WORKBOOK. MATH 32. CALCULUS AND ANALYTIC GEOMETRY II.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

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PL, 2015 (Version 1)

This version has been reformatted by Kerry Ojakian (August 2015)

Contents

1. Recall from MTH 31

(1) State the definiton of the area A of the region under the graph of a continuous functon using limit Riemann sums. Draw an illustration to explain this procedure.

(2) Draw an illustration of four rectangles to estimate the area under the parabola $y = x^2$ from $x = 1$ to $x = 3$ using
  (a) left endpoints;
  (b) right endpoints;
  (c) midpoints;
  (d) Guess the actual area.

(3) What is the Definite Integral of a function f from a to b?

(4) The symbol $\int$ was introduced by ________________________ and is called an ________________________. It is an elongated S and is chosen because an integral is a ____________________________________.

(5) In the notation $\int_a^b f(x)dx$,
  $f(x)$ is called ____________________________________,
  a and b are called ____________________________________,
  a is the ____________________________________,
  and b is the ____________________________________.

(6) The symbol dx simply indicates that ____ ____.

(7) The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called ____________________________________, named after the German mathematician ____________________________________.

(8) **Theorem:** If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuites on $[a, b]$, then f is ________________________; that is, ________________________ exists.

(9) **Theorem:** If f is integrable on $[a, b]$, then $\int f(x)dx =$ ________________________ where $\Delta x =$ ________________________ and $x_i =$ ________________________.

(10) **Properties of Definite Integrals:**
  (a) $\int_b^a f(x)dx =$____________________________________.
  (b) $\int_a^a f(x)dx =$____________________________________.
  (c) $\int_a^b cf(x)dx =$____________________________________.
  (d) $\int_a^b [f(x) + g(x)]dx =$____________________________________.
  (e) $\int_a^b [f(x) - g(x)]dx =$____________________________________.
  (f) $\int_a^b cdx =$____________________________________.

(11) Fill in the blanks with the appropriate inequality.
  (a) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx$________________________0.
  (b) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx$________________________$\int_a^b g(x)dx$.
  (c) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a)$__________$\int_a^b f(x)dx$__________$M(b-a)$.

(12) State the **Fundamental Theorem of Calculus**.

(13) Find the derivative of:
  (a) $g(x) = \int_0^x \sqrt{4 + t^2}dt =$
  (b) $h(x) = \int_0^x \sin\left(\dfrac{\pi t^2}{2}\right) dt =$

(c) $k(x) = \int_0^{x^6} \cos(t^2+5)dt =$

(Hint: Do not forget the Chain rule).

(14) Evaluate the integral:

(a) $\int_0^3 r^4 + r^2 + 5\ dr$

(b) $\int_1^8 \left(\frac{1}{u^2} + 3u^2\right) du$

(c) $\int_1^3 \frac{r^4 + r^2 + 5}{r^9} dr$

(d) $\int_0^8 (\sqrt[3]{u} + 3u^2)\, du$

(e) $\int_{-\pi/4}^{\pi/4} \sec(\theta)\tan(\theta)\ d\theta$

(f) $\int_1^8 \left(\sqrt[3]{\frac{5}{u}}\right) du$

(g) $\int_{-\pi/2}^{\pi} f(t)dt$ where $f(t) = \begin{cases} \cos(t) & \textit{for } -\pi/2 \le t \le \pi/2 \\ \sin(t) & \textit{for } \pi/2 \le t \le \pi. \end{cases}$

(15) Complete the table:

$\int cf(x)dx$	
$\int [f(x) + g(x)]dx$	
$\int [f(x) - g(x)]dx$	
$\int kdx$	
$\int x^n dx$	
$\int \sin(x)dx$	
$\int \cos(x)dx$	
$\int \sec(x)\tan(x)dx$	
$\int \sec^2(x)dx$	
$\int \csc(x)\cot(x)dx$	
$\int \csc^2(x)dx$	

(16) Find the general indefinite integral:

(a) $\int (\sqrt{x^5} + \sqrt[3]{x^7})dx$

(b) $\int (\sqrt{x^5} + \sqrt[3]{x^7})^2 dx$

(c) $\int \sec(\theta)(\sec(\theta) + 3\tan(\theta))d\theta$

(d) $\int \left(u^4 + 3 + \frac{1}{u^4}\right) du$

(e) $\int \left(\frac{\sin(2t)}{\cos(t)}\right) dt$

(f) $\int \left(\frac{x^5 + x^7}{x^3}\right) dx$

(17) Evaluate the following definite integrals:
   (a) $\int_{-2}^{3}(x^5+x^3)^2dx$
   (b) $\int_{-1}^{7}\left(\sqrt{x^5}+x^7\right)dx$
   (c) $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\sec(\theta)(\sec(\theta)+3\tan(\theta))d\theta$
   (d) $\int_{0}^{7}(u^2+3)^2du$
   (e) $\int_{-\pi/2}^{\pi/2}(t^2\sin(t))dt$
   (f) $\int_{-\pi/2}^{\pi}(\sin(t))dt$
   (g) $\int_{-\pi/2}^{\pi}|\sin(t)|dt$
   (h) $\int_{1}^{8}\left(\frac{1}{x^5}+x^7\right)dx$

(18) The acceleration function in km/sec^2 is $a(t)=3t+7$ where time t is in seconds, and $0\leq t\leq 10$. Let $v(t)$ (in km/sec) and $s(t)$ (in km) be the velocity and position functions respectively with the initial velocity, $v(0)=45\ km/sec$ and the initial position $s(0)=4\ km$. Find the velocity function and the position functions. Then, find the total distance covered.

(19) State the Substitution Rule for integration.

(20) Find $\int 4x^3\sqrt{x^4+10}dx$

(21) Find $\int x^3\cos(x^4+10)dx$

(22) Find $\int\sqrt{3x-5}dx$

(23) Explain the Substitution Rule for Definite Integrals.

(24) Find $\int_{1}^{3}\sqrt{3x+5}dx$

(25) Find $\int_{2}^{7}\frac{1}{(3x+5)^5}dx$

(26) Find $\int_{0}^{5}x^5\sqrt{1+x^2}dx$

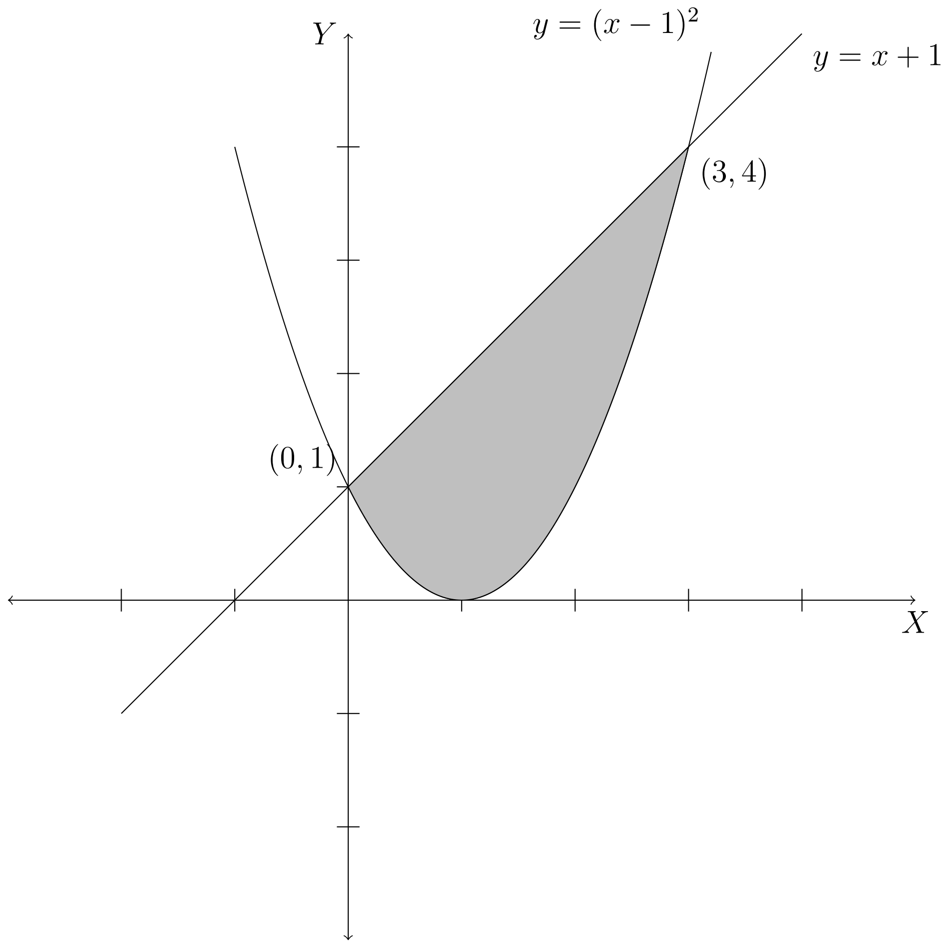
(27) Suppose that$f$ is a continuous function on $[-a,a]$.
   (a) If f is even on $[-a,a]$ then $\int_{-a}^{a}f(x)dx=$
   (b) Find $\int_{-3}^{3}(x^4+2x^2-1)dx$
   (c) If f is odd on $[-a,a]$ then $\int_{-a}^{a}f(x)dx=$
   (d) Find $\int_{-3}^{3}(x^5+2x^3-x)dx$

(28) Find the integral:
   (a) $\int x^7\sqrt{x^8+5}\ dx$
   (b) $\int\frac{1}{(3-7t)^8}dt$
   (c) $\int\frac{\csc^2(1/x)}{x^2}\ dx$
   (d) $\int x^6\cos(x^7+5)dx$
   (e) $\int x^3\sec(x^4+5)\tan(x^4+5)\ dx$
   (f) $\int(5t-7)^{3.55}dt$
   (g) $\int_{0}^{5}\frac{1}{(1+\sqrt{x})^5}\ dx$
   (h) $\int_{-\pi/2}^{\pi/2}x^8\sin(x)dt$
   (i) $\int_{0}^{a}x\sqrt{a^2-x^2}\ dx$
   (j) $\int_{-a}^{a}\sqrt{a^2-x^2}dx$

2. Area between curves

(1) **Theorem:** The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a, x = b$ where f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$ is $A =$ ________________________. Explain with an illustration.

(2) Find the area of the shaded region:

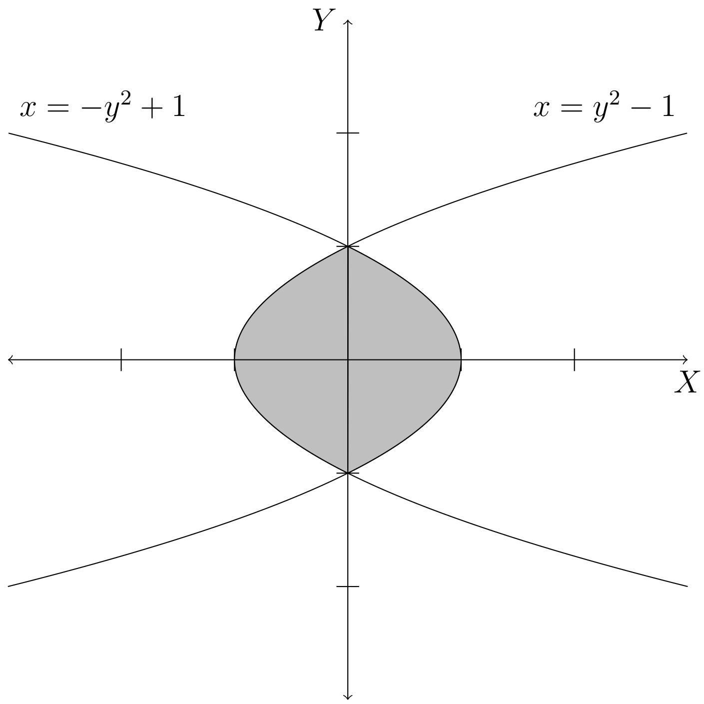


(3) **Theorem:** The area between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $y = b$ is $A =$ ________________________. Explain with an illustration.

(4) Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = \pi/2$, and $x = 3\pi/2$. You will need to graph these two functions on the given domain.

(5) Sometimes we encounter regions bounded by curves obtained when x is a function of y. In this case, if x_R denotes the right hand side curve, and x_L denotes the left hand side curve, then the area $A = \int_c^d (x_R - x_L)dy$ where c, d are limits for y.

(6) Find the area of the shaded region:



(7) Try as many problems (# 1-30) as time permits you from the textbook.

3. Volumes

(1) What is the formula for the volume of a right cyclinder? (Note, the base may be an irregular shape). Draw an illustration.

(2) What is the formula for the volume of a circular cylinder? Draw an illustration.

(3) What is the formula for the volume of a rectangular box (also called, a rectangular parallelepiped). Draw an illustration.

(4) Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \underline{\hspace{6cm}}.$$

Draw an illustration.

(5) Find the volume of a sphere of radius r. Draw an illustration.

(6) The above is an example of the volume of **solids of revolution**.

(7) Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{36 - x^2}$, $y = 0, x = 2, x = 4$, about the x-axis.

(8) Find the volume of the solid obtained by rotating the region bounded by $y = 1 + \sec x$, $y = 3$, about the line $y = 1$. Draw an illustration.

(9) Set up an integration to find the volume and draw an illustration each of the solid obtained by rotating the region bounded by $y = 0, y = \cos^2 x$, $-\pi/2 \leq x \leq \pi/2$

  (a) about the x-axis;

  (b) about the line $y = 1$.

(10) Find the volume of a solid S where the base of S is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis. The cross-sections perpendicular to the y-axis are squares.

(11) Try as many problems as time permits you to from the textbook on this section (# 1-34, 54-60).

4. Volumes by Cylindrical Shells

(1) Sometimes the methods of the previous section may not be feasible. So, we use the method of **cylindrical shells**.
(2) What is the formula for the volume of a cylinder? Draw an illustration.
(3) What is the formula for the volume of a cylindrical shell? Draw an illustration. Write in the form of $V =$ (circumference)$\times$ (height) $\times \Delta r$.
(4) **Theorem:** The volume of the solid obtained by rotating about the y-axis the region under the curve $y = f(x)$ from $x = a$ to $x = b$ is V = ...
Draw an illustration.
(5) Find the volume and draw an illusration of the solid S obtained by rotating the region bounded by
  (a) $y = \cos(x^2)$, $y = 0$, $x = 0$, $x = \sqrt{\frac{\pi}{2}}$ about the y-axis.
  (b) $y = x^2$, $y = 4x - x^2$ about the y-axis.
  (c) $x = y^2 + 1$, $x = 2$, about the line $y = -2$.
  (d) $x^2 - y^2 = 5$, $x = 3$, about the line $y = 5$.
(6) Try as many problems from the textbook as time permits you to (# 1-26).

5. Review Chapter 5

(1) Find the area of the region bounded by the curves and draw an illustration
  (a) $y = x^2$ and $y = 6x - x^2$.
  (b) $x + y = 0$ and $x = y^2 + 2y$.

(2) Find the volume of the solid obtained by rotating the region bounded by the given curves about the specific axis and draw an illustration
  (a) $y = x^2 + 1$ and $y = 4 - x^2$ about the line $y = -2$
  (b) $x^2 - y^2 = 25$, $x = 8$ about the y-axis.

(3) Find the volume of the solid S whose base is a square with vertices $(1, 0), (0, 1), (-1, 0)$, $(0, -1)$ and the cross-section perpendicular to the x-axis is a semicircle.

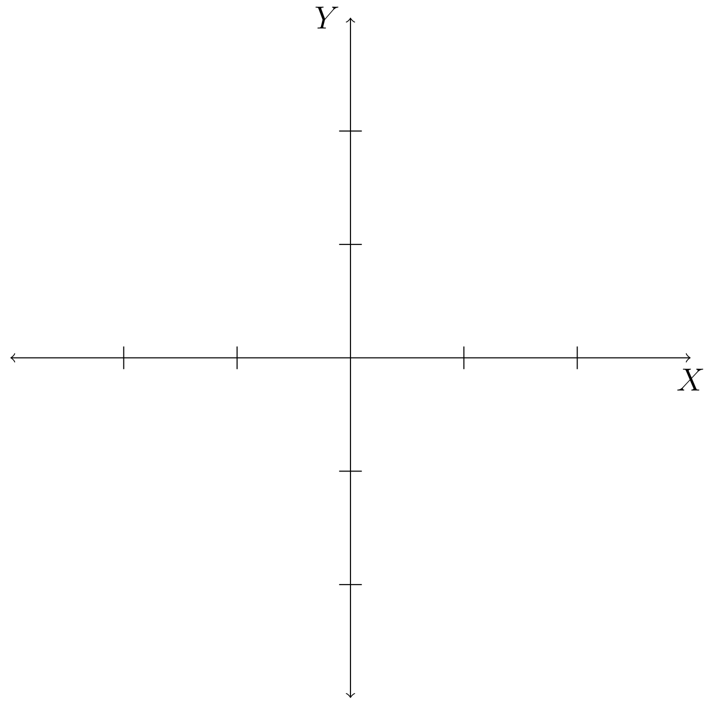
(4) Find the volume of the solid obtained by rotating the region bounded by the curves
  $y = \sqrt{x}, y = x^2$ about the line $y = 2$.

(5) Do as many problems from the textbook as time permits you to (Review section of chapter 5: # 1,2, 7-10,14,15, 22 -26).

6. Inverse Functions

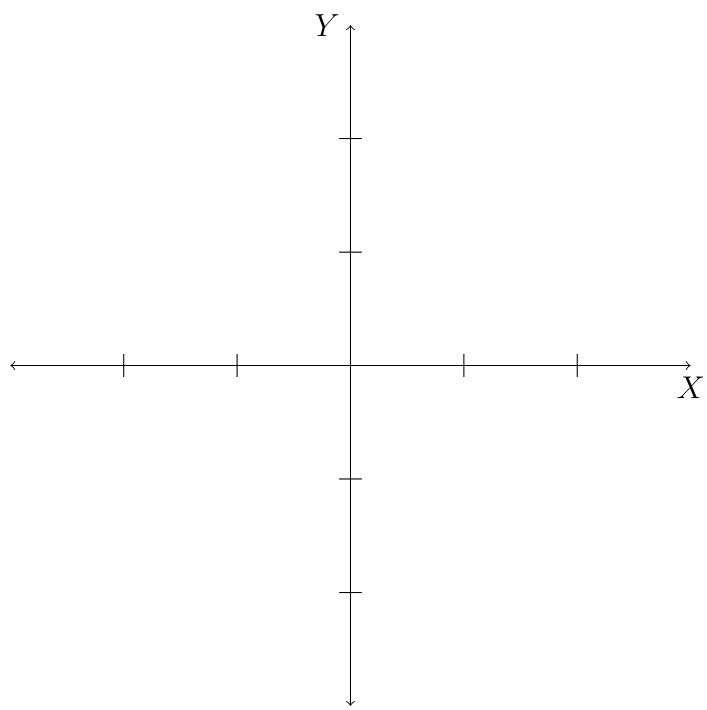
(1) Define a function.

(2) What is the vertical line test for a graph? Draw an illustration.



(3) Define a one-to-one function.

(4) What is the horizontal line test for a graph of a function? Explain with an illustration.



(5) Present an example of a function which is one-to-one. Explain why it is one-to-one.

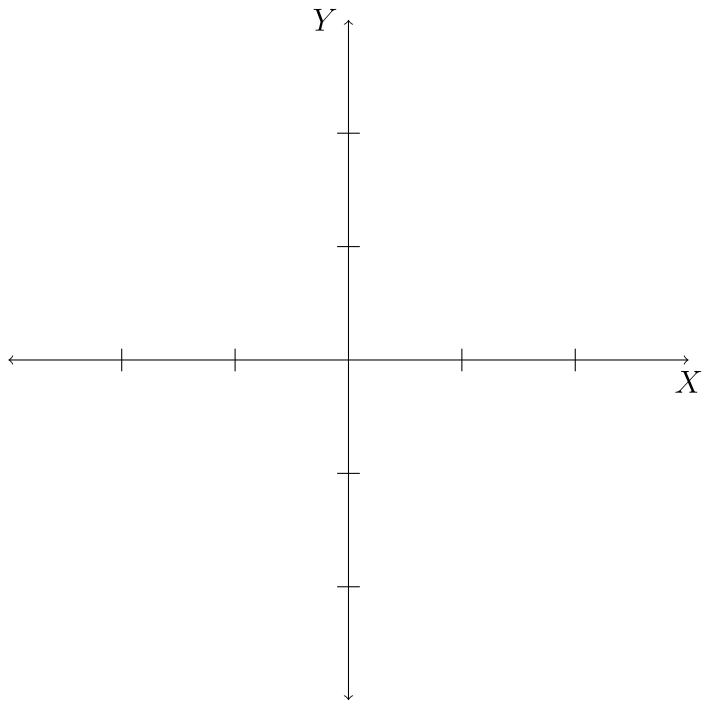
(6) Present an example of a function which is not one-to-one. Explain.

(7) Define the inverse of a one-to-one function. How are the domains and ranges of a one-to-one function and those of its inverse related? What happens if a function is not one-to-one? Why can its inverse be not defined?

(8) What are the cancellation equations for a function and its inverse? Explain with an example each.

(9) Present steps to find the inverse of a one-to-one function.

(10) Find the inverse function of $f(x) = \sqrt[3]{x-5}$. Check that your candidate inverse is indeed the inverse. Draw the graph $y = f(x)$ and the that of the inverse on the same coordinate plane.



(11) Find the inverse function of $f(x) = x^2 + 3$. Check that your candidate inverse is indeed the inverse. Draw the graph $y = f(x)$ and the that of the inverse on the same coordinate plane.

(12) **Theorem:** If f is a one-to-one function defined on an interval, then its inverse function f^{-1} is also continuous.

(13) **Theorem:** If f is a one-to-one differentiable function with the inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

Copy down the proof to this theorem from the textbook.

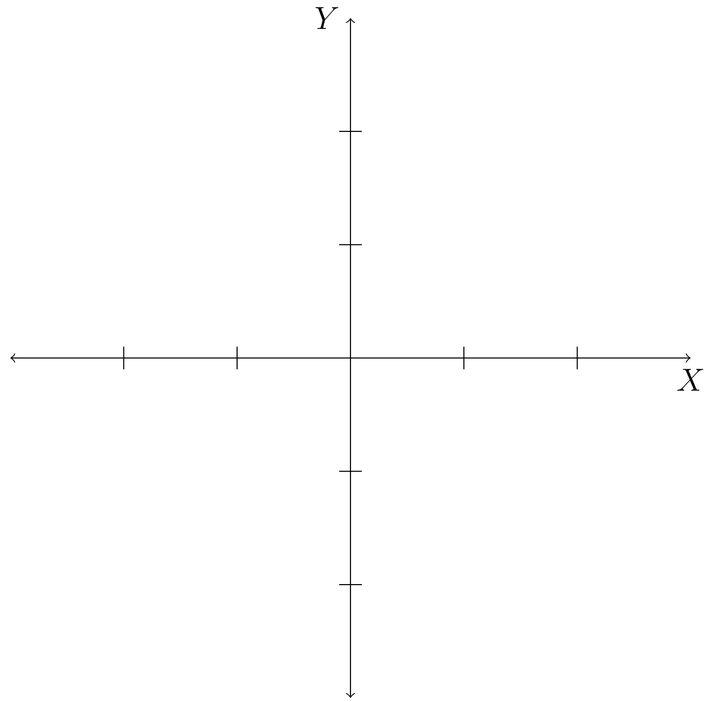
(14) Determine whether the following functions are one-to-one (draw graphs):
  (a) $y = 1 + \sin x$ for $0 \leq x \leq \pi$.
  (b) $y = 1 + \cos x$ for $0 \leq x \leq \pi$.
  (c) $y = |x|$ for $-2 \leq x \leq 2$
  (d) $y = |x|$ for $0 \leq x \leq 2$
  (e) $f(t)$ is your height at age t.

(15) Find the inverse of
  (a) $f(x) = \dfrac{3x-5}{4x+1}$
  (b) $f(x) = x^2 - 10x$ for $x \geq 5$.

(16) Show that f is one-to-one. Find $(f^{-1})'(a)$. Calculate f^{-1}. Find the domain and range for both f and f^{-1}. Calculate $(f^{-1})'$ and check that it agrees with your previous calculation. Sketch the graph of f and f' on the same coordinate plane.

(a) $f(x) = \dfrac{1}{x-2}$, $a = 4$

(b) $f(x) = \sqrt{x-3}$, $a = 4$.



(17) Find $(f^{-1})'(a)$:

(a) $f(x) = 5x^3 - 4x^2 + 11x + 9$, $a = 9$.

(b) $f(x) = \sqrt{3x^3 + 5x^2 + 3x + 5}$, $a = 4$.

(18) Do as many problems from the textbook as time permits you to (section 6.1: # 1-16, 22-28, 34-44).

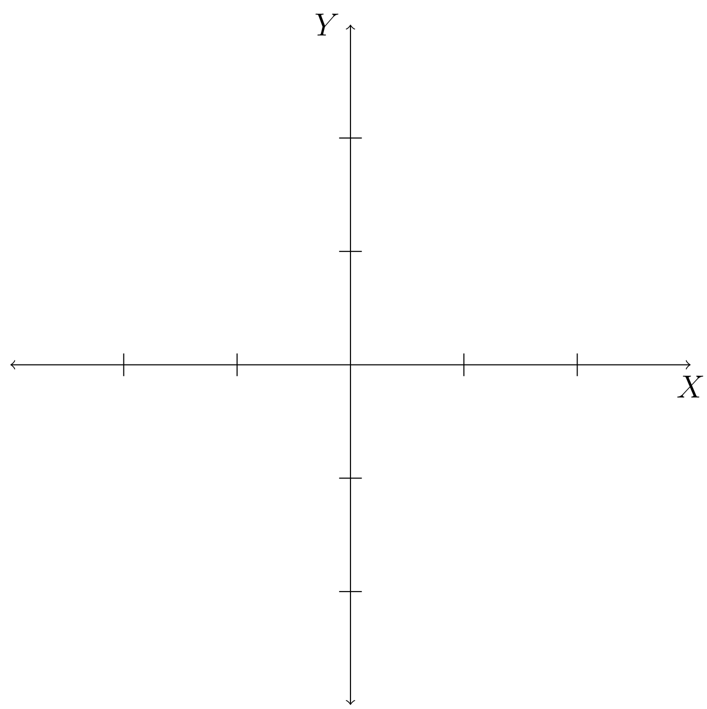
7. Exponential Functions and their derivaties

(1) Recall the definition of an exponential function. First, the basics. Let a be a positive real number. Then
   (a) for n a positive integer, $a^n =$ ________________________;
   (b) for $n = 0$, $a^n =$ ________________________;
   (c) for n a negative integer, say $n = -k$, $a^n =$ ________________________;
   (d) for $x = \frac{p}{q}$ a rational number, $a^x =$ ________________________;
   (e) for x an irrational number, $a^n =$ ________________________. Explain in detail.

(2) Properties of exponents: For any real numbers x, y and a, b positive real numbers,
   (a) $a^{x+y} =$ ________________________;
   (b) $a^{x-y} =$ ________________________;
   (c) $(a^x)^y =$ ________________________;
   (d) $(ab)^x =$ ________________________;
   (e) $\left(\frac{a}{b}\right)^x =$ ________________________;
   (f) $0^0 =$ ________________________.

(3) Let $f(x) = b^x$. Here b is called ________________________, and x is called the ________________________.
   (a) What happens when $b < 0$?
   (b) What happens when $b = 0$?
   (c) What happens with $b = 1$?

(4) Draw the graph of the function $f(x) = 2^x$.



(5) Draw the graph of the function $f(x) = 3^x$.

(6) Draw the graph of the function $f(x) = \left(\frac{1}{2}\right)^x$

(7) Graph the function $f(x) = \left(\frac{1}{3}\right)^x$.

(8) For $b > 1$, the function $f(x) = b^x$ is an ________________________ function.

(9) For $0 < b < 1$, the function $f(x) = b^x$ is a ________________________ function.

(10) The horizontal asymptote for the graph of $f(x) = b^x$, $b > 0, b \neq 1$ is ________________________.

(11) The y-intercept for the graph of $f(x) = b^x$, $b > 0, b \neq 1$ is ________________________.

(12) If $a > 1$, then $\lim_{x\to\infty} a^x =$ ________________________
and $\lim_{x\to-\infty} a^x =$ ________________________.

(13) If $0 < a < 1$, then $\lim_{x\to\infty} a^x =$ ________________________
and $\lim_{x\to-\infty} a^x =$ ________________________.

(14) Find $\lim_{x\to\infty}(3^{-2x} + 2)$.

(15) For the exponential function $f(x) = a^x$, $a > 0, a \neq 1$, use limits to describe
   (a) $f'(0)$
   (b) $f'(x)$

   Therefore, $f'(x) = f(x) \cdot f'(0)$.

   Geometrically, $f'(0)$ is ____
   ____.

(16) Define the Euler number e.

   $e \approx 2.718$.

(17) Let $f(x) = e^x$. Then
   (a) $f'(0) =$ ________________________.
   (b) $f'(x) =$ ________________________.
   (c) That is, $\dfrac{d}{dx}(e^x) =$ ________________________.

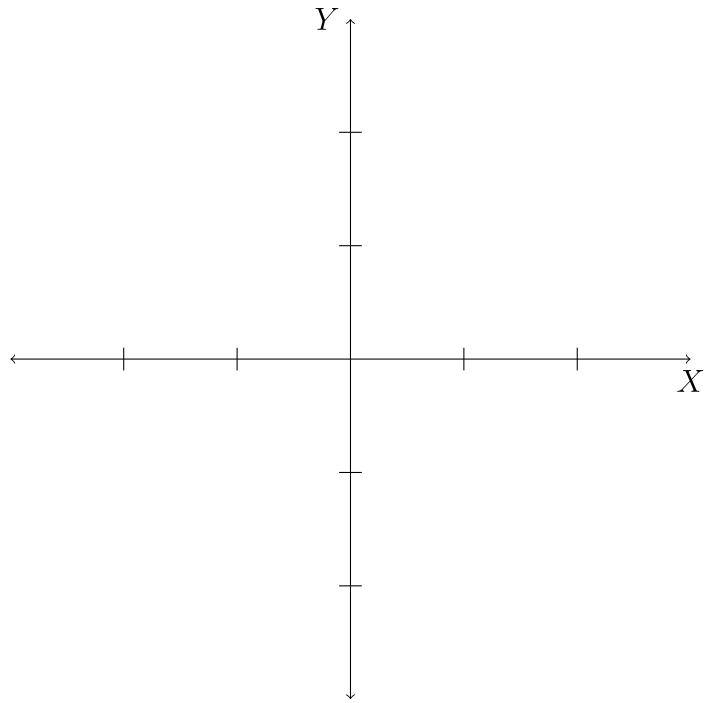
(18) Let $u = u(x)$. Then by the Chain Rule, $\dfrac{d}{dx}(e^u) =$ ________________________.

(19) Find the derivatives of:
   (a) $y = e^{x^2}$
   (b) $y = e^{\cos x}$
   (c) $y = \sin(x)e^{x^3}$
   (d) $y = e^{x^5}e^{x^2}$

(20) Since $\dfrac{d}{dx}(e^x) = e^x$, we have $\int e^x dx =$ ________________________.

(21) Find $\int \cos(x)e^{\sin(x)}dx$.

(22) Graph $f(x) = e^x$.



(23) Let $f(x) = e^x$. Then,

(a) it is an ________________________ function.

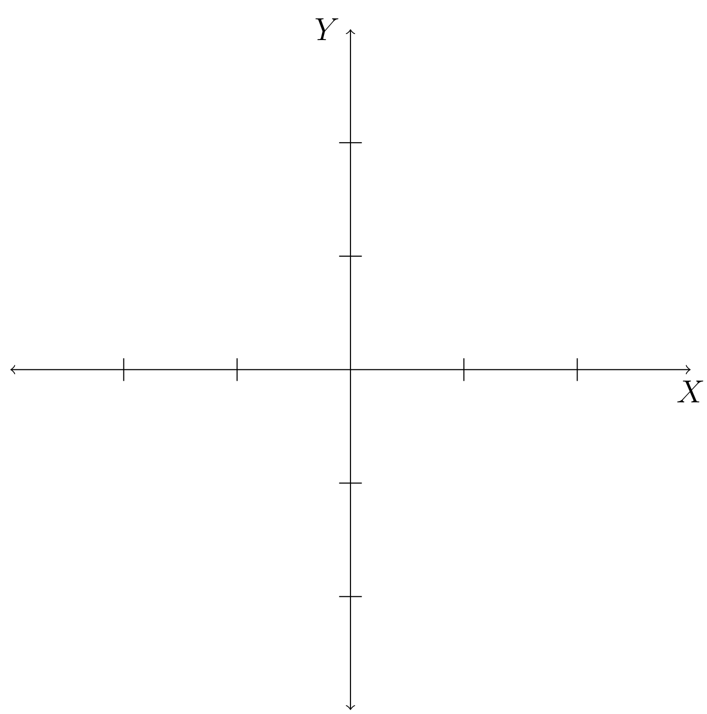
(b) $\lim_{x\to\infty} f(x) =$ ________________________

and $\lim_{x\to-\infty} f(x) =$ ________________________.

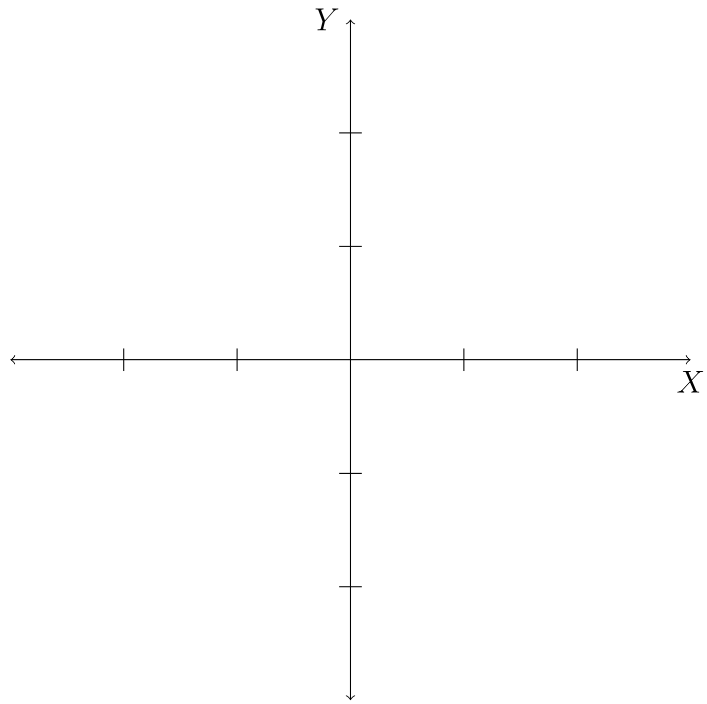
(24) Find $\lim_{x\to\infty} \dfrac{e^{3x}}{1+e^{3x}}$ and $\lim_{x\to-\infty} \dfrac{e^{3x}}{1+e^{3x}}$.

(25) Use transformations to graph:

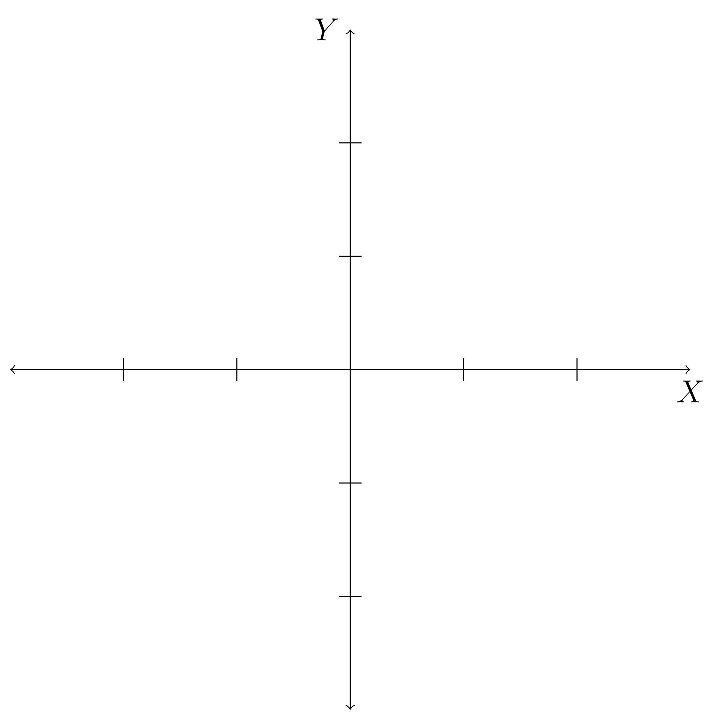
- $f(x) = e^x - 1$



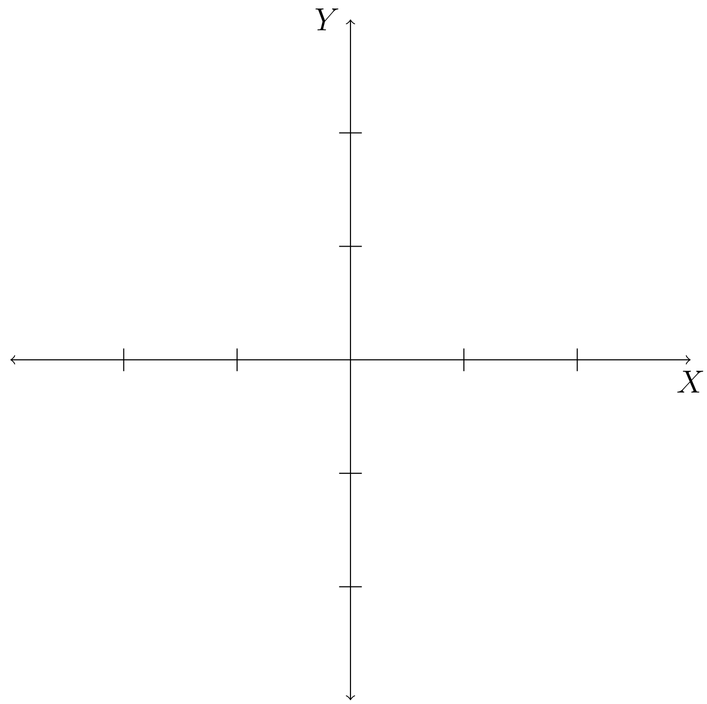
- $f(x) = e^{x-1}$



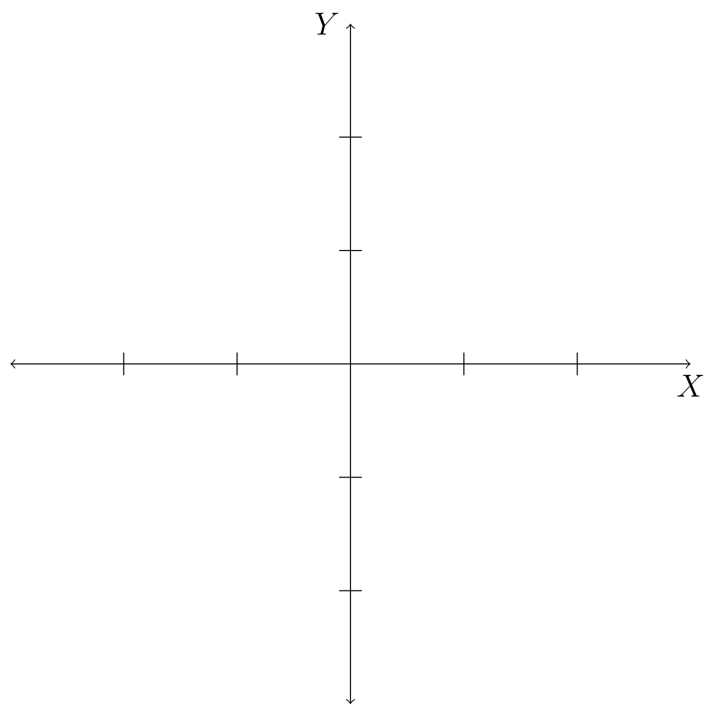
- $f(x) = e^{-x}$



- $f(x) = -e^x$



- $f(x) = e^{|x|}$



(26) Start with the graph of $f(x) = 3^x$. Write the equation of the graph that results from

(a) shifting 4 units downward;

(b) shifting 4 units upward;

(c) shifting 4 units left;

(d) shifting 4 units right;

(e) reflecting about the x-axis;

(f) reflecting about the y-axis.

(27) Find the following limits:
 (a) $\lim_{x\to\infty}(0.99)^x$
 (b) $\lim_{x\to 3^-} e^{\frac{1}{3-x}}$
 (c) $\lim_{x\to 3^+} e^{\frac{1}{3-x}}$
 (d) $\lim_{x\to\infty} \dfrac{e^{4x}-e^{-4x}}{e^{4x}+e^{-4x}}$

(28) Find the derivative of:
 (a) $f(x) = e^{4x^2+8x}$
 (b) $f(x) = e^x + x^e$
 (c) $f(x) = \dfrac{e^x}{1+e^x}$
 (d) $f(x) = e^{2x}\sin(x)$
 (e) $f(x) = \sqrt{x^2+4x+e^{3x}}$

(29) Find the integrals
 (a) $\int x^3 e^{x^4} dx$
 (b) $\int \dfrac{(1+e^x)^2}{e^{4x}} dx$
 (c) $\int_0^1 \dfrac{\sqrt{1+e^{-x}}}{e^x} dx$

(30) Find the volume of the solid obtained by rotating the region bounded by the curves $y = e^x, y = 0, x = 0, x = 1$ about
 (a) the x-axis;
 (b) the y-axis.

(31) Do as many problems from the textbook as time permits you to (section 6.2: # 1,2, 6-12, 22-50, 78-90).

8. Logarithmic functions

(1) The exponential function $f(x) = a^x$, $a > 0, a \neq 1$ is a one-to-one function. The inverse function of this exponential function is called, the logarithmic function, denoted $\log_a(x)$. That is,
$\log_a(x) = y \iff$ ____________________.

(2) The domain of $f^{-1}(x) = \log_a(x)$ is ____________________.

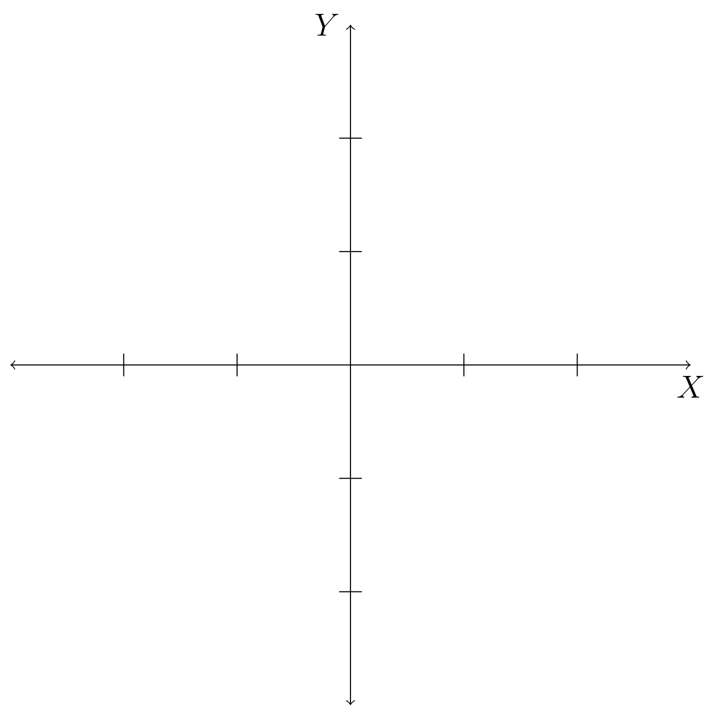
(3) The range of $f^{-1}(x) = \log_a(x)$ is ____________________.

(4) $f \circ f^{-1}(x) = x \Longrightarrow$ ____________________.

(5) $f^{-1} \circ f(x) = x \Longrightarrow$ ____________________.

(6) Prove the following:
   (a) $\log_a(xy) = \log_a(x) + \log_a(y)$
   (b) $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
   (c) $\log_a(x^r) = r \log_a(x)$

(7) Fix $a > 1$. Graph $f(x) = a^x$ and $f^{-1}(x) = \log_a(x)$ on the same coordinate plane.



(8) For $a > 1$, we have $\lim_{x\to\infty} \log_a(x) =$ ____________________ and $\lim_{x\to 0^+} \log_a(x) =$ ____________________.

(9) Find the values:
   (a) $\log_4(64)$
   (b) $\log_{64}(4)$
   (c) $\log_3(405) - \log_3(5)$
   (d) $\log_6(2) + \log_6(3)$

(10) The logarithm function with base e is called the ______________________________ and is denoted by ____________________.

(11) Therefore, $\ln(x) = y \iff$ ______________________________.

(12) We have $\ln(e^x) =$ ____________________ for $x \in$ ____________________.

(13) We have $e^{\ln(x)} =$ ____________________ for $x \in$ ____________________.

(14) $\ln(e) =$ ____________________.

(15) Solve for x:

(a) $\ln(x) = 4$

(b) $\ln(3x + 4) = 7$

(c) $e^{4x+5} = 7$

(16) Express as a single logarithm:

(a) $\ln(x) + 3\ln(y) - \frac{1}{2}\ln(z)$

(b) $4(\frac{1}{3}\ln(a) - \ln(b) + 3\ln(x))$

(17) Use the formula $\log_a(t) = \dfrac{\ln(t)}{\ln(a)}$ to find an approximate value of $\log_3(5)$.

(18) We have $\lim_{x\to\infty}\ln(x) =$ ____________________ and $\lim_{x\to 0^+}\ln(x) =$ ____________________

(19) Find the limits:

(a) $\lim_{x\to\infty}[\ln(5+x) - \ln(4+x)]$

(b) $\lim_{x\to 0}\ln(\sin(x) + 1)$.

(c) $\lim_{x\to 5^+}\ln(3x - 15)$

(20) Solve for x:

(a) $\ln(x) + \ln(x - 3) = \ln 40$

(b) $e^{2x} - 4e^x - 60 = 0$

(21) Express as a single logarithm: $\frac{1}{4}\ln(y+7)^3 - 4\ln(y^2 + 2y - 3)$

(22) Expand the expression: $\ln(a^5\sqrt{b^3\sqrt[4]{c^7}})$.

(23) Do as many problems from the textbook as time permits you to (section 6.3: # 1-18, 26-36, 46-52).

9. Derivatives of Logarithmic Functions

(1) Complete and prove the formula: $\dfrac{d}{dx}(\ln x) =$ ____________.

(2) Find the following derivatives:

(a) $\dfrac{d}{dx}\ln(\cos(x))$

(b) $\dfrac{d}{dx}\ln(4x^2+5x+7)$

(c) $\dfrac{d}{dx}\ln\left(\dfrac{x+4}{x-7}\right)$

(d) $\dfrac{d}{dx}\ln(|x|)$

(3) Since $\dfrac{d}{dx}\ln(|x|) =$ ____________, we have the integration,

____.

(4) Recall the integration formula for $n \neq -1$ (why?) $\int x^n dx =$ ____________.

(5) Find the following integrals:

(a) $\int \dfrac{x^2}{x^3+1}dx$

(b) $\int \dfrac{\ln(x)}{x}dx$

(c) $\int \tan(x)dx$

(6) Complete and prove the following formulae:

(a) $\dfrac{d}{dx}\log_a x =$ ________________________.

(b) For $a \neq 1$, $\dfrac{d}{dx}a^x =$ ________________________. What happens when $a = 1$?

(c) For $a \neq 1$, $\int a^x dx =$ ________________________

(7) Find the derivatives of

(a) $y = \ln(x\sqrt{x^2+5x})$

(b) $y = \ln(\sin(\ln x))$

(8) State the steps in logarithmic differentiation.

(9) Use logarithmic differentiation to find $\frac{dy}{dx}$:

(a) $y = \frac{(x^2+1)^{\frac{3}{4}}\sqrt{4x+5}}{(3x-7)^{\frac{1}{3}}}$
(b) $y = x^{\sqrt{x}}$
(c) $y = \frac{(x^3+4x^2)\sin^2(x)}{e^{-2x}(x+1)}$
(d) $y = x^{\sin(x)}$
(e) $y = (\cos(x))^{\ln(x)}$
(f) $y = (\ln(x))^{\sqrt{x}}$

(10) Find the integrals of

(a) $\int x^3 3^{x^4} dx$
(b) $\int \frac{\cos(\ln x)}{x} dx$
(c) $\int \frac{dx}{3x+8}$
(d) $\int \frac{e^x}{e^x+2} dx$
(e) $\int \frac{\sin(x)}{4+\cos(x)} dx$

(11) Do as many problems from the textbook as time permits you to (section 6.4: # 1-30, 42-54, 70-82).

10. Inverse Trigonometric Function

(1) Define the **Inverse Sine** function (or the arcsine function). Draw the relevant graphs and explain the range and domain for each.

(2) $\sin^{-1} x = y \iff$ ____________________ for __________ $\leq x \leq$ __________ and __________ $\leq y \leq$ __________

.

(3) $\sin^{-1}(\sin(x)) =$ __________ for __________ $\leq x \leq$ __________.

(4) $\sin(\sin^{-1}(x)) =$ __________ for __________ $\leq x \leq$ __________.

(5) State and prove the formula for $\dfrac{d}{dx}(\sin^{-1}(x))$ for __________ $\leq x \leq$ __________.

(6) Let $f(x) = \sin^{-1}(x^2 - 9)$. Find
   (a) Domain of $f(x)$
   (b) The derivative $f'(x)$
   (c) Domain of $f'(x)$

(7) Define the **Inverse Cosine** function (or the arccosine function). Draw the relevant graphs and explain the range and domain for each.

(8) $\cos^{-1} x = y \iff$ ____________________ for __________ $\leq x \leq$ __________ and __________ $\leq y \leq$ __________

.

(9) $\cos^{-1}(\cos(x)) =$ __________ for __________ $\leq x \leq$ __________.

(10) $\cos(\cos^{-1}(x)) =$ __________ for __________ $\leq x \leq$ __________.

(11) State and prove the formula for $\dfrac{d}{dx}(\cos^{-1}(x))$ for __________ $\leq x \leq$ __________.

(12) Let $f(x) = \cos^{-1}(2x - 3)$. Find
   (a) The domain of $f(x)$
   (b) The derivative $f'(x)$
   (c) The domain of $f'(x)$

(13) Define the **Inverse Tangent** function (or the arctan function). Draw the relevant graphs and explain the range and domain for each.

(14) $\tan^{-1} x = y \iff$ ____________________ for __________ $\leq x \leq$ __________ and __________ $\leq y \leq$ __________

.

(15) $\tan^{-1}(\tan(x)) =$ __________ for __________ $\leq x \leq$ __________.

(16) $\tan(\tan^{-1}(x)) =$ __________ for __________ $\leq x \leq$ __________.

(17) State and prove the formula for $\dfrac{d}{dx}(\tan^{-1}(x))$ for __________ $\leq x \leq$ __________.

(18) State and prove the formula for $\dfrac{d}{dx}(\csc^{-1}(x))$ for __________ $\leq x \leq$ __________.

(19) State and prove the formula for $\dfrac{d}{dx}(\sec^{-1}(x))$ for __________ $\leq x \leq$ __________.

(20) State and prove the formula for $\dfrac{d}{dx}(\cot^{-1}(x))$ for __________ $\leq x \leq$ __________.

(21) Fill in the table:

Function $f(x)$	Derivative $f'(x)$	Function $f(x)$	Integral $\int f(x)dx$
x^n			
		$\frac{1}{x}$	
$\cos(x)$		$\sin(x)$	
$\sin(x)$			
$\tan(x)$		$\sec^2(x)$	
$\csc(x)$			
$\sec(x)$			
$\cot(x)$			

Function $f(x)$	Derivative $f'(x)$	Function $f(x)$	Integral $\int f(x)dx$
e^x			
a^x			
$\ln(x)$			
$\log_a(x)$			
$\sin^{-1}(x)$			
$\tan^{-1}(x)$			
$\sec^{-1}(x)$			

(22) Find the exact values:
  (a) $\tan^{-1}(\frac{1}{\sqrt{3}})$
  (b) $\sec^{-1}(\sqrt{2})$
  (c) $\cot^{-1}(-\sqrt{3})$
  (d) $\arcsin(\frac{1}{2})$
  (e) $\tan(\sin^{-1}(\frac{3}{4}))$ (Use right triangle)
  (f) $\tan^{-1}(\cos^{-1}(\frac{1}{4}))$
  (g) $\sin(\arccos(\frac{2}{3}))$
  (h) $\sin(\tan^{-1}(4) + \tan^{-1}(5))$

(23) Find the derivative of:
  (a) $f(x) = \sin^{-1}(\tan^{-1}(x))$
  (b) $f(x) = \cos^{-1}(\sqrt{\sin(x)})$
  (c) $f(x) = \tan^{-1}(\sin^{-1}(\cos^{-1}(x)))$
  (d) $f(x) = \cos^{-1}(4x^3 - 7x^2 + 8x)$

(24) Find the limits
  (a) $\lim_{x\to\infty} \tan^{-1}(4x^3 + 5x + 2)$
  (b) $\lim_{x\to 3^+} \tan^{-1}(\ln(x-3))$

(25) Find the integrals:
  (a) $\int \frac{1}{\sqrt{9-x^2}} dx$
  (b) $\int \frac{1}{x^2+16} dx$
  (c) $\int \frac{1}{x\sqrt{x^2-9}} dx$
  (d) $\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{1+\sin^2(x)} dx$
  (e) $\int \frac{e^{3x}}{\sqrt{1-e^{6x}}} dx$
  (f) $\int \frac{x^2}{1+x^6} dx$

(26) Do as many problems from the textbook as time permits you to (section 6.6: # 4-14, 22-36, 42-46, 58-70).

11. Hyperbolic Functions

(1) Define the **Hyperbolic functions**
   (a) $\sinh(x)$
   (b) $\cosh(x)$
   (c) $\tanh(x)$
   (d) $\operatorname{csch}(x)$
   (e) $\operatorname{sech}(x)$
   (f) $\coth(x)$

(2) Graph the functions $y = \sinh(x)$, $y = \cosh(x)$, $y = \tanh(x)$ on three separate coordinate systems.

(3) Prove the following Hyperbolic identities:
   (a) $\sinh(-x) = -\sinh(x)$
   (b) $\cosh(-x) = -\cosh(x)$
   (c) $\cosh^2(x) - \sinh^2(x) = 1$
   (d) $1 - \tanh^2(x) = \operatorname{sech}^2(x)$
   (e) $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$
   (f) $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

(4) Complete and prove the formulae:
   (a) $\dfrac{d}{dx}(\sinh(x))$
   (b) $\dfrac{d}{dx}(\cosh(x))$
   (c) $\dfrac{d}{dx}(\tanh(x))$
   (d) $\dfrac{d}{dx}(\operatorname{csch}(x))$
   (e) $\dfrac{d}{dx}(\operatorname{sech}(x))$
   (f) $\dfrac{d}{dx}(\coth(x))$

(5) Complete and prove the following formulae:
   (a) $\sinh^{-1}(x) =$ $\qquad$ $x \in$
   (b) $\cosh^{-1}(x) =$ $\qquad$ $x \in$
   (c) $\tanh^{-1}(x) =$ $\qquad$ $x \in$

(6) Complete and prove the following formulae:
   (a) $\dfrac{d}{dx}(\sinh^{-1}(x))$
   (b) $\dfrac{d}{dx}(\cosh^{-1}(x))$
   (c) $\dfrac{d}{dx}(\tanh^{-1}(x))$
   (d) $\dfrac{d}{dx}(\operatorname{csch}^{-1}(x))$
   (e) $\dfrac{d}{dx}(\operatorname{sech}^{-1}(x))$
   (f) $\dfrac{d}{dx}(\coth^{-1}(x))$

(7) Prove the identities:
   (a) $\cosh(x) - \sinh(x) = e^{-x}$
   (b) $\dfrac{1+\tanh(x)}{1-\tanh(x)} = e^{2x}$

(8) Find the limits:
   (a) $\lim_{x\to\infty} \sinh(x)$
   (b) $\lim_{x\to-\infty} \sinh(x)$
   (c) $\lim_{x\to\infty} \cosh(x)$
   (d) $\lim_{x\to-\infty} \cosh(x)$
   (e) $\lim_{x\to\infty} \tanh(x)$
   (f) $\lim_{x\to-\infty} \tanh(x)$
   (g) $\lim_{x\to\infty} \operatorname{csch}(x)$
   (h) $\lim_{x\to-\infty} \operatorname{csch}(x)$
   (i) $\lim_{x\to 0^+} \operatorname{csch}(x)$
   (j) $\lim_{x\to 0^-} \operatorname{csch}(x)$
   (k) $\lim_{x\to\infty} \operatorname{sech}(x)$
   (l) $\lim_{x\to-\infty} \operatorname{sech}(x)$
   (m) $\lim_{x\to\infty} \coth(x)$
   (n) $\lim_{x\to-\infty} \coth(x)$
   (o) $\lim_{x\to 0^+} \coth(x)$
   (p) $\lim_{x\to 0^-} \coth(x)$

(9) Fill in the table:

Function	Derivative	Function	Integral
$x^n, n \neq 1$			
		$\frac{1}{x}$	
$\cos(x)$		$\sin(x)$	
$\sin(x)$			
$\tan(x)$		$\sec^2(x)$	
$\csc(x)$			
$\sec(x)$			
$\cot(x)$			
e^x			
a^x			
$\ln(x)$			
$\log_a(x)$			
$\sin^{-1}(x)$			
$\tan^{-1}(x)$			
$\sec^{-1}(x)$			
$\sinh(x)$			
$\cosh(x)$			
$\tanh(x)$			
$\operatorname{csch}(x)$			
$\operatorname{sech}(x)$			
$\coth(x)$			
$\sinh^{-1}(x)$			
$\cosh^{-1}(x)$			
$\tanh^{-1}(x)$			
$\operatorname{csch}^{-1}(x)$			
$\operatorname{sech}^{-1}(x)$			
$\coth^{-1}(x)$			

(10) Find the derivatives:

(a) $\dfrac{d}{dx}(\cosh(\ln x))$

(b) $\dfrac{d}{dx}(\sinh(1+x^2))$

(c) $\dfrac{d}{dx}(\cosh(x)\sinh(x))$

(d) $\dfrac{d}{dx}(\cosh^{-1}(3x+5))$

(e) $\dfrac{d}{dx}(\sinh^{-1}(4x^2-5))$

(f) $\dfrac{d}{dx}(\tanh^{-1}(e^{2x^3}))$

(11) Find the integrals:

(a) $\int \cosh(3x+4)dx$

(b) $\int \tanh(5x-7)dx$

(c) $\int \dfrac{\cosh(x)}{3+\sinh(x)}dx$

(d) $\int \dfrac{1}{\sqrt{9+x^2}}dx$

(12) Do as many problems as time permits you to from the textbook (section 6.7: # 6 -24, 30-46, 58-68).

12. Intermediate forms and L'Hospital's Rule

(1) State L'Hospital's Rule.

(2) Use L'Hospital's rule when appropriate. When not appropriate, say so.

  (a) $\lim_{x\to 3} \dfrac{x^2+2x-15}{x-3}$

  (b) $\lim_{x\to 0} \dfrac{1-\cos(x)}{\sin(x)}$

  (c) $\lim_{x\to\infty} \dfrac{e^{3x}}{x^3}$

  (d) $\lim_{x\to 0} \dfrac{\sinh(x)-x}{x^2}$

  (e) $\lim_{x\to 0} \dfrac{x}{\tan^{-1}(4x)}$

  (f) $\lim_{x\to\infty} \sqrt{x}e^{-\frac{x}{2}}$

  (g) $\lim_{x\to\infty} x\sin(\frac{1}{x})$

  (h) $\lim_{x\to 0}(\cot(x)-\frac{1}{x})$

  (i) $\lim_{x\to 0}(\csc(x)-\cot(x))$

  (j) $\lim_{x\to 0^+}(\tan(2x))^x$

  (k) $\lim_{x\to\infty}(e^x+1)^{\frac{1}{x}}$

(3) Prove that $\lim_{x\to\infty} \dfrac{e^x}{x^n} = \infty$ for any natural number n.

(4) Do as many problems as time permits you to from the textbook (section 6.7: # 1 -66, 70-74).

13. Review Chapter 6

(1) Graph
  (a) $y = -e^{x+3}$
  (b) $y = \ln(x+1)$

(2) Find the exact value:
  (a) $\log 5 + \log 6 - \log 3$
  (b) $\cot(\sin^{-1}(\sqrt{3}/2))$

(3) Solve for x:
  (a) $\ln(2 + e^{-x}) = 4$
  (b) $\cos(x) = \sqrt{2}/2$

(4) Differentiate:
  (a) $y = (\arcsin(4x^2 + 5x))^3$
  (b) $y = \log_3(1 + x^4)$
  (c) $y = (\sin(x))^x$
  (d) $y = \dfrac{(x^3 + 3x^2)^4}{(3x^2 + 5)^3(2x + 5)^2}$
  (e) $y = x^2 \tanh^{-1}(x)$

(5) Find the limits:
  (a) $\lim_{x\to\infty} \arctan(x^3 - x)$
  (b) $\lim_{x\to\infty} \left(3 + \dfrac{5}{x}\right)^x$
  (c) $\lim_{x\to 0^+} x^2 \ln(x)$

(6) Evaluate the integrals:
  (a) $\int_0^1 ye^{-3y^2} dy$
  (b) $\int \tan(x) \ln(\cos(x)) dx$
  (c) $\int 2^{\tan x} \sec^2(x) dx$

(7) Fill in the table:

Function	Derivative	Function	Integral
x^n			
		$\frac{1}{x}$	
$\cos(x)$			
$\sin(x)$			
$\tan(x)$			
$\csc(x)$			
$\sec(x)$			
$\cot(x)$			
e^x			
a^x			
$\ln(x)$			
$\log_a(x)$			
$\sin^{-1}(x)$			
$\tan^{-1}(x)$			
$\sec^{-1}(x)$			
$\sinh(x)$			
$\cosh(x)$			
$\tanh(x)$			
$\mathrm{csch}(x)$			
$\mathrm{sech}(x)$			
$\coth(x)$			
$\sinh^{-1}(x)$			
$\cosh^{-1}(x)$			
$\tanh^{-1}(x)$			
$\mathrm{csch}^{-1}(x)$			
$\mathrm{sech}^{-1}(x)$			
$\coth^{-1}(x)$			

(8) Do as many problems as time permits you to from the textbook (Review section from Chapter 6: # 4 -48, 62-78, 92-106).

14. Integration by Parts

(1) State and prove the formula for integration by parts.

(2) Find the integrals:

(a) $\int x\cos(x)dx$

(b) $\int x^2\cos(x)dx$

(c) $\int ln(x)dx$

(d) $\int e^x\cos(x)dx$

(e) $\int \sin^{-1}(x)dx$

(f) $\int p^5\ln(p)dp$

(g) $\int t\sinh(5t)dt$

(h) $\int t\sinh(mt)dt$

(i) $\int t^3e^{-t^2}dt$

(3) Prove: $\int x^ne^xdx = x^ne^x - n\int x^{n-1}e^xdx$

(4) Prove: $\int(\ln(x))^ndx = x(\ln(x))^n - n\int(\ln(x))^{n-1}dx$

(5) Prove: $\int\sec^n(x)dx = \dfrac{\tan(x)\sec^{n-2}(x)}{n-1} + \dfrac{n-2}{n-1}\int\sec^{n-2}(x)dx$ for $n \neq 1$.

(6) Prove: $\int\tan^n(x)dx = \dfrac{\tan^{n-1}(x)}{n-1} - \int\tan^{n-2}(x)dx$ (for $n \neq 1$).

(7) Do as many problems as time permits you to from the textbook (Section 7.1: # 1-54).

15. Trigonometric Integrals

(1) Find the integrals (note, one of the exponents is odd – what is the strategy here?)
   (a) $\int \sin^3(x)dx$
   (b) $\int \cos^3(x)\sin^2(x)dx$

(2) When the exponents of sin and cos are both even, use:

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x));$$
$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x));$$
$$\text{Sometimes we will need, } \sin(x)\cos(x) = \frac{1}{2}\sin(2x).$$

Prove these formulae.

(3) Find:
   (a) $\int \sin^2(x)\cos^5(x)dx$
   (b) $\int \cos^2(x)\sin^5(x)dx$
   (c) $\int \sin^2(x)\cos^2(x)dx$
   (d) $\int \cos^4(x)dx$
   (e) $\int \cos^5(x)dx$
   (f) $\int \sin^3(x)\cos^3(x)dx$

(4) Recall the formulae:
   (a) $\tan(x)dx =$
   (b) $\sec(x)dx =$
   (c) $\tan^n(x)dx =$
   (d) $\sec^n(x)dx =$
   (e) $\ln(|x|) =$

(5) Find the integrals:
   (a) $\int \tan^2(x)dx$
   (b) $\int \sec^2(x)dx$
   (c) $\int \tan^3(x)dx$
   (d) $\int \sec^3(x)dx$
   (e) $\int \sec^4(x)dx$
   (f) $\int \sec^5(x)dx$
   (g) $\int \tan(x)\sec(x)dx$
   (h) $\int \tan^2(x)\sec(x)dx$
   (i) $\int \tan(x)\sec^2(x)dx$
   (j) $\int \tan^3(x)\sec^5(x)dx$ (power of tan and sec are both odd, and power of tan is greater than 1. In this case, let $u = \sec(x)$).
   (k) $\int \tan^4(x)\sec^6(x)dx$ (power of sec is even, and power of tan is greater than 0. In this case, let $u = \tan(x)$).
   (l) $\int \tan^5(x)\sec^6(x)dx$

(6) Complete and prove the formulae:
   (a) $\sin(A)\cos(B) =$
   (b) $\sin(A)\sin(B) =$
   (c) $\cos(A)\cos(B) =$
(7) Find $\int \sin(5x)\cos(4x)dx$
(8) Find $\int \sin(5x)\sin(4x)dx$
(9) Find $\int \cos(5x)\cos(4x)dx$
(10) Do as many problems as time permits you to from the textbook (Section 7.2: # 1 -32).

16. Trigonometric Substitutions

(1) Here are some trigonometric substitutions:

Expression	Substitution	Interval	Identity
$\sqrt{a^2 - x^2}$			
$\sqrt{a^2 + x^2}$			
$\sqrt{x^2 - a^2}$			

(2) Find the integrals:

(a) $\int \frac{x^2}{\sqrt{36 - x^2}} dx$

(b) $\int \frac{x^2}{\sqrt{36 + x^2}} dx$

(c) $\int \frac{x}{\sqrt{x^2 - 4}} dx$

(d) $\int \frac{dx}{\sqrt{x^2 - 10x + 26}}$

(e) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1 + \sin^2 x}} dx$

(3) Do as many problems as time permits you to from the textbook (Section 7.3: # 1-30).

17. Integration of Rational Functions by Partial Fractions

(1) Evaluate the integrals:
   (a) $\int \frac{2t-3}{t+5}dt$
   (b) $\int \frac{x^2+x-5}{x^2+2x-35}dx$
   (c) $\int \frac{dx}{x^2(x-2)^2}$
   (d) $\int \frac{3x^2+x+4}{x^4+3x^2+2}dx$
   (e) $\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}dx$
   (f) $\int \frac{\sin(x)}{\cos^2(x)-3\cos(x)}dx$
   (g) $\int \frac{\cosh(t)}{\sinh^2(t)+\sinh^4(t)}dt.$

(2) Do as many problems as time permits you to from the textbook (Section 7.4: # 1-30, 39-50).

18. Strategy for Integration

(1) Write down a complete strategy for integration.

(2) Evaluate the integral:

(a) $\int_0^1 \dfrac{x-1}{x^2-4x-5}dx$

(b) $\int_0^{\frac{\sqrt{2}}{2}} \dfrac{x^2}{\sqrt{1-x^2}}dx$

(c) $\int_{-1}^2 |e^x - 1|dx$

(d) $\int \dfrac{\sqrt{2x-1}}{2x+3}dx$

(e) $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}dx$

(f) $\int (x+\sin(x))^2 dx$

(g) $\int \dfrac{dx}{\sqrt{x}+x\sqrt{x}}$

(h) $\dfrac{dx}{x^2\sqrt{4x^2-1}}$

(3) Do as many problems as time permits you to from the textbook (Section 7.5: # 1 - 60.)

19. Improper Integrals

(1) Describe Improper Integrals of Type 1.

(2) Determine whether each integral is convergent or divergent.

   (a) $\int_0^\infty \dfrac{1}{\sqrt[4]{1+x}}dx$

   (b) $\int_0^\infty \dfrac{1}{\sqrt[4]{(1+x)^5}}dx$

   (c) $\int_{-\infty}^0 2^r dr$

   (d) $\int_{-\infty}^\infty \cos(\pi t)dt$

   (e) $\int_1^\infty \dfrac{1}{x^p}dx$ for p an integer.

(3) Describe Improper Integrals of Type 2.

(4) Determine whether each integral is convergent or divergent.

   (a) $\int_6^8 \dfrac{4}{(x-6)^3}dx$

   (b) $\int_6^8 \dfrac{4}{\sqrt[3]{x-6}}dx$

   (c) $\int_1^3 \dfrac{1}{\sqrt{3-x}}dx$

(5) Describe the Comparison Test for Improper Integrals.

(6) Determine whether each integral is convergent or divergent.

   (a) $\int_1^\infty \dfrac{2+e^{-x}}{x}dx$

   (b) $\int_0^\infty \dfrac{\arctan(x)}{2+e^x}dx$

(7) Do as many problems as time permits you to from the textbook (Section 7.8: # 1,2 5-32, 49-54).

20. Review Chapter 7

Evaluate the integral:

(1) $\int_1^2 x^5 \ln(x)dx$

(2) $\int_0^1 \frac{\sqrt{\arctan(x)}}{1+x^2}dx$

(3) $\int \frac{\sec^2(\theta)}{\tan^2(\theta)}d\theta$

(4) $\int e^x \cos(x)dx$

(5) $\int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$

(6) $\int \frac{1-\tan(\theta)}{1+\tan(\theta)}d\theta$

(7) $\int_1^\infty \frac{\ln(x)}{x^4}dx$

(8) $\int_0^1 \frac{1}{2-3x}dx$

(9) $\int_1^\infty \frac{\tan^{-1}(x)}{x^2}dx$

(10) Do as many problems as time permits you to from the textbook (Review section from Chapter 7: # 1- 26, 41-50).

21. Arc Length

(1) Describe the formula for calcuating the arc length of a curve. Explain how you arrived at the formula.

(2) Find the length of the curve:

(a) $y = xe^{-x}$, $0 \leq x \leq 2$ (Just set up the integral).

(b) $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$, $1 \leq y \leq 3$

(c) $x = \ln(\cos(y))$, $0 \leq y \leq \pi/3$

(d) $y = \sqrt{x - x^2} + \sin^{-1}(\sqrt{x})$

(e) $x = 1 - e^{-y}$, $0 \leq y \leq 2$

(3) Do as many problems as time permits you to from the textbook (Section 8.1: # 1- 18).

22. Area of surface of revolution

(1) Give a formula for the surface area of a right circular cylinder. Explain with an illustration.
(2) Give a formula for the surface area of a circular cone. Explain with an illustration.
(3) Give a formula for the surface area of a band (frustum of a cone). Explain with an illustration.
(4) Give a formula for the surface area of the surface obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis. Explain with an illustration.
(5) Set up an integral to calculate the area of the surface obtained by rotating the curve:
   (a) $y = x^{-2}$, $1 \leq x \leq 2$ about the x-axis
   (b) $y = x^{-2}$, $1 \leq x \leq 2$ about the y-axis
(6) Find the area of the surface obtained by rotating the curve about the x-axis:
   (a) $y = \sqrt{1 + e^x}$, $0 \leq x \leq 1$
   (b) $y = \dfrac{x^2}{6} + \dfrac{1}{2x}$, $\dfrac{1}{2} \leq x \leq 1$.
(7) Find the area of the surface obtained by rotating the curve about the y-axis:
   (a) $y = 1 - x^2$, $0 \leq x \leq 1$
   (b) $y = \dfrac{x^2}{4} - \dfrac{\ln(x)}{2}$, $1 \leq x \leq 2$
(8) Do as many problems as time permits you to from the textbook (Section 8.2: # 1- 16).

23. Curves defined by parametric equations

(1) Explain the term **parametric curve.** Explain using the parametric equation $x = t^2, y = t^3 - 4t$. Identify the initial point and the terminal point.
(2) Explain the parametric curve $x = \cos(t), y = \sin(t)$.

24. Calculus with parametric curves

(1) Explain when a parametric curve has a
   (a) horizontal tangent
   (b) vertical tangent
(2) Find area under the curve: $x = \cos(2t), y = \sin(2t)$, $0 \le t \le \pi/4$.
(3) Develop the formula for the length of a curve C parametrized by $x = f(t)$, $y = g(t)$, $\alpha \le t \le \beta$ where f', g' are coninuous on $[\alpha, \beta]$and C is traversed exactly once as t increases from α to β.
(4) Find the length of the parametric curve $x = e^t + e^{-t}$, $y = 5 - 2t$ for $0 \le t \le 3$.
(5) Develop the formula for the formula for the surface area of the surface obtained by rotating about the x-axis, the parametric curve $x = f(t), y = g(t)$, for $\alpha \le t \le \beta$, $g(t) \ge 0$, and f', g' continuous on $[\alpha, \beta]$.
(6) Find the area of the surface area obtained by rotating $x = t^3, y = t^2$ for $0 \le t \le 1$.

25. Polar Coordinates

(1) Complete the table:

Rectangular Coordinates	Polar coordinates
Given x y	 $r =$ $\theta =$
 $x =$ $y =$	Given r θ

(2) Explain the correspondence above with illustrations (one each for $r > 0$ and $r < 0$).

(3) Find the rectangular coordinates of
- (a) $(3, 3\pi/4)$
- (b) $(-3, 3\pi/4)$
- (c) $(2, 7\pi/6)$
- (d) $(-2, 7\pi/6)$
- (e) $(-2, 8\pi/3)$

(4) Find polar coordinates with $0 \leq \theta < 2\pi$ and
- (a) (a) $r > 0$ and (b) $r < 0$.
- (b) $(-3, -3)$
- (c) $(1, -\sqrt{3})$
- (d) $(3, 3)$
- (e) $(-\sqrt{3}, 1)$

(5) Sketch the region:

(6) $2 \leq r \leq 4, \pi/3 < \theta < 5\pi/3$

(7) $r \geq 2, \pi \leq \theta \leq 2\pi$

(8) Identify the curve by finding a Cartesian equation.
- (a) $r = 4\sec(\theta)$
- (b) $\theta = \pi/3$
- (c) $r = \tan(\theta)\sec(\theta)$

(9) Find a polar equation:
- (a) $y = x$
- (b) $4y^2 = x$
- (c) $xy = 4$

(10) Sketch the curve with the given polar equation by first sketching the graph of r as a function of θ in cartesian coordinates:
- (a) $r = 1 + 2\cos(\theta)$
- (b) $r = 3 + \sin(\theta)$
- (c) $r = 2\cos(4\theta)$
- (d) $r = \ln(\theta)$, $\theta \geq 1$

(11) Do as many problems as time permits you to from the textbook (Section 10.3: # 1- 12, 15-26, 29-46).

26. Areas and Lengths in Polar Coordinates

(1) Let $\mathcal{R}$ be the region bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$ where f is a positive continuous function and where $0 < b - a \leq 2\pi$. Draw a possible such region.
(2) Draw a small sector in the region above, and describe its area.
(3) Find an approximate area of the area of the region $\mathcal{R}$.
(4) Using Riemann sums to present a formula for the area of the region $\mathcal{R}$. .
(5) Find the area of the region bounded by the curve
  (a) $r = \tan(\theta)$, $\pi/6 \leq \theta \leq \pi/3$
  (b) $r = 1 - \sin(\theta)$ for $0 \leq \theta \leq \pi$
  (c) $r = 1 - \sin(\theta)$ for $0 \leq \theta \leq 2\pi$
  (d) $r^2 = 9\sin(2\theta)$ for one loop.
(6) Find the area of the region that lies inside the first curve and outside the second curve:
  $r = 2 + \sin(\theta), r = 3\sin(\theta)$
(7) Find the area of the region that lies inside both curves: $r = 3 + 2\cos(\theta), r = 3 + 2\sin(\theta)$.
(8) Describe the formula for the length of a curve with polar equation, $r = f(\theta), a \leq \theta \leq b$.
(9) Find the length of the polar curve $r = 5^\theta$ for $0 \leq \theta \leq 2\pi$.
(10) Find the length of the polar curve $r = 2(1 + \cos(\theta))$ for $0 \leq \theta \leq 2\pi$.
(11) Do as many problems as time permits you to from the textbook (Section 10.4: # 1- 32, 45-48).

27. Conic Sections

(1) Define a parabola. Explain with an illustration. Identify the **focus**, **directrix**, **axis** and **vertex**.
(2) Write an equation and draw illustrations of the parabola with focus $(0, p)$ and directrix $y = -p$
(3) Write an equation and draw illustrations of the parabola with focus $(p, 0)$ and directrix $x = -p$
(4) Find the focus and directrix of the parabola $y^2 + 12x = 0$ and sketch.
(5) Find the focus and directrix of the parabola $x^2 + 5x = 0$ and sketch.
(6) Define an ellipse. Explain with an illustration. Identify the **foci**, **vertices**, the **major axis**, and the **minor axis**.
(7) Write an equation of the ellipse with $a \geq b > 0$ and foci $(\pm c, 0)$ for $c^2 = a^2 - b^2$. Explain with an illustration.
(8) Write an equation of the ellipse with $a \geq b > 0$ and foci $(0, \pm c)$ for $c^2 = a^2 - b^2$. Explain with an illustration.
(9) Sketch the graph of $4x^2 + 9y^2 = 121$. Identify the foci.
(10) Sketch the graph of $9x^2 + 4y^2 = 36$. Identify the foci.
(11) Sketch the graph of $2x^2 + 3y^2 = 6$. Identify the foci.
(12) Find an equation of the ellipse with foci $(0, \pm 3)$ and vertices $(0, \pm 4)$
(13) Define a hyperbola. Explain with an illustration. Identify the **foci** and the **asymptotes**.
(14) Write an equation and draw an illustration of a hyperbola with foci $(\pm c, 0)$ where $c^2 = a^2 + b^2$ and vertices $(\pm a, 0)$.
(15) Sketch the graph of $4x^2 - 9y^2 = 121$. Identify the foci and the asymptotes.
(16) Sketch the graph of $3x^2 - 2y^2 = 6$. Identify the foci and the asymptotes.
(17) Write an equation and draw an illustration of a hyperbola with foci $(0, \pm c)$ where $c^2 = a^2 + b^2$ and vertices $(0, \pm a)$.
(18) Find the foci and equation of the hyperbola with vertices $(\pm 1, 0)$ and asymptotes $y = \pm 3x$.
(19) Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptotes $y = \pm 3x$.
(20) Find an equation of the ellipse with foci $(2, 2), (4, 2)$ and vertices $(1, 2), (5, 2)$.
(21) Find the conic $9x^2 - 4y^2 - 36x + 8y - 4 = 0$. Identify all the important characteristics of this conic.
(22) Do as many problems as time permits you to from the textbook (Section 10.5: # 1- 48).

28. Conic Sections in Polar Coordinates

(1) State and explain the **Theorem 1** from your text.
(2) Present polar equation for a conic. Explain when you obtain an ellipse, a parabola, or a hyperbola.
(3) Sketch the conic $r = \dfrac{6}{3 - 6\sin(\theta)}$.
(4) If the conic in your previous question is rotated through an angle $\pi/3$ about the origin, then find a polar equation and graph the resulting conic.
(5) State Kepler's laws of planetary motion.
(6) Write the polar equation of an ellipse. Explain the parameters being used
(7) Explain the terms **perihelion**, **aphelion**, **perihelion distance** and **aphelion distance**.
(8) Do as many problems as time permits you to from the textbook (Section 10.6: # 1- 16).

29. Review Chapter 10

(1) Sketch the polar curve:
   (a) $r = 3 + 3\cos(3\theta)$
   (b) $r = \dfrac{3}{2 - 2\cos(\theta)}$

(2) Find the area enclosed by the inner loop of the curve $r = 1 - 2\sin(\theta)$.

(3) Find the area of the region that lies inside the curve $r = 2 + \cos(2\theta)$ but outside the curve $r = 2 + \sin(\theta)$.

(4) Find the length of the curve:
   (a) $x = 2 + 3t, y = \cosh(3t)$, $0 \le t \le 1$
   (b) $r = \sin^3(\theta/3)$, $0 \le \theta \le \pi$

(5) Find an equation of the parabola with focus $(2, 1)$ and directrix $x = -4$.

(6) Find an equation of the ellipse with foci $(3, \pm 2)$ and major axis with length 8.

(7) Do as many problems as time permits you to from the textbook (Review section of Chapter 10: # 9-16, 31-40,45-56).

30. Practice Problems

Below is a collection of problems from all chapters in no particular order.

(1) Assume that f is a continuous function and that
$$\int_0^x tf(t)dt = \frac{2x}{4+x^2}.$$
 (a) Determine $f(0)$.
 (b) Find the zeros of f, if any.

(2) Apply the Mean Value Theorem to the function
$$F(x) = \int_a^x f(t)dt \quad \text{on } [a,b],$$
to obtain the *Mean Value Theorem for Integrals:* If f is continuous on $[a,b]$, then there is at least one number c in (a,b) such that
$$\int_a^b f(x)dx = f(c)(b-a).$$

(3) Let f be continuous and let the functions F and G be defined by
$$F(x) = \int_c^x f(t)dt, \quad \text{and} \quad G(x) = \int_d^x f(t)dt,$$
where $c, d \in [a,b]$. Show that F and G differ by a constant.

(4) (a) Sketch the graph of the function $f(x) = \begin{cases} x^2+x, & 0 \le x \le 1 \\ 2x, & 1 < x \le 3. \end{cases}$
 (b) Find the function $F(x) = \int_0^x f(t)dt$, $0 \le x \le 3$, and sketch its graph.
 (c) What can you say about f and F at $x = 1$?

(5) A rectangle has one side on the x-axis and the upper two vertices on the graph of $y = 1/(1+x^2)$. Where should the vertices be placed so as to maximize the area of the rectangle?

(6) Find the vertex, focus, axis and directrix and then sketch the parabola.
 (a) $y^2 = 2(x-1)$
 (b) $y - 3 = 2(x-1)^2$
 (c) $y = x^2 + x - 1$

(7) Compare:
 (a) $\dfrac{d}{dx}\left[\int_a^x f(t)dt\right]$ to $\int_a^x \dfrac{d}{dt}[f(t)]dt$.
 (b) $\dfrac{d}{dx}\left[\int f(x)dx\right]$ to $\int \dfrac{d}{dx}[f(x)]dx$.

(8) Let $f(x) = x^3 - x$. Evaluate $\int_{-2}^2 f(x)dx$ without computing the integral.

(9) As a particle moves about the plane, its x-coordinate changes at the rate of $t-2$ units per second and its y coordinate changes at a rate of $\sqrt{t}$ units per second. If the particle is at the point $(3,1)$ when $t = 4$ seconds, where is the particle 5 seconds later?

(10) Let f be continuous and define F by
$$F(x) = \int_0^x \left[t\int_1^t f(u)du\right] dt.$$
Find (a) $F'(x)$. (b) $F''(1)$.

(11) Let f be a continuous function. Show that

(a) $\int_{a+c}^{b+c} f(x-c)dx = \int_a^b f(x)dx$.

(b) $\frac{1}{c}\int_{ac}^{bc} f\left(\frac{x}{c}\right)dx = \int_a^b f(x)dx$, if $c \neq 0$.

(12) Prove that the hyperbolic cosine function is even and the hyperbolic sine function is odd.

(13) Use integration to find the area of the trapezoid with vertices $(-2,-2),(1,1),(5,1),(7,-2)$.

(14) Sketch the region bounded by the curves and find the volume of the solid generated by revolving the region about the x- or y- axis, as specified below.

(a) $y = 1-|x|$, $y = 0$, revolved around the x-axis

(b) $x = \sqrt{9-y^2}$, $x = 0$, revolved around the y-axis.

(15) Sketch the region Ω bounded by the curves and use the shell method to find the volume of the solid generated by revolving Ω about the y-axis.

(a) $x = y^2$, $x = 2-y$.

(b) $x = |y|$, $x = 2-y^2$.

(c) $f(x) = \cos\frac{1}{2}\pi x$, $y = 0$, $x = 0$, $x = 1$.

(16) Find the rectangular coordinates of the given point.

(a) $[3, \frac{1}{2}\pi]$

(b) $[2, 0]$

(c) $[-1, \frac{1}{4}\pi]$

(17) A ball of radius r is cut into two pieces by a horizontal plane a units above the center of the ball. Determine the volume of the upper piece by using the shell method.

(18) Determine the exact value of the given expressions:

(a) $\tan^{-1} 0$

(b) $\sin[\arccos(-\frac{1}{2})]$

(c) $\arctan(\sec 0)$

(19) Write the equation in rectangular coordinates and identify the curve.

(a) $r\sin\theta = 4$

(b) $\theta^2 = \frac{1}{9}\pi^2$

(c) $r = \frac{4}{1-\cos\theta}$

(20) The region bounded between the graphs $y = \sqrt{x}$ and the x-axis for $0 \leq x \leq 4$ is revolved around the line $y = 2$. Find the volume of the solid that is generated.

(21) Show that for $a \neq 0$,

$$\int \frac{dx}{a^2+(x+b)^2} = \frac{1}{a}\arctan^{-1}\left(\frac{x+b}{a}\right) + C$$

(22) Assume that $f(x)$ has a continuous second derivative. Use integration by parts to derive the identity

$$f(b) - f(a) = f'(a)(b-a) - \int_a^b f''(x)(x-b)dx$$

(23) Find the area under the graph of $y = \frac{\sqrt{x^2-9}}{x}$ from $x = 3$ to $x = 5$.

(24) Find values of a and b such that

$$\lim_{x\to 0} \frac{\cos ax - b}{2x^2} = -4.$$

(25) Find the center, the vertices, the foci, the asymptotes, and the length of the transverse axis of the given hyperbola. Then sketch the figure.

(a) $\frac{(x-1)^2}{16} - \frac{(y-3)^2}{16} = 1$

(b) $y^2/9 - x^2/4 = 1$

(c) $4x^2 - 8x - y^2 + 6y - 1 = 0.$

(26) An object starts at the origin and moves along the x-axis with velocity

$$v(t) = 10t - t^2, \quad 0 \le t \le 10.$$

(a) What is the position of the object at any time t, $0 \le t \le 10$?

(b) When is the object's velocity a maximum, and what is its position at that time?

(27) (a) Calculate the area A of the region bounded by the graph of $f(x) = \dfrac{1}{x^2}$ and the x-axis for $x \in [1, b]$.

(b) The results in part (a) depends on b. Calculate the limit of $A(b)$ as $b \to \infty$.

(28) Find the indicated limit

(a) $\lim_{x\to 0} \frac{\ln(\sec x)}{x^2}$

(b) $\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{x}+\sin\sqrt{x}}$

(c) $\lim_{x\to 1} \frac{x^{1/2}-x^{1/4}}{x-1}$

(d) $\lim_{x\to\infty} (\sqrt{x^2+2x} - x)$

(e) $\lim_{x\to 0} \left(\frac{1+2^x}{2}\right)^{1/x}$

(29) Plot the given points in polar coordinates.

(a) $[-2, 0]$

(b) $[-1, \frac{1}{3}\pi]$

(c) $[\frac{1}{3}, \frac{2}{3}, \pi]$

(30) Sketch the region bounded by the curves and find its area

(a) $4x = 4y - y^2$, $4x - y = 0$.

(b) $y = e^x$, $y = e$, $y = x$, $x = 0$.

(c) $4y = x^2$ and $y = \frac{8}{x^2+4}$.

(d) $y = \sin^2 x$, $y = \tan^2 x$, $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

(e) The region in the first quadrant bounded by the x-axis, the parabola $y = \frac{x^2}{3}$, and the circle $x^2 + y^2 = 4$.

(31) The base of a solid is the region bounded by $x = y^2$ and $x = 3 - 2y^2$. Find the volume of the solid given that the cross sections perpendicular to the x-axis are

(a) rectangles of height h.

(b) equilateral triangles.

(32) Find an equation for the ellipse that satisfies the given conditions.

(a) foci at $(3, 1)$, $(9, 1)$; minor axis 10.

(b) center at $(2, 1)$; vertices at $(2, 6)$, $(1, 1)$.

(33) Find the arc length of the following graphs and compare it to the straight line distance between endpoints of the graph.

(a) $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $x \in [1, 5]$.

(b) $f(x) = \ln(\sin x)$, $x \in [\pi/6, \pi/2]$

(34) Find a formula for the distance between $[r_1, \theta_1]$ and $[r_2, \theta_2]$. (*Hint:* Recall the Cosine Law of triangles).

(35) Let $f(x)$ be a continuous function on $[0, \infty)$. The Laplace transform of f is the function F defined by

$$F(s) = \int_0^\infty e^{-sx} f(x)dx.$$

The domain of F is the set of all real numbers s such that the improper integral converges. Find the Laplace transform of each of the following functions and give the domain of F.

(a) $f(x) = \cos 2x$.

(b) $f(x) = e^{ax}$.

(36) At time t a particle has position

$$x(t) = 1 + \arctan t, \quad y(t) = 1 - \ln(\sqrt{1+t^2}).$$

Find the total distance traveled from $t = 0$ to $t = 1$.

(37) Write the equation in polar coordinates.

(a) $y = x$

(b) $(x^2 + y^2)^2 = 2xy$

(38) Find the area of the given region.

(a) $r = 2\tan\theta$ and the rays $\theta = 0$ and $\theta = \frac{1}{8}\pi$.

(b) $r = a(4\cos\theta - \sec\theta)$ and the rays $\theta = 0$ and $\theta = \frac{1}{4}\pi$.

(39) Calculate the derivative:

(a) $\frac{d}{dx}\left(\int_{x^2}^{3} \frac{\sin t}{t} dt\right)$

(b) $\frac{d}{dx}\left(\int_{\tan x}^{2x} t\sqrt{1+t^2}dt\right)$.

(c) $H'(3)$ given that $H(x) = \frac{1}{x}\int_3^x [2t - 3H'(t)]dt$.

(40) Show that the set of all points $(a\cos t, b\sin t)$ with real t lie on an ellipse.

(41) Evaluate the integrals:

(a) $\int e^{-kx}dx$
(b) $\int_{\pi/6}^{\pi/2} \frac{\cos x}{1+\sin x}dx$
(c) $\int_0^3 \frac{r}{\sqrt{r^2+16}}dr$
(d) $\int \frac{x+1}{x^2}dx$
(e) $\int \frac{\sin(e^{-2x})}{e^{2x}}dx$
(f) $\int \sinh 2x\, e^{\cosh\, 2x}dx$
(g) $\int \frac{\ln(x+a)}{x+a}dx$
(h) $\int e^{\ln x}dx$
(i) $\int_{-a}^0 y^2 \left(1-\frac{y^3}{a^2}\right)^{-2} dy$
(j) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$
(k) $\int \frac{1}{\sin^2 x}dx$
(l) $\int \tan x \ln(\sec x)dx$
(m) $\int \frac{1+\tanh}{\cosh^2 x}dx$
(n) $\int \frac{dx}{x\ln[1+(\ln x)^2]}$
(o) $\int_5^8 \frac{dx}{x\sqrt{x^2-16}}$
(p) $\int_0^1 \cos^2 \frac{\pi}{2}x \sin\frac{\pi}{2}x dx$
(q) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})}dx$
(r) $\int_{\ln 2}^{\ln 3} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx$

(a) $\int x\sqrt{x^2+6x}dx$
(b) $\int_0^\infty e^{-px}dx,\ p>0$
(c) $\int \cos\sqrt{x}dx$
(d) $\int \tan^2(2x)dx$
(e) $\int x10^{-x^2}dx$
(f) $\int \tanh dx$
(g) $\int_0^1 \frac{5p^{\sqrt{x+1}}}{\sqrt{x+1}}dx$
(h) $\int \frac{\log_5 x}{x}dx$
(i) $\int_0^{\ln(\pi/4)} e^x \sec e^x dx$
(j) $\int \sin(\ln x)dx$
(k) $\int \frac{\sinh ax}{\cosh^2 ax}dx$
(l) $\int \frac{\ln(x+1)}{\sqrt{x+1}}dx$
(m) $\int \frac{dx}{(x^2-4x+4)^{3/2}}$
(n) $\int \frac{x-3}{x^3+x^2}dx$
(o) $\int \tan^2 x \sec^2 x dx$
(p) $\int_0^{\pi/4} \sin 5x \cos 2x dx$
(q) $\int \frac{dx}{x^4-16}$
(r) $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}dx$

(42) Evaluate the derivative:
(a) $y=\frac{e^{2x}-1}{e^{2x}+1}$
(b) $y=\ln(\cos(e^{2x}))$
(c) $g(x)=\sqrt{\tan^{-1}(2x)}$
(d) $f(x)=x^2\sec^{-1}\left(\frac{1}{x}\right)$
(e) $\theta(r)=\sin^{-1}(\sqrt{1-r^2})$
(f) $y=(\tan x)^{\sec x}$
(g) $f(x)=x\sin^{-1}(2x)$
(h) $f(x)=2^{5x}3^{\ln x}$
(i) $f'(e)$ where $f(x)=\log_3(\log_2 x)$
(j) $y=x\sqrt{c^2-x^2}+c^2\sin^{-1}\left(\frac{x}{c}\right),\ c>0$
(k) $y=\ln(\cosh x)$
(l) $f(x)=\arctan(\sinh x)$
(m) $y=\sec h(3x^2+1)$
(n) $y=\frac{\cosh x}{1+\sec hx}$
(o) $f(x)=(\sinh x)^x$