 3.** *Suppose $L$ is the line segment in $R^2$ connecting the points $u$ and $v$. Suppose $T$ is a linear transformation. Then $T(L)$ is the line segment with endpoints $T(u)$ and $T(v)$.*

**PROBLEM 13.**

*Consider the linear transformation corresponding to the following matrix:* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

*What does it do to the line segment connection $(0,0)$ to $(3,1)$? What does it do in general?*

**PROBLEM 14.** *Find a matrix which we can use to get reflection across the y-axis.*

**PROBLEM 15.** *Find a matrix which we can use to triple the length of a line segment.*

**PROBLEM 16.** *See Section 3.1 (page 91)*

(a) *Find a matrix which we can use to rotate a line segment 90 degrees counter clockwise.*

(b) *Find a matrix which we can use to rotate a line segment 90 degrees clockwise.*

**PROBLEM 17.** *Prove the above theorem.*

7. Final Problems

**PROBLEM 18.** *From Section 3.1 (page 94): Exercises 61, 66, 69, 75*