WORKBOOK. MATH 31. CALCULUS AND ANALYTIC GEOMETRY I.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE

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Contents

1. Tangent and Velocity

(1) A bottle is leaking water. The table below gives the amount of water left (V in cc) in the bottle after t minutes.

t min.	0	5	10	15	20	25	30
V cc.	1000	966	834	659	532	448	362

(a) Let $P = (10, 834)$ be a point on the curve of V. Find the slopes of the secant lines PQ when Q is the point on the graph with $t = 0, 5, 15, 20, 25$, and 30.

(b) Estimate the slope of the tangent to the curve of V at point P by taking average of two nearby secants.

(c) The slope at P represents the rate at which water is leaking out of the bottle at time $t = 10$ min. Can you suggest how one would find this slope?

(2) Draw the graph of the curve $y = x^3$.

(a) Using a calculator, find the slopes (m) of secants passing through $P = (1, 1)$ and the points given by the given x- values.

x	y	m		x	y	m
0				2		
0.5				1.5		
0.75				1.25		
0.9				1.1		
0.99				1.01		

(b) Do you see a pattern to these various slopes? What number do these slopes tend to?

(c) Draw these various secants in your graph above. What line do these secants tend to?

(d) We have used the phrase "tend to" in the above two questions. Explain what they mean in your own words.

(3) What is the average velocity at a specific time $t = t_0$ and $t = t_1$ of a particle in motion where its position (y) is a real valued function of time (t); that is, $y = y(t)$? What is the instantaneous velocity at $t = t_0$? Explain with the example $y = t^2 + 2t$ for $t_0 = 1$ second and $t_1 = 2, 1.5, 1.1, 1.01$ seconds.

(4) If a ball is thrown into the air with a velocity of 60 ft/sec, its height in feet is given by $y = 60t - 16t^2$ where t denotes time in seconds.

(a) Find the average velocity for the time period beginning when $t = 3$ seconds and lasting 0.5, 0.1, 0.05, and 0.01 seconds.

(b) Estimate the instantaneous velocity when $t = 3$ seconds.

2. The Limit of a Function

(1) What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is defined near a for some $a \in \mathbb{R}$?

Explain with an example and a graph when

(a) f is defined at a

(b) f is not defined at a.

(2) What does it mean to say that the limit of $f(x)$ equals L as x approaches a.

This is denoted by $\lim_{x\to a} f(x) = L$ or $f(x) \to L$ as $x \to a$. Explain with an example and a graph when

(a) f is defined at a

(b) f is not defined at a.

(3) Consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$.

(a) Is the function defined at $x = 1, 2, 3$? Explain.

(b) Is the function defined near $x = 1$? Explain.

(c) What is the limit of $f(x)$ as x approaches 1? That is, find $\lim_{x \to 1} f(x)$. Explain.

(d) Draw a graph illustrating this example.

(4) Consider the function $f(x) = \dfrac{x-3}{x^2-9}$.

(a) Is the function defined at $x = 1, 2, 3$? Explain.

(b) Is the function defined near $x = 3$? Explain.

(c) What is the limit of $f(x)$ as x approaches 3? That is, find $\lim_{x\to 3} f(x)$. Explain.

(d) Draw a graph illustrating this example.

(5) Consider the function $f(x) = \dfrac{\sqrt{x} - 3}{x - 9}$.

(a) Is the function defined at $x = 9, 9.1, 9.05, 9.005$? Explain.

(b) Is the function defined near $x = 9$? Explain.

(c) What is the limit of $f(x)$ as x approaches 9? That is, find $\lim_{x\to 9} f(x)$. Explain. Use a calculator to understand its graph.

(6) Consider the function $f(x) = \dfrac{\sin x}{x}$ where x is in radians. You will need a calculator for the following.

(a) Is the function defined at $x = 0, 0.5, 0.2, 0.1, -0.5, -0.2, -0.1$? Explain.

(b) Is the function defined near $x = 0$? Explain.

(c) Guess the limit of $f(x)$ as x approaches 0? That is, guess $\lim_{x\to 0} \dfrac{\sin x}{x}$. Explain. Use a calculator to understand its graph.

(7) Consider the function $f(x) = \sin\left(\frac{\pi}{x}\right)$. Discuss its limit at $x = 0$. Use your calculator to understand its graph.

(8) The Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined by

$$H(t) = \begin{cases} 0 & \textit{if } t < 0, \\ 1 & \textit{if } t \geq 0. \end{cases}$$

Discuss its limit for $t = 0$. Draw a graph to understand the one-sided limits of this function.

(9) What is the left hand limit of a function f as x approaches a? Explain using an illustration.

(10) What is the right hand limit of a function f as x approaches a? Explain using an illustration.

(11) Draw the graph of a

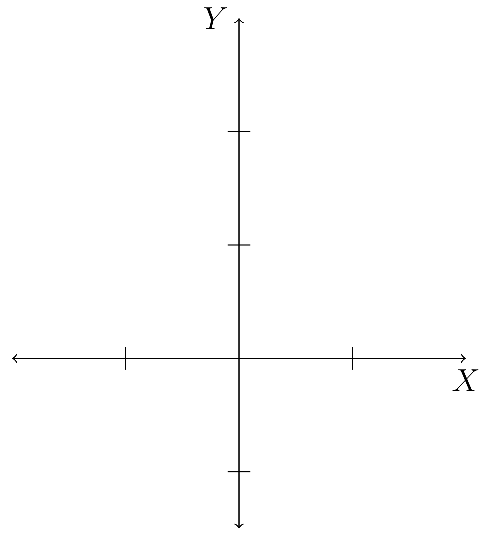
(a) function f satisfying

$f(1) = 2$

$\lim_{x\to 1^+} f(x) = 3$

$\lim_{x\to 1^-} f(x) = -1$

What is $\lim_{x\to 1} f(x)$?



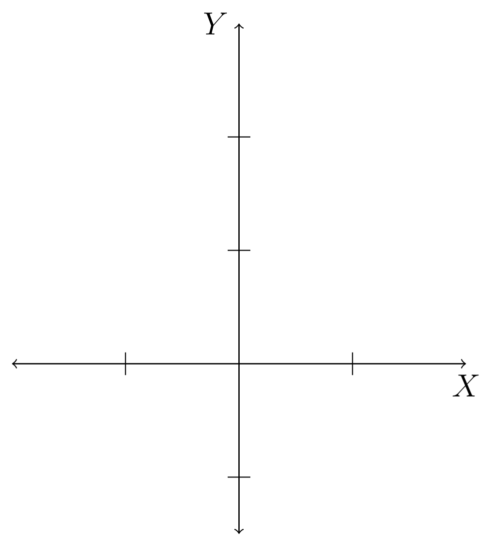
(b) function f satisfying

$f(1) = 2$

$\lim_{x\to 1^+} f(x) = -1$

$\lim_{x\to 1^-} f(x) = 3$

What is $\lim_{x\to 1} f(x)$?



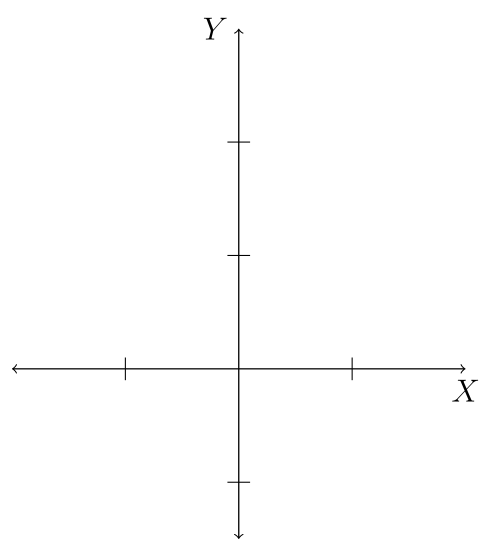
(c) function f satisfying

$f(1) = 2$

$\lim_{x\to 1^+} f(x) = 2$

$\lim_{x\to 1^-} f(x) = 3$

What is $\lim_{x\to 1} f(x)$?



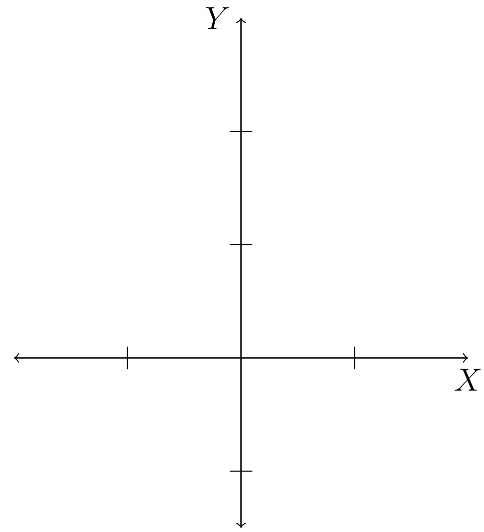
(d) function f satisfying

$f(1) = 2$

$\lim_{x\to 1^+} f(x) = 3$

$\lim_{x\to 1^-} f(x) = 2$

What is $\lim_{x\to 1} f(x)$?



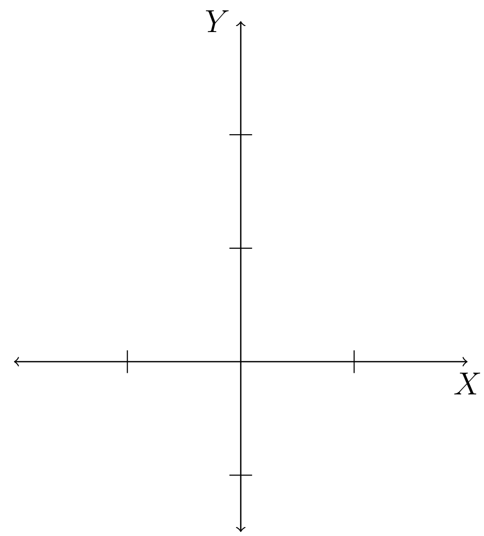
(e) function f satisfying

$f(1) = 2$

$\lim_{x\to 1^+} f(x) = 3$

$\lim_{x\to 1^-} f(x) = 3$

What is $\lim_{x\to 1} f(x)$?



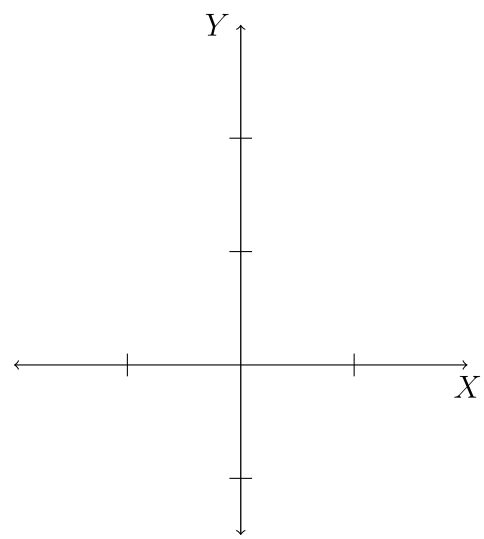
(f) function f satisfying

$f(1) = 2$

$\lim_{x\to 1^+} f(x) = 2$

$\lim_{x\to 1^-} f(x) = 2$

What is $\lim_{x\to 1} f(x)$?



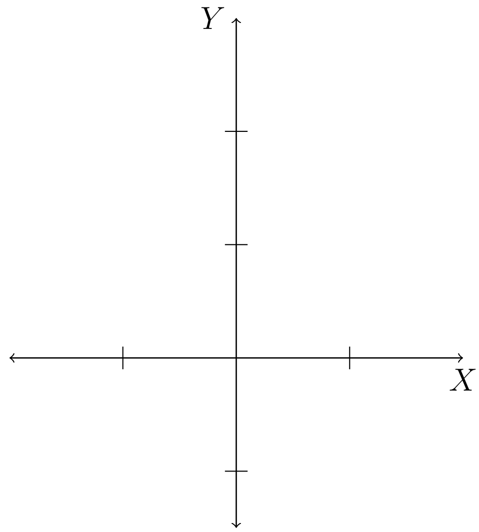
(g) function f satisfying

$f(1) = 2$

$\lim_{x\to 1^+} f(x) = \infty$

$\lim_{x\to 1^-} f(x) = 3$

What is $\lim_{x\to 1} f(x)$?



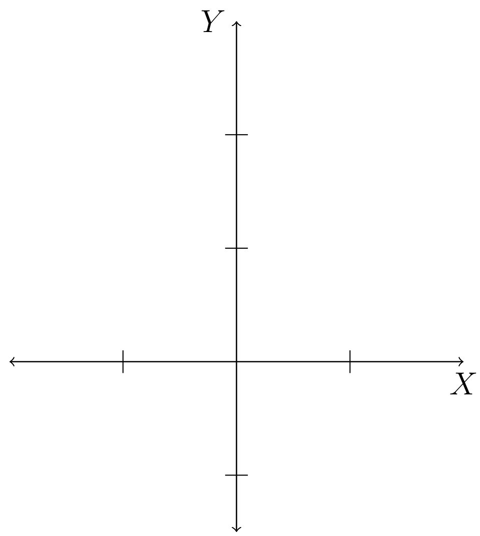
(h) function f satisfying

$f(1) = 2$

$\lim_{x\to 1^+} f(x) = -1$

$\lim_{x\to 1^-} f(x) = \infty$

What is $\lim_{x\to 1} f(x)$?



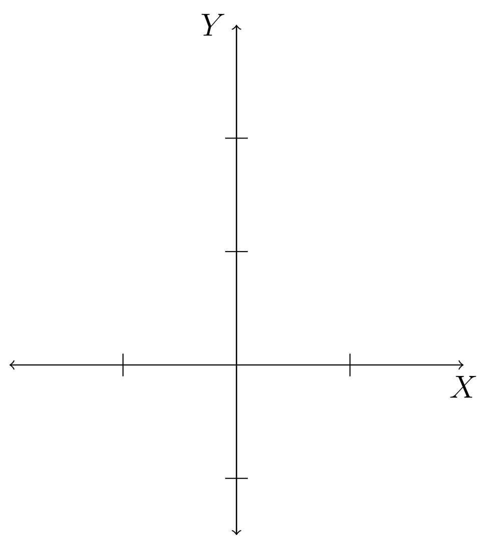
(i) function f satisfying

$f(1) = 2$

$\lim_{x\to 1^+} f(x) = \infty$

$\lim_{x\to 1^-} f(x) = \infty$

What is $\lim_{x\to 1} f(x)$?



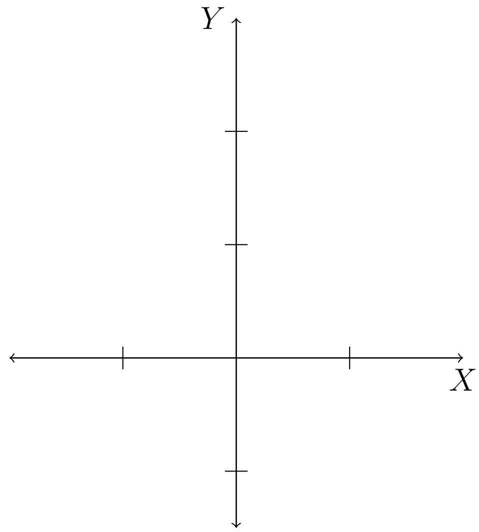
(j) function f satisfying

$f(1) = 2$

$\lim_{x\to 1^+} f(x) = -\infty$

$\lim_{x\to 1^-} f(x) = -\infty$

What is $\lim_{x\to 1} f(x)$?

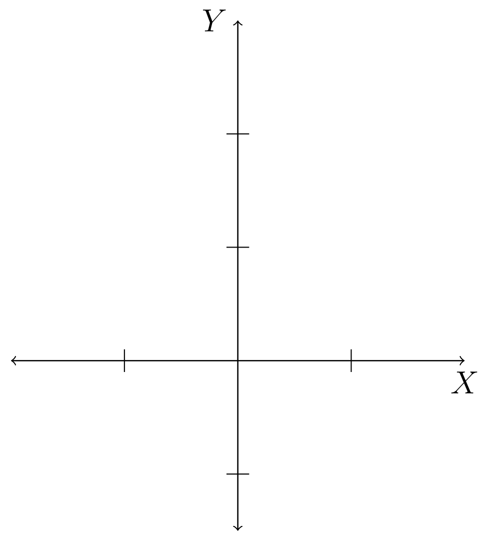


(12) When do you say that $\lim_{x\to a} f(x) = \infty$? Explain with an illustration.

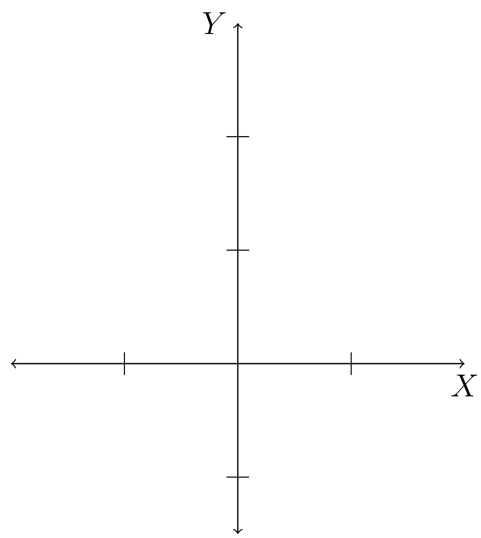
(13) When do you say that $\lim_{x\to a} f(x) = -\infty$? Explain with an illustration.

(14) For each of the following functions discuss $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1} f(x)$, $f(1)$ and draw a graph illustrating your conclusions.

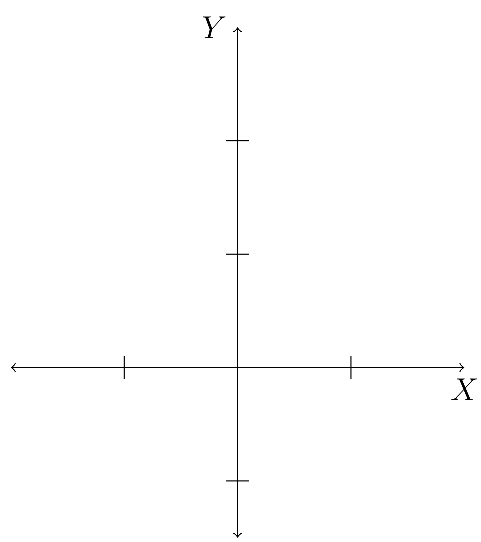
(a) $f(x) = \dfrac{1}{x-1}$



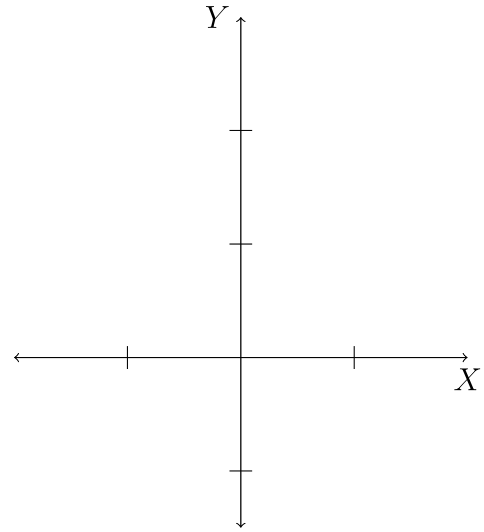
(b) $f(x) = \dfrac{-2}{x-1}$



(c) $f(x) = \dfrac{1}{(x-1)^2}$



(d) $f(x) = \dfrac{-2}{(x-1)^2}$



(15) When is the line $x = a$ said to be a vertical asymptote of the curve $y = f(x)$? Explain with several illustrations.

(16) Find the values a for which $\lim_{x\to a} f(x)$ exist. Explain using a graph.

$$f(x) = \begin{cases} 1+\cos x & \textit{if } x \le 0; \\ \sin x & \textit{if } 0 < x \le \pi; \\ \cos x & \textit{if } x > \pi. \end{cases}$$

(17) Find $\lim_{x\to 1} \dfrac{x^7-1}{x^{10}-1}$.

(18) Find $\lim_{x\to 0} \dfrac{\sin 5x}{\sin 4x}$.

(19) Find $\lim_{x\to 0} \dfrac{\cos 4x}{x+1}$.

(20) Find $\lim_{x\to 0} \dfrac{\cos 5x}{\cos 4x}$.

(21) Find $\lim_{x\to 0} \dfrac{\tan 5x}{\tan 4x}$.

3. Calculating Limits Using Limit Laws

We first present the **Limit Laws**. Let $c \in \mathbb{R}$. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are two functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ **exist** for a fixed $a \in \mathbb{R}$.

- $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
- $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
- $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$.
- $\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
- If $\lim_{x\to a} g(x) \neq 0$ then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

These laws result in the following laws:

- $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ for n a positive integer.
- $\lim_{x\to a} c = c$.
- $\lim_{x\to a} x = a$.
- $\lim_{x\to a} x^n = a^n$ for n a positive integer.
- $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ for n a positive integer. When n is even, we assume $a > 0$.
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ for n a positive integer. When n is even, we assume $\lim_{x\to a} f(x) > 0$.
- If $f(x)$ is a polynomial or a rational function and a is in the domain of f then $\lim_{x\to a} f(x) = f(a)$.

We now present a few theorems:

Theorem (Two-sided limits)**.** *A two-sided limit exists if and only if both of the one-sided limits exist and are equal. That is,*

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a+} f(x) = L = \lim_{x\to a^-} f(x).$$

Theorem. *If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then*

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

Theorem (The Squeeze theorem)**.** *If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and*

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

then $\lim_{x\to a} g(x) = L$.

Find the following limits:

(1) $\lim_{x\to 2} 3x^2 - 5x + 6$

(2) $\lim_{x\to 2} \dfrac{x^2-5x+6}{x^2+4}$

(3) $\lim_{x\to 2} \dfrac{x^2-5x+6}{x^2-4}$

(4) $\lim_{x\to -3} h(x)$ where $h(x) = \begin{cases} \dfrac{2x^2-5}{x+5} & \textit{if } x \neq -3, \\ 23 & \textit{if } x = -3. \end{cases}$.

(5) $\lim_{x\to -3} f(x)$ where $f(x) = 4$ for all $x \in \mathbb{R}$.

(6) $\lim_{x\to 4} 7$.

(7) $\lim_{h\to 0} \dfrac{(4+h)^2-16}{h}$.

(8) $\lim_{x\to 0} \dfrac{\sqrt{x^2+16}-4}{x^2}$.

(9) $\lim_{x\to 0^+} \dfrac{x}{|x|}$.

(10) $\lim_{x\to 0^-} \dfrac{x}{|x|}$.

(11) $\lim_{x\to 0} \dfrac{x}{|x|}$.

(12) $\lim_{x\to 0} |x|$.

(13) $\lim_{x\to 2} g(x)$ where $g(x) = \begin{cases} x^2+3x-5 & \textit{if } x > 2, \\ 7-x & \textit{if } x < 2. \end{cases}$.

What is $g(2)$?

(14) $\lim_{x\to 3} \dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}$.

(15) Given that $2x \leq g(x) \leq x^4 - x^2 + 2$ for all $x \in \mathbb{R}$, evaluate $\lim_{x\to 1} g(x)$.

(16) The greatest integer function is defined by $||x||$ = the largest integer that is less than or equal to x. Graph this function and discuss its one-sided limits.

4. Continuity

(1) Recall: When do we say that limit of $f(x)$ as x approaches a is L?

(2) When do we say that a function f is continuous at a number a? Illustrate with a graph.

(3) When do we say that a function f is continuous from the right at a? Illustrate with a graph.

(4) When do we say that a function f is continuous from the left at a? Illustrate with a graph.

(5) Relate continuity with left and right continuity.

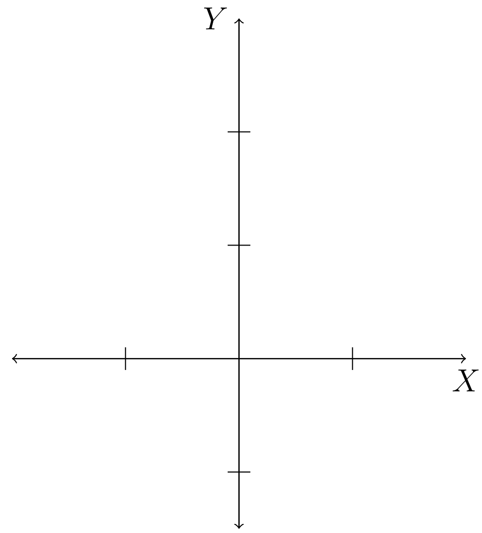
(6) Draw the graph of a

(a) function f satisfying

$f(1) = 2$,

f is continuous at 1.

What is $\lim_{x\to 1} f(x)$?



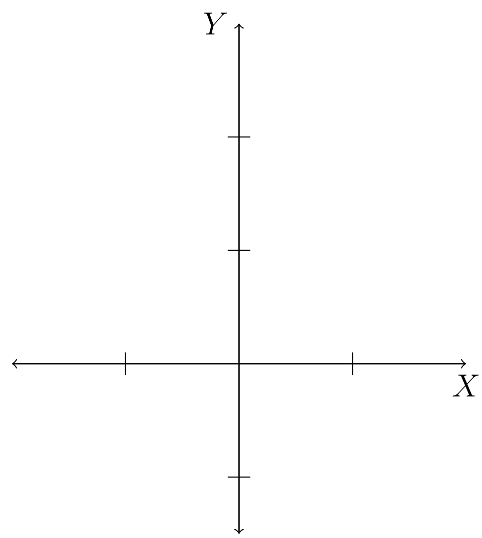
(b) function f satisfying

$f(1) = 2$,

f is continuous from right at 1,

f is not continuous from left at 1,

What is $\lim_{x\to 1} f(x)$?



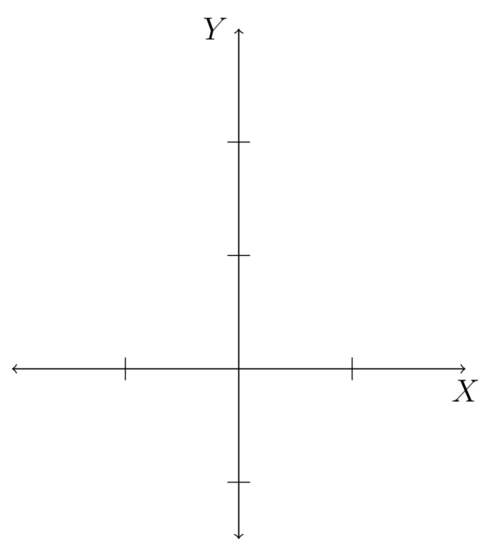
(c) function f satisfying

$f(1) = 2$,

f is continuous from left at 1,

f is not continuous from right at 1,

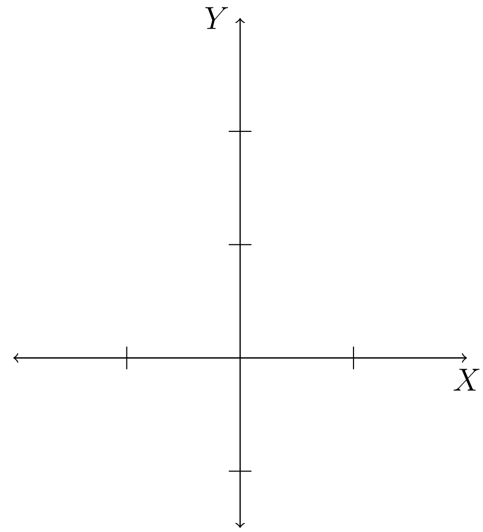
What is $\lim_{x\to 1} f(x)$?



(d) function f satisfying

$f(1)$ *is undefined*,
$\lim_{x\to 1} f(x) = 2.$

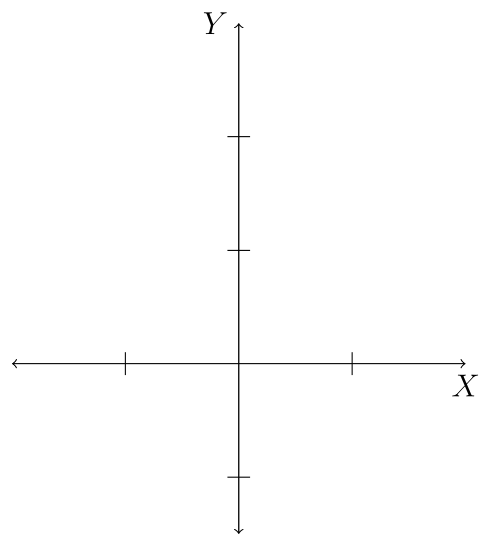
Is f continuous at 1 ?



(e) function f satisfying

$f(1) = 2,$
$\lim_{x\to 1^+} f(x) = 2$
$\lim_{x\to 1^-} f(x) = \infty$

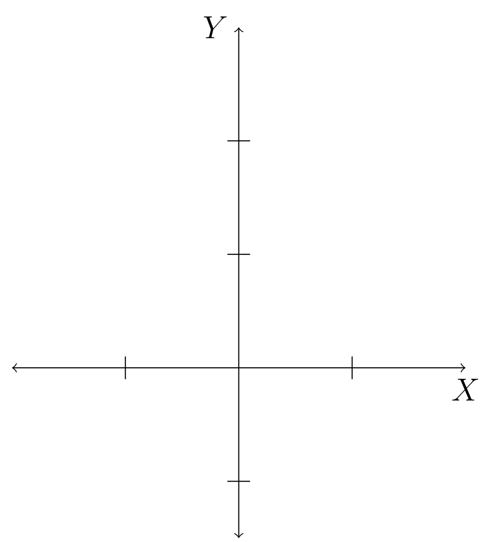
Discuss continuity of f at 1



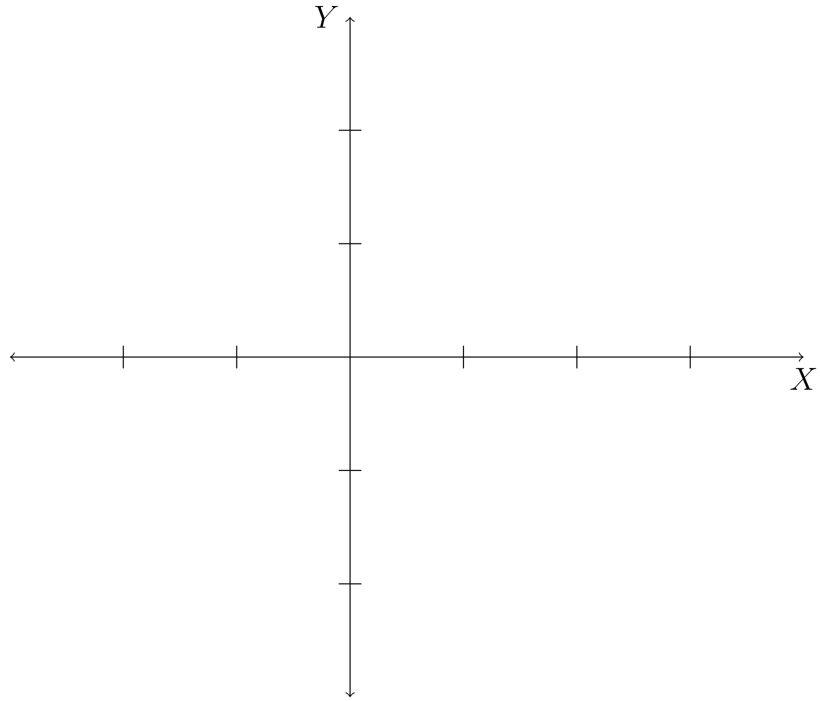
(f) function f satisfying

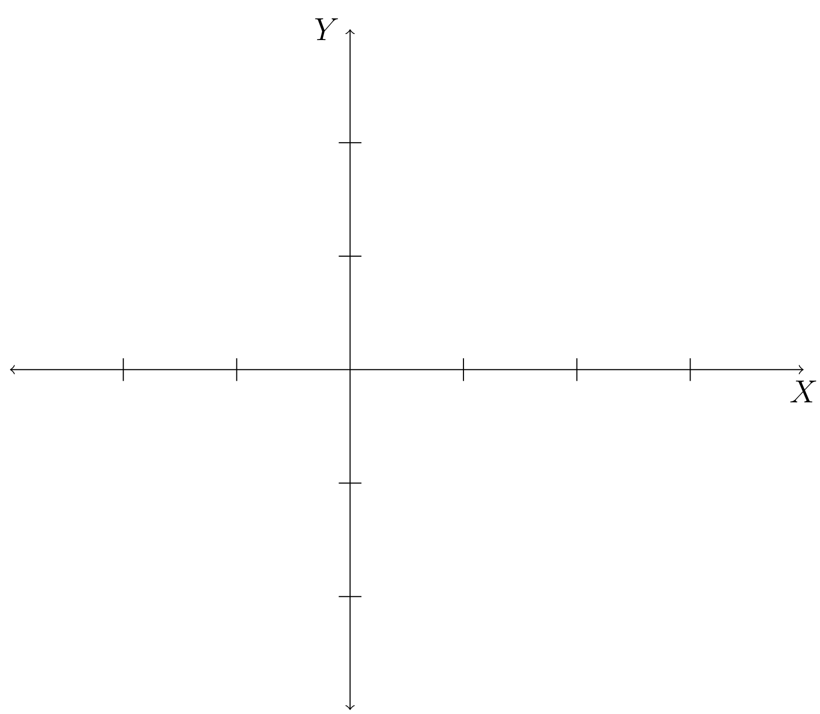
$f(1) = 2,$
$\lim_{x\to 1} f(x) = -\infty$

Discuss continuity of f at 1



(7) Consider the functions $f(x) = \dfrac{x^2 - 5x + 6}{x - 2}$ and $g(x) = x - 3$. Draw their respective graphs, and discuss their continuity.





(8) When is a function said to be continuous on an interval? What happens at the end points if the interval is a closed interval?

Theorem. *Suppose functions f and g are continuous at $a \in \mathbb{R}$. Let c be a real number. Prove that*

(1) $f + g$ *is continuous at* a.

(2) $f - g$ *is continuous at* a.

(3) *$c \cdot f$ is continuous at a.*

(4) *$f \cdot g$ is continuous at a.*

(5) *If $g(a) \neq 0$ then $\dfrac{f}{g}$ is continuous at a.*

Theorem. (1) *A polynomial function is continuous on* $\mathbb{R}$.

(2) *A rational function is continuous on its domain.*

(3) *The root functions are continuous on their domains.*

(4) *The trigonometric functions are continuous on their domains.*

Assume the following theorem without proof:

Theorem. *If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$.*

Using this result, prove the following:

Theorem. *If g is continuous at a and f is continuous at $g(a)$, then the composition function $f \circ g$ is continuous at a.*

The following theorem will be accepted without proof. Draw an illustration to explain it.

Theorem (Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.*

(1) Let $f(x) = x^{10} + 3x + 1$. Show that f has a zero in $[-1, 1]$.

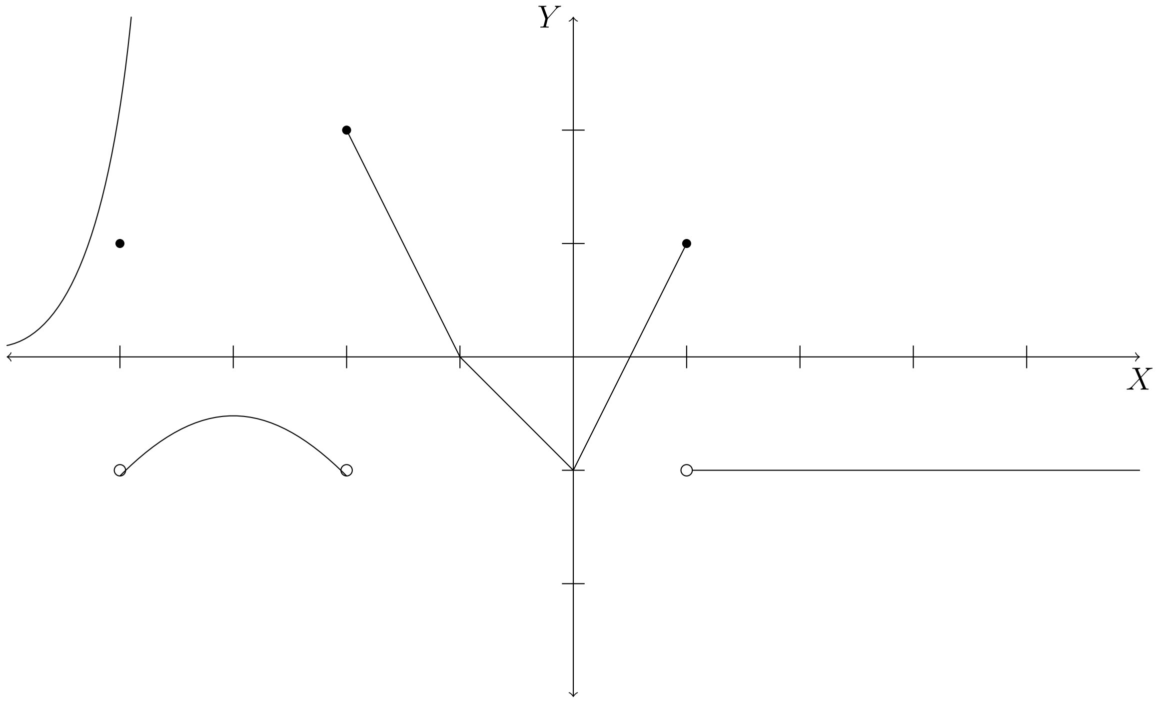
(2) Show that $3x^2 - 17x + 11 = 0$ has a root in $[0, 1]$.

(3) Given the graph of the function f,

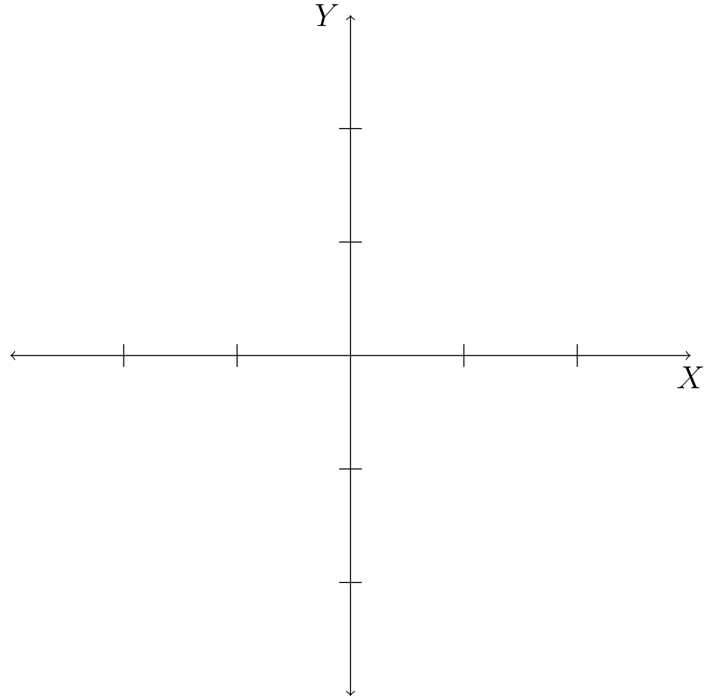
(a) on which intervals is f continuous?

(b) for which values of x is f not continuous?

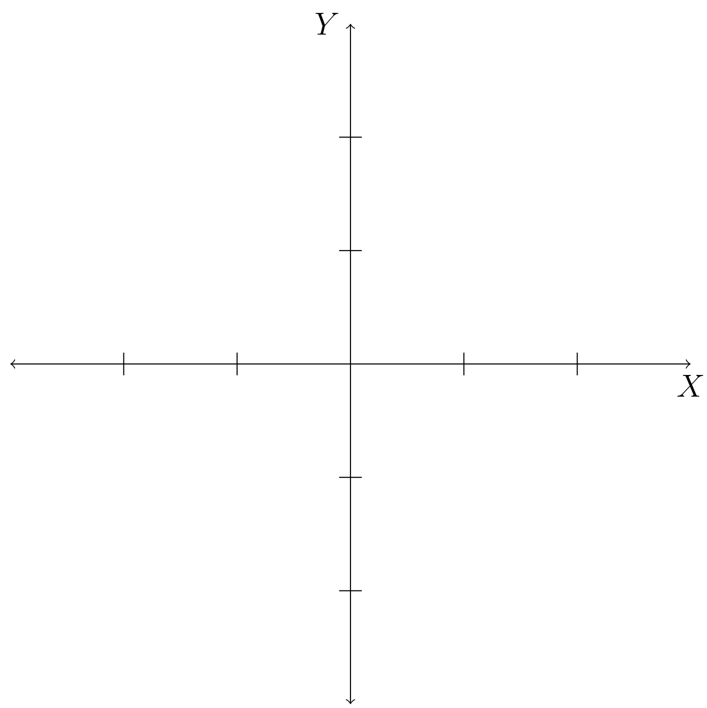
(c) for values of discontinuity, discuss one-sided limits, and one-sided continuity.



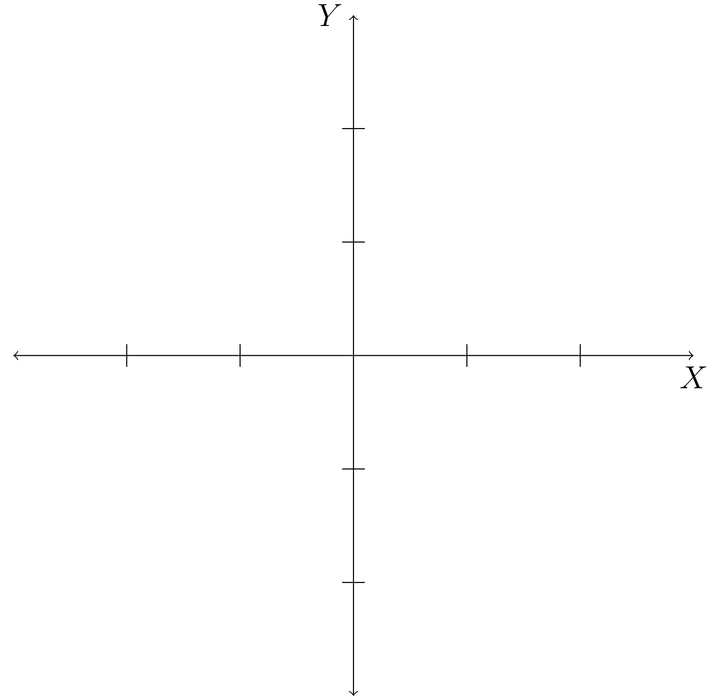
(4) What is a removable discontinuity? Explain with an illustration.



(5) What is a jump discontinuity? Explain with an illustration.

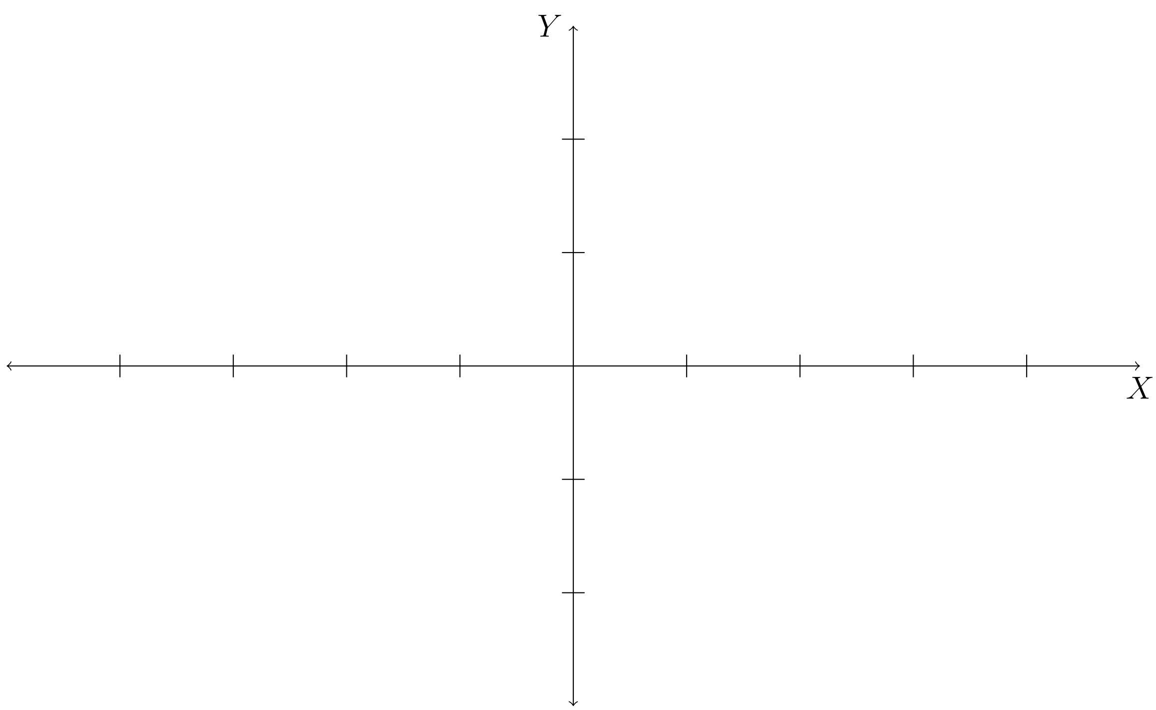


(6) What is an infinite discontinuity? Explain with an illustration.



(7) Sketch the graph of a function f that is continuous except for the stated discontinuity:

- Discontinuous at $x = -4$ but left continuous.
- Has a vertical asymptote but is right continuous at $x = -2$.
- Has a removable discontinuity at $x = 1$.
- Neither left nor right continuous at $x = 2$.
- Has a jump discontinuity at $x = 3$.
- Has an infinite discontinuity at $x = 4$.



(8) Suppose that a factory produces widgets. The set-up costs of the factory is \$ 20,000.00 per month. After that, for every 10 hours the factory runs, the cost goes up by \$ 200.00. If you graph the cost with respect to the number of hours the factory runs, what kind of a graph would you get? What kind of discontinuities do you encounter?

(9) Suppose f and g are continuous functions such that $f(3) = 4$ and $\lim_{x\to 3}[f(x)g(x)+7g(x)] = 12$. Find $g(3)$.

(10) Explain why the following functions are continuous on the given open intervals:

(a) $f(x) = \dfrac{3x^2+7}{x-4}$ on $(4, \infty)$.

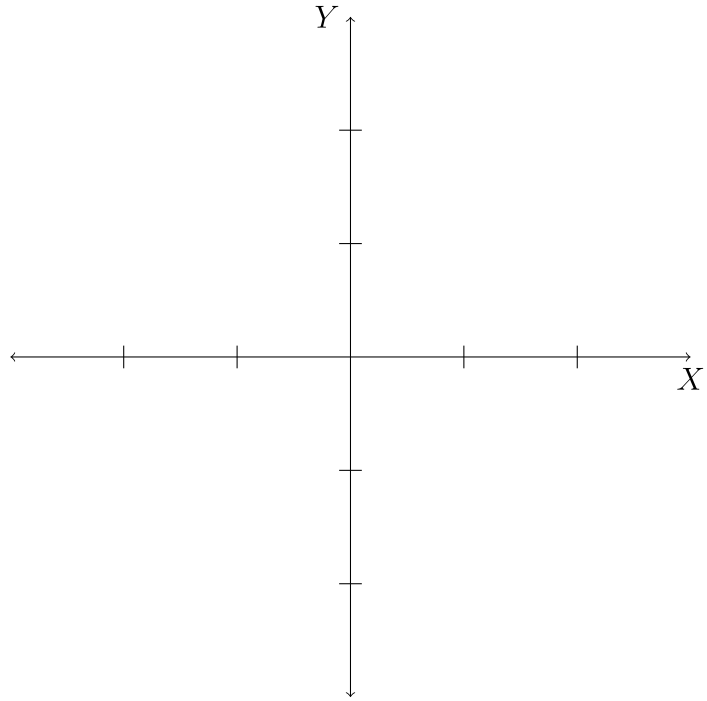
(b) $g(x) = \dfrac{4}{\sqrt{5-x}}$ on $(-\infty, 5)$.

(c) $h(x) = -\sqrt{3}x^5 - 100x^4 + \frac{3}{4}x^3 + 11x^2 - 7x + 8$ on $\mathbb{R} = (-\infty, \infty)$.

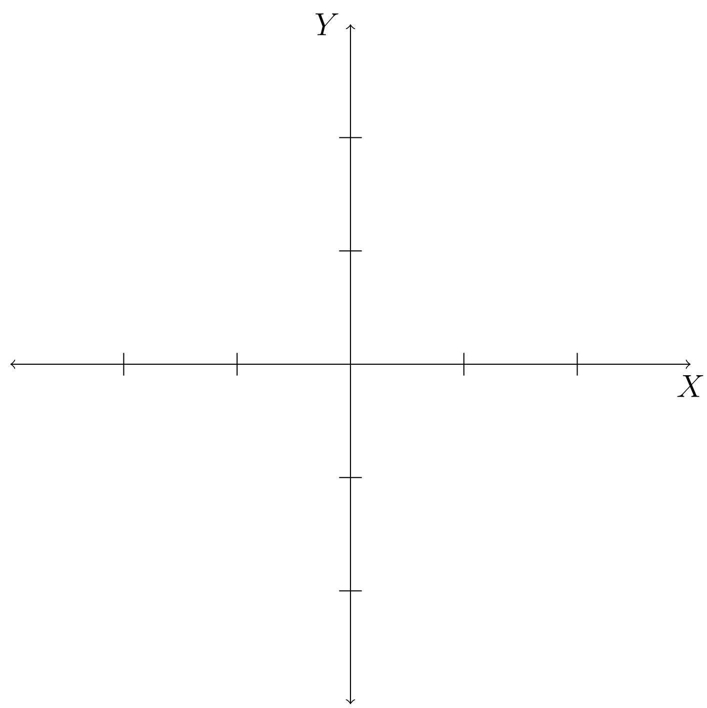
(d) $j(x) = \tan(x) + \sin(x)$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

(11) Explain whether the given function is continuous or discontinuous. If discontinuous at any point, then explain which kind of discontinuity you encounter. Sketch the graph in each case.

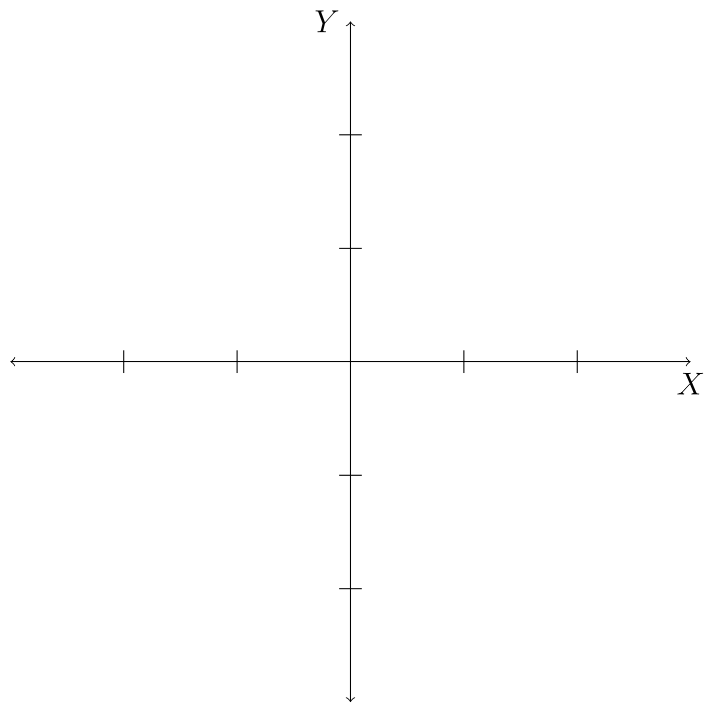
(a) $f(x) = \begin{cases} x^2 - 1 & \textit{for } x \leq 2 \\ x - 1 & \textit{for } x > 2 \end{cases}$



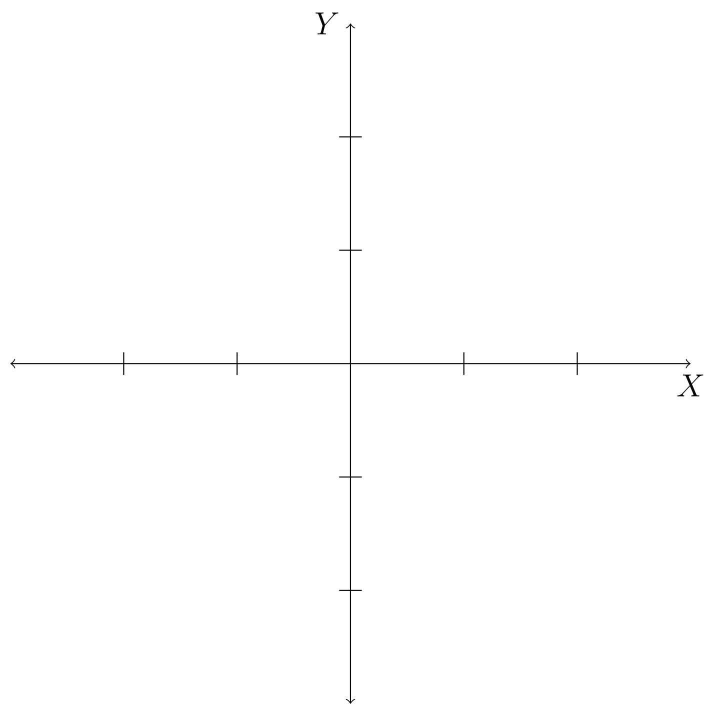
(b) $f(x) = \dfrac{1}{x+3} \quad \textit{for } x \neq -3$



(c) $f(x) = \begin{cases} \cos(x) & \textit{for } x < \pi, \\ -1 & \textit{for } x \geq \pi \end{cases}$



(d) $f(x) = \dfrac{3x^2 + 7x + 2}{x + 2} \quad \textit{for } x \neq -2$



(12) Find the domain and explain whether the function is continuous on its domain.

(a) $f(x) = \dfrac{x}{1 + \cos(x)}$.

(b) $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$.

(c) $f(x) = \sqrt{5 - x}$.

(d) $f(x) = \tan(\sqrt{x})$.

(13) Use continuity to evaluate the limit. State the rules you use.

(a) $\lim_{x \to \frac{\pi}{2}} \sin(\cos(x))$.

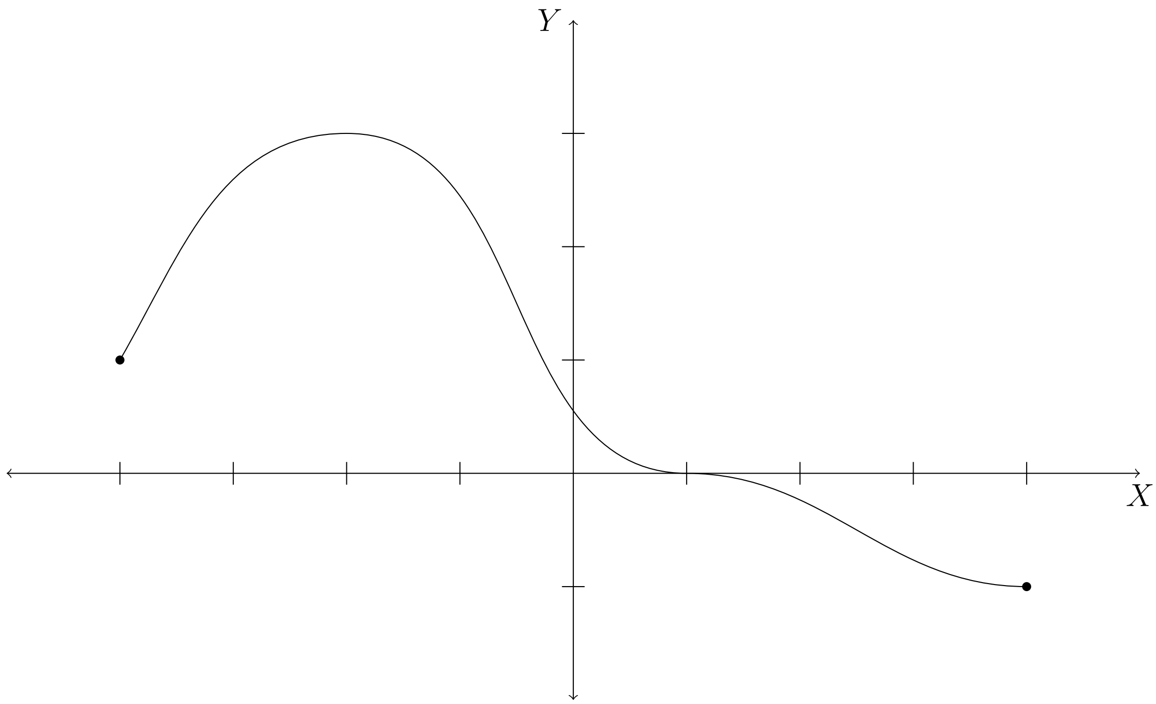
(b) $\lim_{x \to \frac{\pi}{4}} x \sin^2(x)$.

(c) $\lim_{x \to -3} \sqrt{x^2 + 7}$.

(d) $\lim_{x \to 5} (x^2 - 2x - 14)^{1000}$.

5. Review Chapter 1

(1) The graph of function f is given.
   (a) What is the value of $f(1)$?
   (b) For which x is $f(x) = 3$?
   (c) What is the domain of f?
   (d) What is the range of f?
   (e) On which set is f increasing?
   (f) On which set is f decreasing?
   (g) Is f even, odd, or neither even nor odd? Explain.
   (h) Is f one-to-one? Explain.



(2) What is the Vertical Line Test for a graph? Explain its use with illustrations.

(3) Let $f(x) = x^2 + 3x - 6$. Find and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

(4) Find the domain and the range of the function. Write your answer in interval notation.

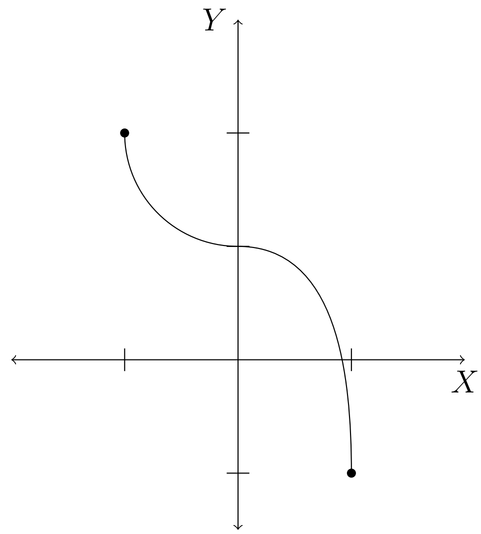
(a) $f(x) = \dfrac{2}{4x - 5}$.

(b) $f(x) = \sqrt{7 - x}$.

(c) $f(x) = \cos(x) + 2$.

(d) $f(x) = \sqrt{9 - x^2}$.

(5) The graph $y = f(x)$ is given.



Draw the graphs

- $y = f(x+1)$
- $y = f(x) + 1$
- $y = f(x-1)$
- $y = f(x) - 1$

- $y = f(-x)$
- $y = -f(x)$
- $y = f(2x)$
- $y = 2f(x)$

- $y = f(\frac{1}{2}x)$
- $y = \frac{1}{2}f(x)$
- $y = |f(x)|$
- $y = f(|x|)$

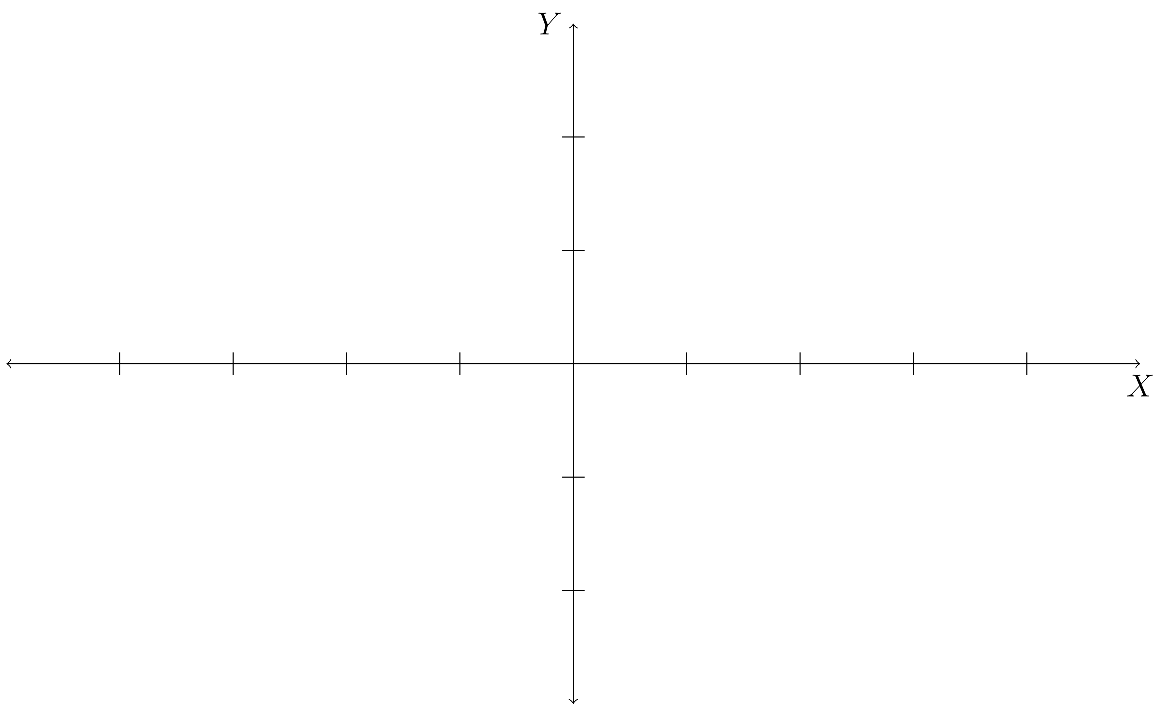
(6) What is an even function? Give five examples of even functions. What can you say about the graph of an even function?

(7) What is an odd function? Give five examples of odd functions. What can you say about the graph of an odd function?

(8) Recall from your course in Pre-Calculus some of these important functions. Make sure that you can graph every one of these graphs without using your calculators. Classify them into even, odd, and neither even nor odd functions: $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$, $f_4(x) = x^3$, $f_5(x) = \dfrac{1}{x}$, $f_6(x) = \dfrac{1}{x^2}$, $f_7(x) = |x|$, $f_8(x) = \sqrt{x}$, $f_9(x) = \sqrt[3]{x}$, $f_{10}(x) = \sin(x)$, $f_{11}(x) = \cos(x)$, and $f_{12}(x) = \tan(x)$.

(9) Define composition of two functions. Explain using several examples (at least three examples).

(10) In the coordinate plane given below draw the graph of a function with several discontinuities. At each point of discontinuity, discuss one-sided limits, existence of limit, one-sided continuity, and possible vertical asymptotes.



(11) Find the limit:

(a) $\lim_{x\to 3} \dfrac{2x^2 - x - 15}{x - 3}$

(b) $\lim_{x\to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

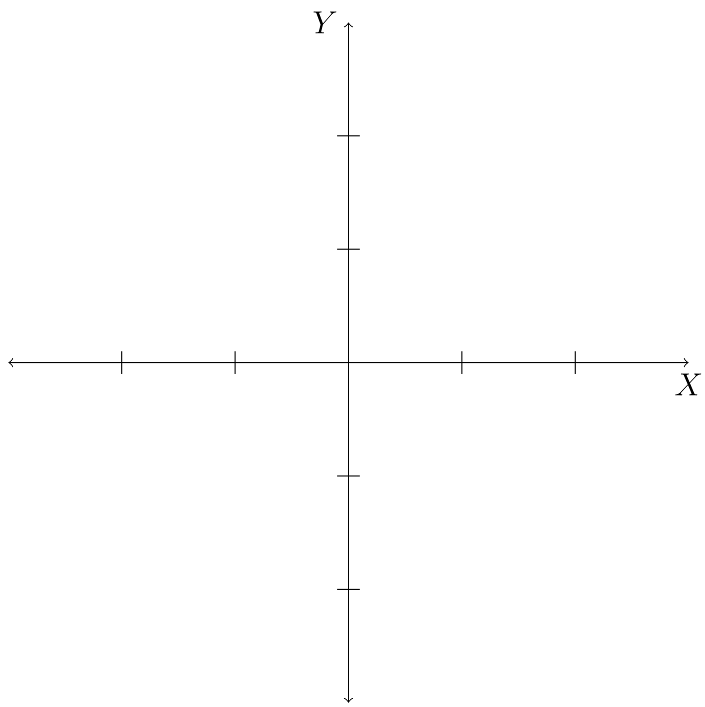
(c) $\lim_{h\to 0} \dfrac{(h+2)^3 - 8}{h}$

6. Derivatives

(1) What is the tangent line to the curve $y = f(x)$ at point $P = (a, f(a))$? Explain with an illustration.

(2) Find the slope of the tangent to the curve $y = x^3$ at the point $P = (1, 1)$ by using the table given below, and then by using the definition.

x	y	m
0		
0.5		
0.8		
0.9		
0.99		
2		
1.5		
1.1		
1.01		



(3) Give an alternative formula for the slope of the tangent line to the curve $y = f(x)$ at the poin $P = (a, f(a))$.

(4) Use this alternative formula for finding the slope of the tangent line to the curve $y = \dfrac{2}{x}$ at the point $P = (1, 2)$.

(5) Let $f(t)$ denote the position function of a moving object with respect to time t. Define average velocity (after h seconds) and instantaneous velocity at time $t = a$ seconds.

(6) Find the derivative of the function $f(x) = x^2 + 5x - 3$ at $x = a$.

The derivative of f at $x = a$ is denoted by $f'(a)$. So, the equation of the tangent line passing through $(a, f(a))$ is given by ________________________________.

(7) Find the equation of the tangent line to the parabola given by the equation $y = 3x^2 - 4x + 5$ at point $P = (2, 9)$.

(8) Let $y = f(x)$ for a function f. Then

- the **increment** in x when x changes from x_1 to x_2 is ____________________;
- the corresponding change in y is denoted by ____________________;
- the **average rate of change of** y **with respect to** x is ____________________;
- the **instantaneous rate of change of** y **with respect to** x **at** $x = x_1$ is ____________________;

(9) The cost of producing x widgets is f dollars.

- What is the meaning of the derivative $f'(x)$? What are its units?

- What is the meaning of the expression $f'(100) = 4$?

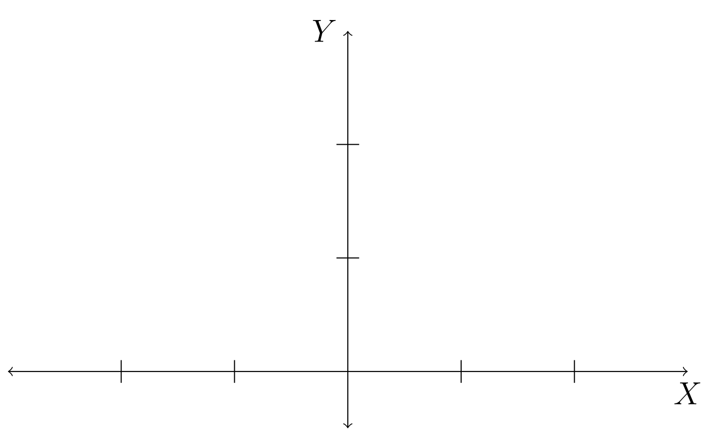
- Explain why the following may happen: $f'(1000) > f'(10) > f'(100)$?

(10) Find an equation of the tangent line to the curve $y = \sqrt{x-1}$ at the point $P = (2, 1)$.

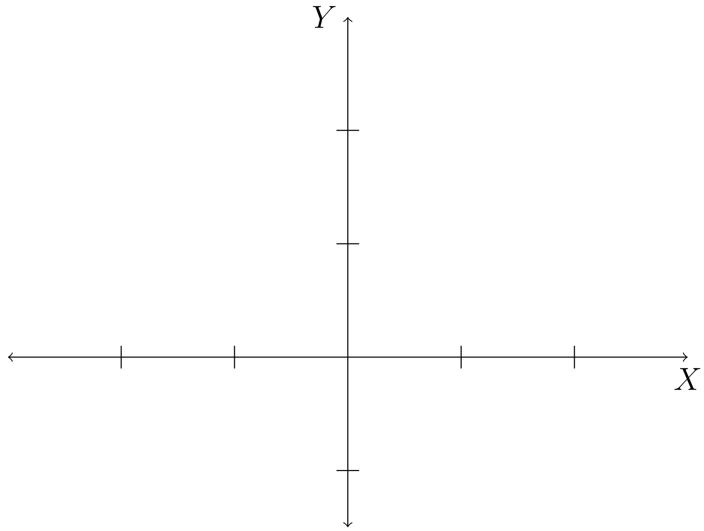
(11) Find an equation of the tangent line to the curve $y = g(x)$ at $x = 3$ if $g(3) = -1$ and $g'(3) = 2$.

(12) If the tangent line to the curve $y = f(x)$ at (0,1) passes through $(2, 3)$, then find $f(0)$ and $f'(0)$.

(13) Sketch the graph of a function for which $f(0) = 2$, $f'(0) = -1$, $f'(1) = 0$, $f'(2) = 3$, $f'(-1) = 0$, $f'(-2) = 2$.



(14) Sketch the graph of $f(x)$ where $f(-2) = 1$, $\lim_{x\to-2^-} f(x) = -\infty$, $\lim_{x\to-2^+} f(x) = \infty$, $f(-1) = 0$, $f'(-1) = -1$, $f(0) = -1$, $f'(0) = 0$, $f(1) = 0$, $f'(1) = 1$, $f(2) = 1$, $f'(2) = -1$.



(15) Let $f(x) = 4x^2 - x$. Find $f'(3)$ and use this to find an equation of the tangent to the curve $y = 4x^2 - x$ at the point $(3, 33)$.

(16) Let $f(x) = \dfrac{2x}{x^2-1}$. Find $f'(2)$ and use this to find an equation of the tangent to the curve $y = \dfrac{2x}{x^2-1}$ at $(2, \frac{4}{3})$.

(17) Find $f'(a)$ when $f(x) = 3x^2 - 4x + 5$.

(18) Find $f'(a)$ when $f(x) = \sqrt{x-2}$.

(19) Find $f'(a)$ when $f(x) = \dfrac{2-x}{3+x}$.

(20) Each limit represents the derivative of some function f at some number a. State f and a:

- $\lim_{h\to 0} \dfrac{\sin(\frac{\pi}{2}+h)-1}{h}$. Here $f(x) =$ ____________________, $a =$ __________.
- $\lim_{h\to 0} \dfrac{\sin(\pi+h)}{h}$. Here $f(x) =$ ____________________, $a =$ __________.
- $\lim_{x\to 3} \dfrac{4^x-64}{x-4}$. Here $f(x) =$ ____________________, $a =$ __________.
- $\lim_{x\to 0} \dfrac{\sqrt{4+x}-2}{x}$. Here $f(x) =$ ____________________, $a =$ __________.
- $\lim_{t\to 1} \dfrac{t^2+t-2}{t-1}$. Here $f(t) =$ ____________________, $a =$ __________.

(21) A particle moves along a straight line with equation of motion $s(t) = 2t^{-1} - t$ where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 4$ seconds.

(22) The number of widgets N used by people in the Bronx is shown in the table below:

t	1991	1993	1995	1997	1999	2001
N	56	88	100	153	80	40

(a) Find the rate of growth of widget use from 1991 to 1995. Do not forget to mention the units.

(b) Find the rate of growth of widget use from 1997 to 2001. Do not forget to mention the units.

(c) Estimate the instantaneous rate of widget use in 1995 by using the average of average velocities from 1991 to 1995 and 1993 to 1995. What are its units?

(23) Let $f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right) & \textit{for } x \neq 0 \\ 0 & \textit{for } x = 0. \end{cases}$. Find $f'(0)$.

(24) Let $f(x) = \begin{cases} x^2\sin\left(\dfrac{1}{x}\right) & \textit{for } x \neq 0 \\ 0 & \textit{for } x = 0. \end{cases}$. Find $f'(0)$.

(25) Let $f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \textit{for } x \neq 0 \\ 0 & \textit{for } x = 0. \end{cases}$. What can you say about $f'(0)$?

7. The Derivative as a Function

(1) Recall the definition of the derivative of function f at $a \in \mathbb{R}$ (both definitions).

(2) Use the appropriate definition from above to write function f' with variable x.

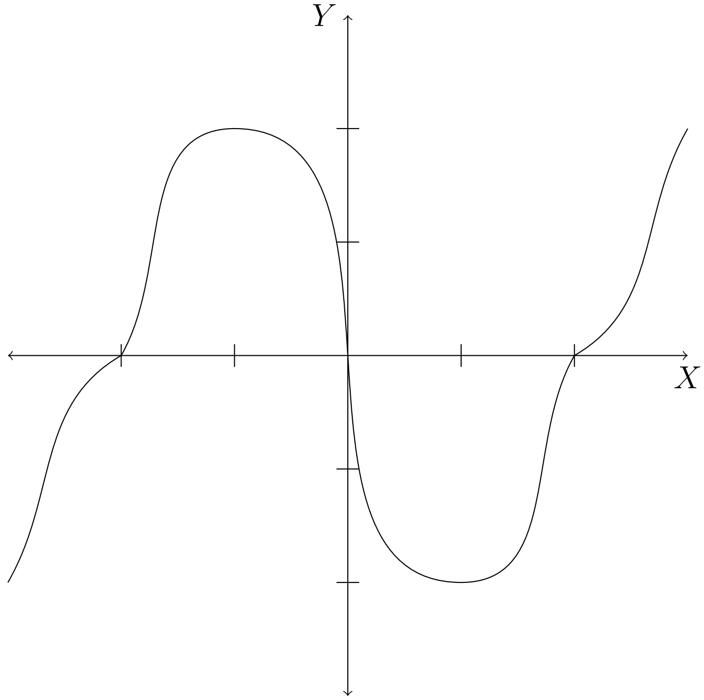
(3) Let $f(x) = x^3 - 3x$. For a fixed $a \in \mathbb{R}$, find $f'(a)$.

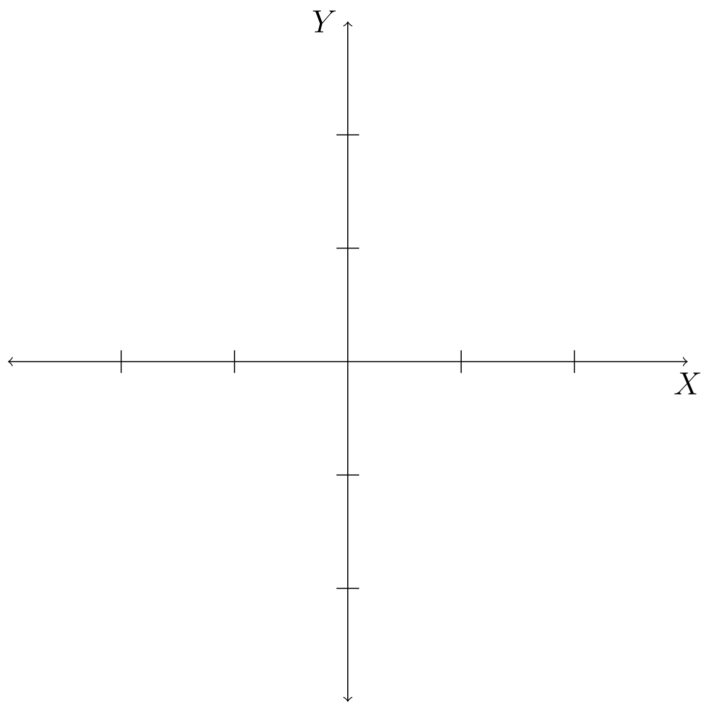
(4) Write down the function f' from the previous question in variable x.

(5) Using your graphing calculator, graph the function f from the previous question on a coordinate plane. Notice the intervals on which the graph is rising, the intervals where the graph is falling, and the points where the graph is horizontal.

(6) Graph the function f' on a coordinate plane. Understand the significance of the graph of f'.

(7) The graph of a function f is given below. Draw a possible graph of f'.





(8) Let $f(x) = \sqrt{x-1}$. Find the domain of f. Find f' and find its domain. Draw the graphs of the two functions.

(9) Let $f(x) = \dfrac{2-x}{1+x}$. Find the domain of f. Find f' and find its domain. Draw the graphs of the two functions.

(10) Explain the Leibnitz notation for the derivative of a function at x and the derivative at a value, $x = a$. Explain with an example.

(11) A function f is said to be differentiable at $x = a$ if ______________________________.
A function f is said to be differentiable on an open interval (a, b) if ______________________________
____.

(12) Where is the function $f(x) = |x|$ differentiable? Explain with graphs.

(13) Prove: If function f is differentiable at a, then f is continuous at a.

(14) Prove or disprove: If function f is continuous at a, then f is differentiable at a.

(15) Explain using illustrations the various ways a function may not be differentiable at a number $x = a$.

(16) Let $f(x) = x^3 + x$. Find $f'(x), f''(x), f'''(x), f^{(4)}(x), f^{(5)}(x)$.

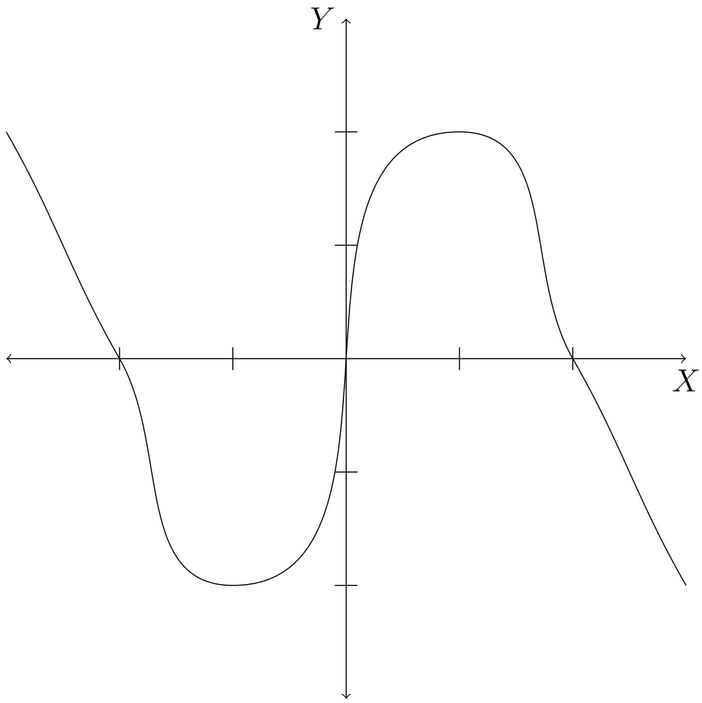
(17) Let s denote the position function of a particle in motion with respect to time t in seconds where $s(t)$ is measured in meters. The following are named

$s'(t) =$ ________________________ with units ________________________.

$s''(t) =$ ________________________ with units ________________________.

$s'''(t) =$ ________________________ with units ________________________.

(18) Use the given graph to estimate the value of each derivative. Then sketch the graph of f'.



$f'(-3) =$

$f'(-2) =$

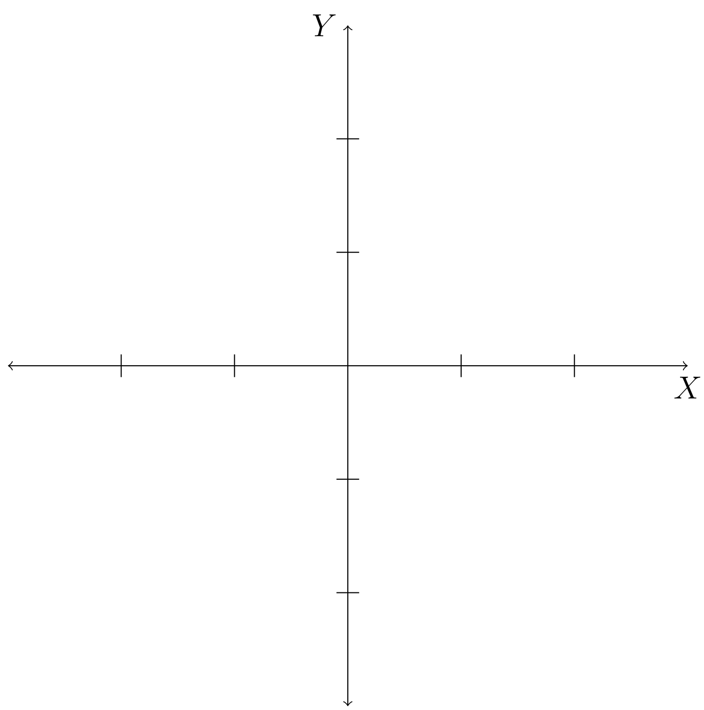
$f'(-1) =$

$f'(0) =$

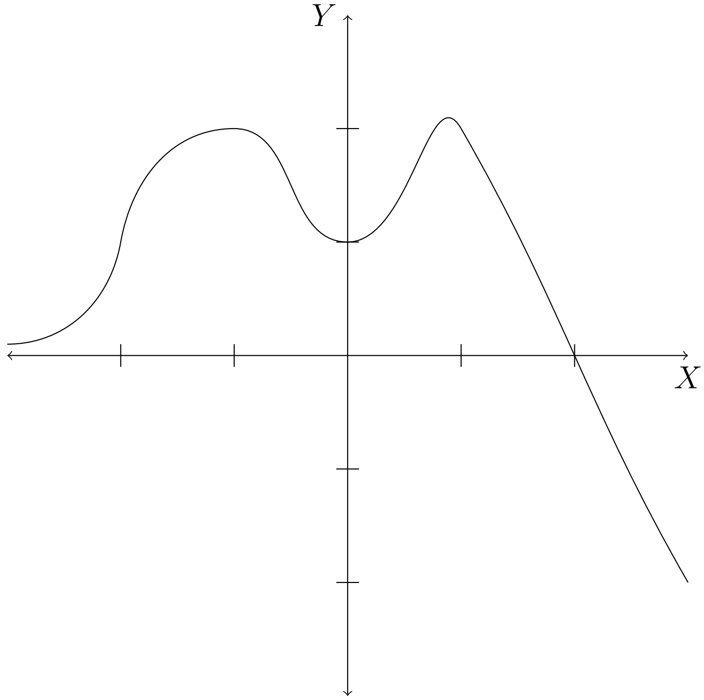
$f'(1) =$

$f'(2) =$

$f'(3) =$



(19) Use the given graph to estimate the value of each derivative. Then sketch the graph of f'.



$f'(-3) =$

$f'(-2) =$

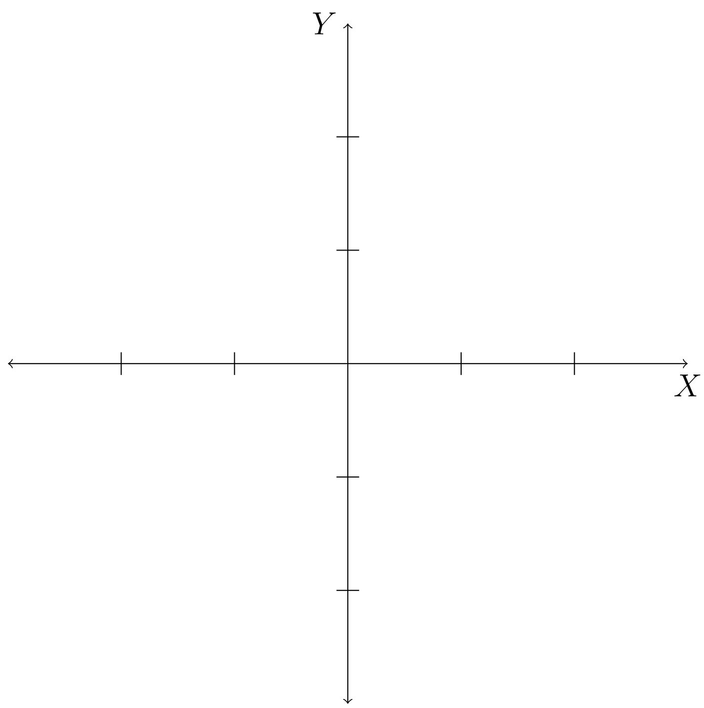
$f'(-1) =$

$f'(0) =$

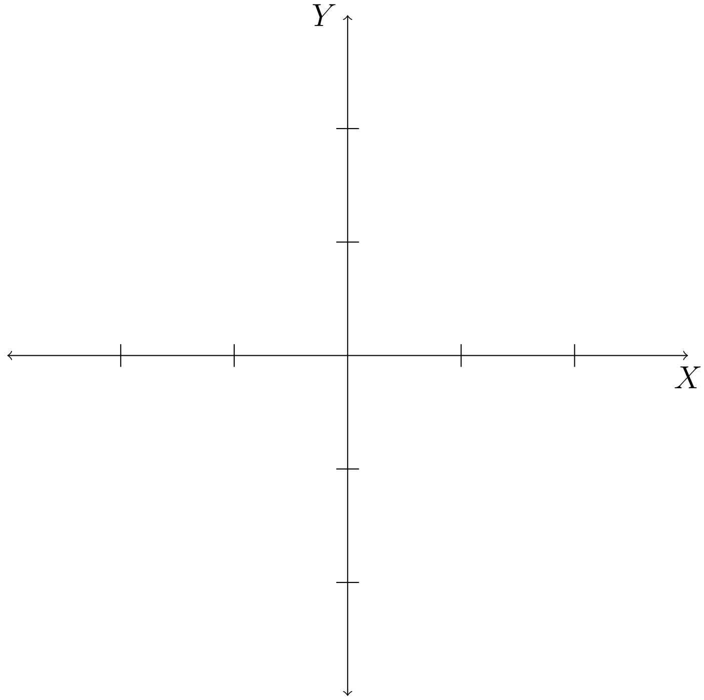
$f'(1) =$

$f'(2) =$

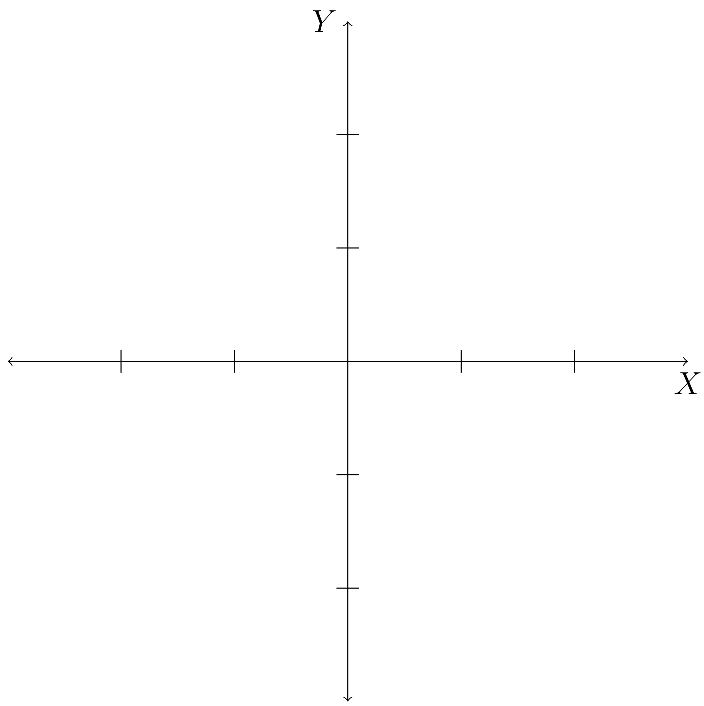
$f'(3) =$



(20) Draw the graph of any function f. Then sketch the graph of f'.



$$\begin{aligned} f'(-3) &= \\ f'(-2) &= \\ f'(-1) &= \\ f'(0) &= \\ f'(1) &= \\ f'(2) &= \\ f'(3) &= \end{aligned}$$



(21) Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

(a) $f(x) = \dfrac{2}{3}x - \dfrac{1}{2}$.

(b) For a fixed $m, b \in \mathbb{R}$, $f(x) = mx + b$.

(c) $f(x) = 3x^2 - 4x + 5$

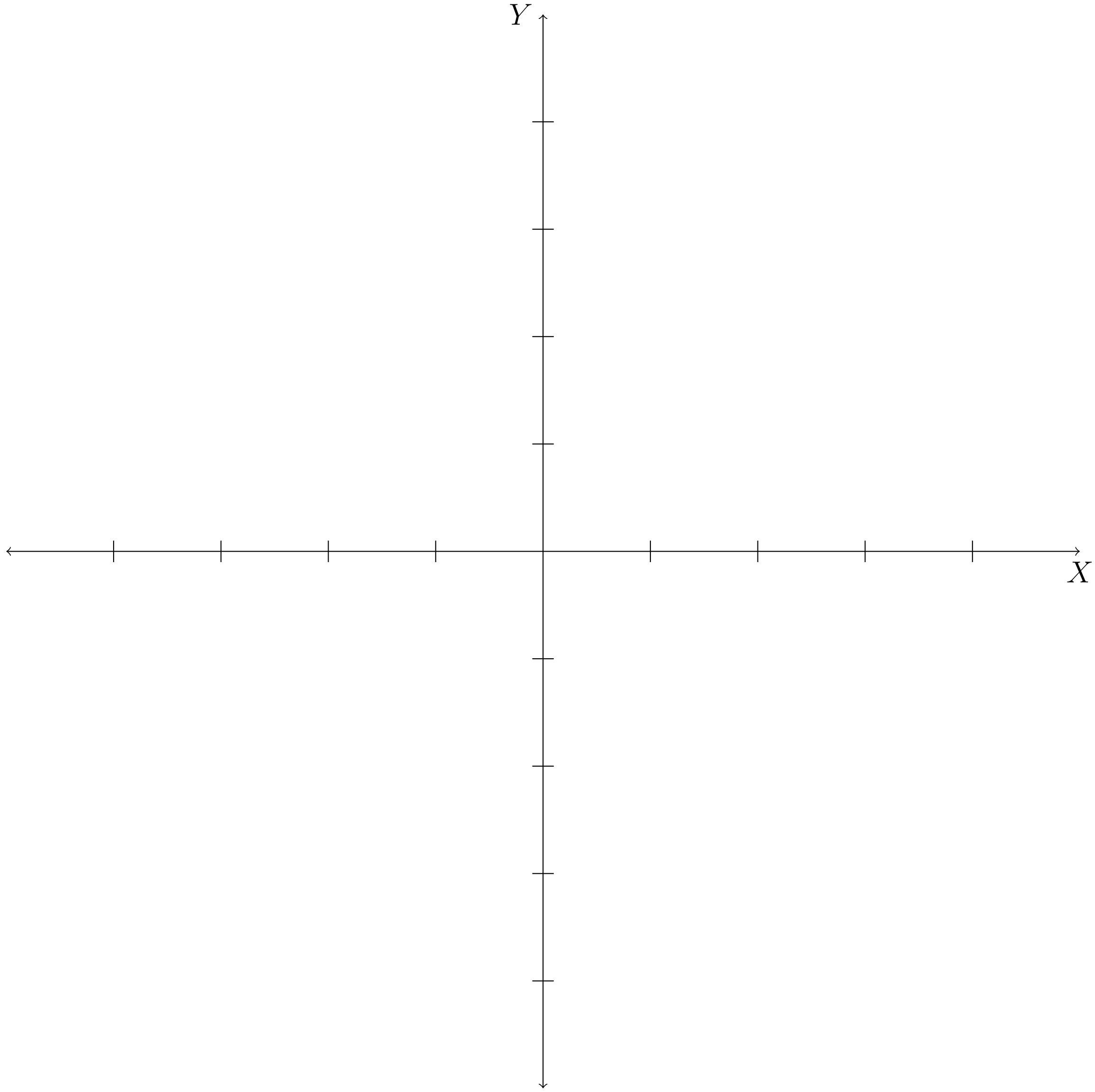
(d) $f(x) = \sqrt{5 - x}$

(e) $f(x) = \dfrac{1}{\sqrt{x}}$

(f) $f(x) = \dfrac{4-x}{x-2}$

(22) Find higher derivatives $f', f'', f''', f^{(4)}$ for $f(x) = 3x^4 - 5x + 4$.

(23) In the same coordinate plane below, draw the graphs of a function f and then possible graphs of its higher derivatives, $f', f'', f''', f^{(4)}$.



8. Differentiation Formulae

(1) Prove the differentiation formula for the constant function. That is, prove $\frac{d}{dx}(c) = 0$.

(2) Check the formula $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + a^{n-1})$ for any positive integer n.

(3) If n is a positive integer, then prove that $\frac{d}{dx}(x^n) = nx^{n-1}$.

(4) Find the derivatives:
- $\frac{d}{dx}(x^{100})$
- $\frac{d}{dx}(4^{100})$

(5) Prove that if f is a differentiable function and $c \in \mathbb{R}$ then $\dfrac{d}{dx}(cf(x)) = c\dfrac{d}{dx}(f(x))$.

(6) Find the derivatives:

- $\dfrac{d}{dx}(3x^{52})$
- $\dfrac{d}{dx}(3 \cdot 4^{52})$

(7) Prove that if f and g are differentiable functions then $\dfrac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$.

(8) Find the derivative $\dfrac{d}{dx}(3x^{52} + 3 \cdot 4^{52})$

(9) Prove that if f and g are differentiable functions then $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$.

(10) Find the derivative $\frac{d}{dx}(3x^{52} - 3 \cdot 4^{52})$

(11) (Product Rule) Prove that if f and g are differentiable functions then $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$.

(12) Find the derivatives:

- $\frac{d}{dx}(7x^6 \cdot 4x^9)$
- $\frac{d}{dx}(28x^{15})$

(13) (Quotient Rule) Prove that if f and g are differentiable functions then $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$.

(14) Find the derivative of

- $\dfrac{2x^3 - 4x + 6}{6x^2 + 7x - 1}$
- $\dfrac{3x^2 - 5x + 9}{x - 7}$

(15) For n a positive integer, prove that $\dfrac{d}{dx}(x^{-n}) = -nx^{-n-1}$. (Hint: Use Quotient Rule)

In fact, we have the General Power Rule: For **any real number** n, $\dfrac{d}{dx}(x^n) =$ ____________.

(16) Find the derivatives of:

(a) • $f(x) = x^{\pi}$ • $f(x) = x^{e}$.

(b) • $f(x) = \sqrt[3]{x}$ • $f(x) = \dfrac{1}{\sqrt{x}}$.

(c) • $f(x) = x^{5}$ • $f(x) = \pi^{5}$.

(d) • $f(x) = \sqrt{x}(x^2 - 5x + 6)$ • $f(x) = \pi^{5}(\sqrt[5]{x} + x^3)$.

(e) • $f(x) = \dfrac{\sqrt[3]{x}}{(x^2 - 5x + 6)}$ • $f(x) = \dfrac{x^5}{(\sqrt[5]{x} + x^3)}$.

(17) What is the normal line to a curve at a point P on the curve? Explain using an illustration.

(18) Find equations of the tangent and normal lines to the curve $y = \dfrac{\sqrt[3]{x}}{x-5}$ at the point $(1, -\frac{1}{4})$.

(19) Find the equations of the tangent and normal lines to the curve $xy = 3$.

(20) At which point(s) on the curve $xy = 4$ is the tangent line parallel to the line $4x + y = 3$?

(21) At which point(s) on the curve $y = 4x^2 - 6x + 7$ is the tangent line perpendicular to the line $4x - y = 3$?

(22) List all the differentiation formulae you learnt so far.

(23) Differentiate the functions:

(a) • $f(x) = x^{400}$ • $f(x) = 400^e$.

(b) • $f(x) = \sqrt[7]{x}$ • $f(x) = \dfrac{x}{\sqrt{x}}$.

(c) • $f(x) = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$ • $f(x) = \left(\sqrt{5} + \dfrac{1}{\sqrt{5}}\right)^2$.

(d) • $f(x) = \sqrt{x}(x^2 + 5x - 5)$ • $f(x) = (\sqrt{3x} + \sqrt{3}x)$.

(e) • $f(x) = \dfrac{\sqrt{x}}{(x^2 - 6x + 6)}$ • $f(x) = \dfrac{x^5}{(\sqrt[3]{x} + x^5)}$.

(f) • $f(x) = (2 - x + x^2)(x^5 + 7x^3 - 10)$ • $f(x) = \dfrac{x^3 - 4}{x^2 - 5x + 7}$.

(g) • $f(y) = \left(\dfrac{3}{y^2} - \dfrac{2}{y^3}\right)(y^{20} - 5y^3)$ • $f(y) = \dfrac{cy + 4}{\sqrt{y} + y}$.

(h) • $f(t) = \dfrac{5 + 6t}{7 - 3t}$ • $f(t) = \dfrac{t}{t + \frac{4}{t}}$.

(24) Find an equation of the tangent line and the normal line to the curve $y = 3x^2 - 4x + 5$ at $P = (0, 5)$.

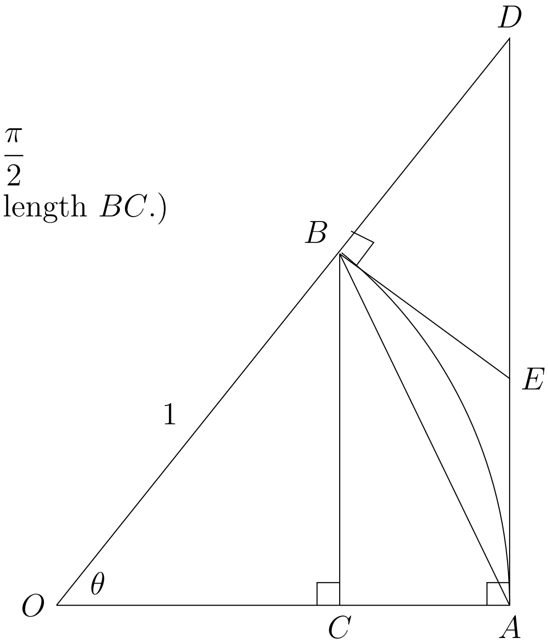
(25) Suppose $f(4) = 3, f'(4) = 2, g(4) = 7, g'(4) = -1$ then find

- $(fg)'(4)$
- $\left(\dfrac{f}{g}\right)'(4)$
- $\left(\dfrac{g}{f}\right)'(4)$

(26) Find the points on the curve $y = 2x^3 + 3x^2 - 4x + 7$ that has a tangent with slope 2.

9. Derivatives of Trigonometric Functions

(1) Use the figure given below to draw the following conclusions:



- $\sin(\theta) < \theta$ and so, $\dfrac{\sin(\theta)}{\theta} < 1$ for $0 < \theta < \dfrac{\pi}{2}$
(Hint: Show $\theta =$ arc length AB and $\sin(\theta) =$ length BC.)

- Use the fact that arc length $AB < |AE| + |EB|$ to show that $\theta < \tan(\theta)$.

- Use the above two items to show that $\cos(\theta) < \dfrac{\sin(\theta)}{\theta} < 1$. Use limits and the Squeeze theorem to find $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta}$.

We have, $\lim_{\theta\to 0} \dfrac{\sin(\theta)}{\theta} =$ ____________.

(2) Evaluate $\lim_{\theta\to 0} \dfrac{\cos(\theta) - 1}{\theta}$.

(3) Find the derivative of the function $f(x) = \sin(x)$ using the definition of the derivative.

(4) Find the derivative of the function $f(x) = \cos(x)$ using the definition of the derivative.

(5) Find the derivative of the function $f(x) = \tan(x)$.

(6) Find the derivative of the functon $f(x) = \csc(x)$.

(7) Find the derivative of the functon $f(x) = \sec(x)$.

(8) Find the derivative of the functon $f(x) = \cot(x)$.

(9) List the derivative formulae for the six trigonometric functions.

(10) Find the 28th derivative of $f(x) = \sin(x)$.

(11) Find the 44th derivative of $f(x) = \cos(x)$.

(12) Find $\lim_{x \to 0} x \cot(x)$.

(13) Find $\lim_{x \to 0} \dfrac{\sin(9x)}{2x}$.

(14) Find the derivative of

(a) • $f(x) = \sqrt{x}(x + \cos(x))$ • $f(x) = 3\csc(x) - \sec(x)$

(b) • $f(x) = \tan(x) + \frac{3}{4}\sin(x)$ • $f(x) = \cos(x) + \cot(x)$

(c) • $f(x) = \dfrac{x}{1 - \sin(x)}$ • $f(x) = \dfrac{\sin(x)}{1 - \cos(x)}$

(d) • $f(x) = \dfrac{\tan(x)}{(1 - \cos(x))}$ • $f(x) = x^2 \csc(x)$

(15) Find an equation of the tangent line to the curve $y = 3x\cos(x)$ at the point $P = (\pi, -3\pi)$. Illustrate by graphing the curve and its tangent.

(16) Find $f'(\theta), f''(\theta), f'''(\theta)$ for $f(\theta) = \theta\cos(\theta)$.

(17) Find the limits:

(a) • $\lim_{t\to 0} \dfrac{\sin(5t)}{3t}$ • $\lim_{t\to 0} \dfrac{\sin(7t)}{12t}$

(b) • $\lim_{t\to 0} \dfrac{\sin(5t)}{\tan(3t)}$ • $\lim_{t\to 0} \dfrac{\cos(t)-1}{\sin(t)}$

(c) • $\lim_{t\to 0} \dfrac{\sin(5t)\sin(2t)}{3t^2}$ • $\lim_{t\to 0} \dfrac{\sin(7t)}{t^2-t}$

(d) • $\lim_{t\to 0} \dfrac{2-\tan(t)}{\sin(t)-2\cos(t)}$ • $\lim_{t\to 0} \dfrac{\sin(t^2)}{12t}$

(18) Find the derivatives:

• $\dfrac{d^{87}}{dx}(\sin(x))$ • $\dfrac{d^{87}}{dx}(\cos(x))$

10. The Chain Rule

(1) State the Chain Rule for differentiation, using both Newton's and Leibnitz's notations.

(2) Find $f'(x)$ if

- $f(x) = \sqrt{x^3 + 5x}$
- $f(x) = \sin(x^2 - 5)$
- $f(x) = (x^3 + 5x^2 - 3x + 5)^{120}$
- $f(x) = \sin(\cos(\tan(x^2 - 5)))$

- $f(x) = \sin(x^{12})$
- $f(x) = \sin^{12}(x)$
- $f(x) = (x^3 + 5x^2 - 3x + 5)^2(x^2 + 5)^4$
- $f(x) = x^3 \tan(x^2 - 3)$
- $f(x) = \left(\dfrac{2x^2 - 3x + 4}{x + 5}\right)^3$
- $f(x) = \cos(x \sec(x))$
- $f(x) = \sqrt[7]{x^3 - \dfrac{5}{x^2}}$
- $f(x) = \cot(\csc(\sec(x^2 - 5)))$

(3) Find the first and second derivative of $f(x) = \cos^3(x)$.

(4) Find the first and second derivative of $f(x) = \cos(x^3)$.

(5) Find the equation of the tangent line to the curve $y = \sqrt{3 + x^2}$ at $(1, 2)$.

(6) Find the equation of the tangent line to the curve $y = \cos(x) + \cos^2(x)$ at $(0, 2)$.

(7) Find the equation of the tangent line to the curve $y = \cot(\frac{\pi x^2}{4})$ at $(1, 1)$.

(8) Let $r(x) = f(g(h(x)))$ with $h(2) = 3$, $g(3) = -7$, $f(-7) = 2$, $h'(2) = 1$, $g'(3) = 5$, $f'(-7) = 12$. Find $r'(2)$.

(9) Let $F(x) = 3f(xf(2x))$ with $f(1) = 2$, $f(2) = 4$, $f(4) = 7$, $f'(1) = 3$, $f'(2) = 1$, $f'(4) = 2$. Find $F'(1)$.

11. Implicit Differentiation

(1) Consider the curve $xy = 5$.

- Solve for y and find $\frac{dy}{dx}$ explicitly.
- Find $\frac{dy}{dx}$ using implicit differentiation.
- Explain using another example the difference between implicit and explicit differentiation.

(2) Draw the curve given by the equation $x^2 + y^2 = 9$.

Does this graph represent a function? Explain.

Find equations of the tangent and the normal to this circle at the point $(2, \sqrt{5})$.

Draw the tangent and normal in the diagram above.

(3) Consider the curve $x^5 + y^5 = 10$. Find y', y''.

(4) Find y' if $\cos(x+y) = y^2 \sin(x)$.

(5) Find y':

- $3x^2 - x + xy = 2$
- $\sin(x) + \sqrt{y} = 2$
- $3\sqrt{x} + \sqrt{y} = 7$
- $3x^4 - x^3 y + xy^3 = 7$

- $x\sin(y) + y\sin(x) = 1$
- $\tan(x-y) = \dfrac{y}{1+x^2}$

(6) Find an equation of the tangent line to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ at the point (x_0, y_0). [Compare with $\dfrac{x_0 x}{a^2} + \dfrac{y y_0}{b^2} = 1$.]

(7) Find an equation of the tangent line to the curve
- $y^2(y^2-4) = x^2(x^2-5)$ at $(0,-2)$.

- $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ at $(-3\sqrt{3}, 1)$.

- $\sin(x + y) = 2x - 2y$ at (π, π).

12. Rates of Change in Natural and Social Sciences

(1) Let $s(t) = t^3 - 8t^2 + 13t$ be the position function in feet of a moving particle where t is in seconds.

- Find the velocity $v(t)$ of the particle at time t. What is the velocity when $t = 3$ seconds?
- When is the particle at rest?
- When is the particle moving forward? When is it moving backward?
- Draw a diagram to represent the motion of the particle.
- Find the total distance travelled by the particle in the first 10 seconds.
- Find the acceleration $a(t)$ of the particle at time t. What is the acceleration when $t = 3$ seconds?
- Draw the graphs of $v(t)$ and $a(t)$ in the same coordinate plane.
- When is the particle speeding up?

(2) If a ball is thrown vertically upward with a velocity of 96 ft/s then its height after t seconds is $s = 96t - 16t^2$ feet.

- What is the maximum height reached by the ball?

- What is the velocity of the ball when it is 128 ft above the ground on its way up? On its way down?

(3) A paricle moves with position function $s(t) = t^4 - 4t^3 - 20t^2 + 20t$, $t \geq 0$. At what time does the particle have a velocity of 20 m/s? At what time is the acceleration 0? What is the significance of this value of t?

(4) A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s. Find the rate at which the area within the circle is increasing after 1 second, 2 seconds, 3 seconds, and 5 seconds?

(5) The volume of a growing spherical cell is $V = \frac{4}{3}\pi r^3$ where the radius r is measured in micrometers ($1\ \mu\text{m} = 10^{-6}$ m). Find the average rate of change of V with respect to r when r changes from

- 5 to 8 μm
- 5 to 6 μm
- 5 to 5.1 μm

Find the instantaneous rate of change of V with respect to r when $r = 5\ \mu$m

13. Related Rates

(1) Air is being pumped into a spherical balloon so that its volume increases at a rate of 120 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 30 cm?

(2) A ladder 12 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

(3) A water-tank has the shape of a circular cone with base radius 5 m and height 7 m. If water is leaking out of the tank at a rate of 1 m^3/min, find the rate at which the water level is changing when the water is 6 m deep.

(4) Car A is travelling east at 60 mi/h and car B is travelling north at 65 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.5 mi and car B is 0.4 mi from the intersection?

(5) A man walks along a straight path at a speed of 5 ft/s. A searchlight is located on the ground 30 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 12 ft from the point on the path closest to the searchlight?

14. Linear Approprixamations and Differentials

(1) What is the **linear approximation** or **tangent linear approximation** of a function f at real number a? Explain with an illustration.

(2) Find the linearization of the function $f(x) = \sqrt{x-2}$ at $a = 6$. Use this to find the approximate values of $\sqrt{3.96}$ and $\sqrt{4.01}$. Are these estimates overestimates or underestimates?

(3) For what values of x is the linear approximation $\sqrt{x-2} \approx \dfrac{x}{4} + \dfrac{1}{2}$ accurate to within 0.5? What about accuracy to within 0.1? Use your calculator.

(4) What is the differential dy for $y = f(x)$? How is it different from Δy, the incremental change in y? Explain with an example.

(5) Find the linear approximation of

- $f(x) = \sin(x)$ at $a = \pi/3$

- $f(x) = x^{\frac{2}{3}}$ at $x = 125$

(6) Find approximate values of $\sqrt{15.95}$, $\sqrt{15.99}$, $\sqrt{16.05}$, $\sqrt{16.01}$.

(7) Find approximate values of $\sqrt[3]{7.95}$, $\sqrt[3]{7.99}$, $\sqrt[3]{8.05}$, $\sqrt[3]{8.01}$.

(8) Determine the values of x for which the linear approximation is accurate to within 0.1:

- $(1+x)^{-3} \approx 1-3x$

- $\tan(x) \approx x$

(9) Find the differential dy and evaluate dy for the given values of x and dx.

- $y = \sin(\pi x)$, $x = \frac{1}{3}$, $dx = -0.02$

- $y = \dfrac{x+2}{x-1}$, $x = 2$. $dx = 0.05$

(10) Find approximate values of

- $\cos(1^o)$.

- $(2.3)^7$.

(11) The radius of a circular disk is 24 cm with a maximum error in measurement of 0.2 cm. Use differentials to estimate the maximum error in the calculated area of the disk. What is the relative error? What is the percentage error?

(12) Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

15. Review Chapter 2

(1) Find $f'(x)$ using the definition of a derivative of

- $f(x) = \dfrac{x+2}{x-3}$

- $f(x) = x^3 - 2x^2 + 4x - 5$

(2) Draw the graph of a differentiable function f. Then draw the possible graphs of f', f'', f'''.

(3) Find the derivative of the following:

- $\left(x+\frac{1}{x}\right)^{\sqrt{3}}$
- $(x^2+\sin(x)+\cos(x)+\tan(x))^{251}$
- $\sec(x^2-3)$
- $\cot(\frac{2}{x^2})$
- $\sqrt{x^2-3}\sin(x+5)$
- $\dfrac{\sqrt{x^2+5}}{\cos(x+5)}$

(4) Find $f^{(4)}(x)$ where $f(x)=\dfrac{1}{3-x}$

(5) Find the limits:

- $\lim_{x\to 0} \dfrac{1-\cos(x)}{\sin(x)}$
- $\lim_{t\to 0} \dfrac{t^5}{\tan^5(3t)}$

(6) Find equations of the tangent line and the normal line of to the curve $y = \sqrt{x}\sqrt{x-5}$ at the point $(6, \sqrt{6})$.

(7) A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm^3/s how fast is the water level rising when the water is 5 cm deep?

Try as many problems from this chapter as time permits you to.

16. Maximum and Minimum Values

(1) Let c be a real number in the domain D of a function f. When is the value $f(c)$ an absolute maximum on D? Explain with an example.

(2) Let c be a real number in the domain D of a function f. When is the value $f(c)$ an absolute minimum on D? Explain with an example.

(3) Let c be a real number in the domain D of a function f. When is the value $f(c)$ a local maximum on D? Explain with an example where $f(c)$ is a local maximum but not an absolute maximum.

(4) Let c be a real number in the domain D of a function f. When is the value $f(c)$ a local minimum on D? Explain with an example where $f(c)$ is a local minimum but not an absolute minimum.

(5) Given an example of a function which does not have any extremal points, local or absolute and draw the graph.

(6) State the **Extremal Value Theorem**. Explain with illustrations.

(7) Give an example illustrating why continuity of the function on a closed interval is necessary for the Extremal Value Theorem to hold true.

(8) Give an example illustrating why the domain of a continuous function needs to be a closed interval for the Extremal Value Theorem to hold true.

(9) State and prove Fermat's Teorem. Explain with an illustration.

(10) Give an example of a function f and a number a in the domain of f such that

- $f(a) = 2$ but $f'(a)$ does not exist;
- $f(a)$ is the absolute maximum;
- Draw the graph of f.

(11) Give an example of a function f and a number a in the domain of f such that

- $f(a) = 2$ but $f'(a)$ does not exist;
- $f(a)$ is the absolute minimum;
- Draw the graph of f.

(12) Define a critical number of a function f. Explain using three different examples.

(13) Find critical numbers of $f(x) = (x-1)\sqrt{x+2}$.

(14) If f has a local maximum or minimum at c, then ____.

(15) List the steps to find the absolute maximum and absolute minimum values of a continuous function f on a closed interval $[a, b]$.

(16) Find the absolute maximum and minimum values of the function $f(x) = 2x^3 + x^2 - 20x + 2$ for $-3 \leq x \leq 3$.

(17) Find the absolute maximum and minimum values of the function $f(\theta) = \theta - 2\cos(\theta)$ for $0 \leq \theta \leq 2\pi$.

(18) Sketch the graph of a function f with domain $[1, 4]$ satisfying

- $f'(2) = 0$ but $f(2)$ is not a local maximum or minimum.

- f has absolute maximum at $x = 1$, absolute minimum at $x = 2$ and local minimum at $x = 4$.

17. The Mean Value Theorem

(1) State and prove the Rolle's theorem. Explain with illustrations.

(2) Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's theorem.
- $f(x) = x^3 - x^2 - 6x + 2$ on $[0, 3]$.

- $f(x) = \cos(2x)$ on $[\pi/8, 7\pi/8]$.

(3) Let $f(x) = \dfrac{1}{x^2}$. Show that $f(-2) = f(2)$, but there is no c in $(-2, 2)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?

(4) State and prove the Mean Value Theorem. Explain with illustrations.

(5) Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c which satisfy the conclusion of the Mean Value Theorem.

- $f(x) = x^3 - 3x + 2$ on $[-2, 2]$

- $f(x) = \dfrac{1}{x}$ on $[1, 3]$

(6) Let $f(x) = 2 - |2x - 1|$. Show that there is no value c such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Value Theorem?

(7) Show that the equation $2x - 1 - \sin(x) = 0$ has exactly one root.

(8) Show that the equation $x^2 - 15x + c = 0$ has at most two real roots.

18. How Derivatives Affect the Shape of a Graph

(1) Prove the following **Increasing Test**, and explain with an illustration: If $f'(x) > 0$ on an interval then f is increasing on that interval.

(2) Prove the following **Decreasing Test**, and explain with an illustration: If $f'(x) < 0$ on an interval then f is decreasing on that interval.

(3) Draw the graph of a function. Find the intervals on which f is increasing and the intervals on which f is decreasing.

(4) For the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals on which f is increasing or decreasing.

(5) Complete the following **First Derivative Test** and draw appropriate illustrations: Suppose that c is a critical number of a continuous function f:

- If f' changes from positive to negative at c, then f has ________________________ at c.

- If f' changes from negative to positive at c, then f has ________________________ at c.

- If f' does not change sign at c then, f __________________________ at c.

(6) Find the local minimum and maximum values of the function $f(x) = x^3 - 3x^2 - 9x + 2$.

(7) Find the local maximum and minimum values of the function $f(x) = \dfrac{x}{x^2+1}$.

(8) The graph of f said to be concave upward on an interval I if the graph of f lies ____________ all of its tangents on the interval I. Explain using an illustration.

(9) The graph of f said to be concave downward on an interval I if the graph of f lies ____________ all of its tangents on the interval I. Explain using an illustration.

(10) **Concavity Test:** Explain the following using illustrations

- If $f''(x) > 0$ for all x in I, then the graph of f is concave ____________ on I.

- If $f''(x) < 0$ for all x in I, then the graph of f is concave __________ on I.

(11) A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from ____

____.

(12) Give an example of a function where concavity changes from upward to downward at $x = 2$ and the curve has an inflection point. Draw its graph.

(13) Give an example of a function where concavity changes from downward to upward at $x = 2$ and the curve has an inflection point. Draw its graph.

(14) Give an example of a function where concavity changes from upward to downward at $x = 2$ and the curve does not have an inflection point. Draw its graph.

(15) Give an example of a function where concavity changes from downward to upward at $x = 2$ and the curve does not have an inflection point. Draw its graph.

(16) **The Second Derivative Test:** Draw illustrations for each of the following. Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then ______________________________________.

- If $f'(c) = 0$ and $f''(c) < 0$, then ______________________________________.

(17) Find the local maximum and minimum values of f using both the First and Second Derivative tests. $f(x) = x^4 - 2x^2$.

(18) For the given functions,

- find the x, y-intercepts whenever the calculations seem reasonable;
- find the intervals on which f is increasing or decreasing;
- find the local maximum and minimum values of f;
- find the intervals of concavity and the inflection points.

(a) $f(x) = \cos^2(x) - 2\sin(x)$ for $0 \leq x \leq 2\pi$.

(b) $f(x) = 2x^3 - 9x^2 - 60x - 20$

(19) Sketch the graph of the following functions using the given steps:

- Find the intervals of increase or decrease.
- Find the local maximum and minimum values.
- Find the intervals of concavity and the inflection points.

(a) $f(x) = 36x - 3x^2 - 2x^3$

(b) $f(x) = x^4 + 6x^3 + 10$

(c) $f(x) = 4x^{\frac{2}{3}} - 3x^{\frac{5}{3}}$

19. Limits at Infinity; Horizontal Asymptotes

(1) Let f be a function defined on some interval (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that

____

____.

(2) Let f be a function defined on some interval $(-\infty, a)$. Then $\lim_{x\to-\infty} f(x) = L$ means that

____

____.

(3) The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if

____.

(4) Draw the graph of a function satisfying

• $f(0) = 1$, • $\lim_{x\to-\infty} f(x) = 2$, • $\lim_{x\to-3^-} f(x) = \infty$, • $\lim_{x\to-3^+} f(x) = -\infty$,
• $\lim_{x\to2} f(x) = -\infty$, • $\lim_{x\to\infty} f(x) = -3$.

List all the asymptotes you see in this graph.

(5) What are the following limits:

- $\lim_{x\to\infty} \dfrac{1}{x} =$ ____.
- $\lim_{x\to\infty} \dfrac{1}{x^2} =$ ____.
- $\lim_{x\to\infty} \dfrac{1}{x^3} =$ ____.
- $\lim_{x\to\infty} \dfrac{1}{x^{\frac{1}{2}}} =$ ____.
- $\lim_{x\to\infty} \dfrac{1}{x^{\frac{3}{8}}} =$ ____.
- $\lim_{x\to\infty} \dfrac{1}{x^r}$ for any rational number $r > 0$ is __________________________.

(6) Is $\dfrac{1}{x^r}$ defined for $x < 0$ for any r? Explain.

(7) If $r > 0$ is a rational number such that x^r is defined for all x, then $\lim_{x\to-\infty} \dfrac{1}{x^r} =$ __________________________.

(8) Find the limits and explain your steps:

- $\lim_{x\to\infty} \dfrac{5x^2 - 7x + 8}{3x^2 + 2x + 9}$

- $\lim_{x\to\infty} \dfrac{-7x + 8}{3x^2 + 2x + 9}$

- $\lim_{x\to\infty} \dfrac{5x^2-7x+8}{2x+9}$

- $\lim_{x\to-\infty} \dfrac{4x^2+8x-3}{3x^2-3x+5}$

- $\lim_{x\to-\infty} \dfrac{8x-3}{3x^2-3x+5}$

- $\lim_{x\to-\infty} \dfrac{4x^2+8x-3}{-3x+5}$

(9) Find the horizontal and vertical asymptotes of $f(x) = \dfrac{\sqrt{3x^2+4}}{2x-9}$.

(10) Find:

- $\lim_{x\to\infty}(\sqrt{x^2+5}-x)$

- $\lim_{x\to\infty} \sin\left(\dfrac{1}{x}\right)$.

- $\lim_{x\to-\infty} \sin\left(\frac{1}{x}\right)$.

- $\lim_{x\to\infty} \sin(x)$.

- $\lim_{x\to-\infty} \sin(x)$.

- $\lim_{x\to\infty} x^3 - 4x^2$.

(11) State the precise definitions of

- $\lim_{x\to\infty} f(x) = L$
- $\lim_{x\to-\infty} f(x) = L$
- $\lim_{x\to\infty} f(x) = \infty$
- $\lim_{x\to\infty} f(x) = -\infty$
- $\lim_{x\to-\infty} f(x) = \infty$
- $\lim_{x\to-\infty} f(x) = -\infty$

(12) Find the limit or show it does not exist:

- $\lim_{x\to\infty} \dfrac{3x^3 - 5x^2 + 6x - 7}{7x^3 + 8x - 3}$

- $\lim_{x\to\infty} \dfrac{2x - 3}{5 - x}$

- $\lim_{x\to\infty} \dfrac{x - x^2\sqrt{x}}{x^{\frac{5}{2}} - x^{\frac{3}{2}} + x}$

- $\lim_{x\to-\infty} \dfrac{\sqrt{3x^2 + 4}}{x + 3}$ (Note: Here, x tends to $-\infty$).

- $\lim_{x\to\infty} \cos(x)$

- $\lim_{x\to\infty}(x + \sqrt{x^2 + 3x})$

- $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$.

20. Summary of Curve Sketching

(1) Here are guidelines for sketching a curve. Explain each step in your own words.

(a) **Domain:**

(b) X**-intercepts (tangential and transversal):**

(c) Y**-intercept:**

(d) **Symmetry (even, odd, periodic):**

(e) **Asymptotes (Vertical, Horizontal, Slant):**

(f) **Intervals of increase or decrease:**

(g) **Local maximum/minimum:**

(h) **Concavity and Points of inflection:**

(2) Using the guidelines which you have built, sketch the curve:

- $y = \dfrac{2x^2}{x^2 - 4}$

- $y = \dfrac{2x^2}{\sqrt{x-4}}$

- $y = \dfrac{\sin(x)}{2+\cos(x)}$

- $y = \dfrac{x^3}{x^2 - 1}$

- $y = 2 + 3x^2 - x^3$

- $y = x^5 - 16x$

- $y = \dfrac{x^2 - 9}{x^2 - 3x}$

- $y = \dfrac{x^2}{x^2+4}$

- $y = 1 + \dfrac{1}{x} + \dfrac{1}{x^2}$

- $y = \sqrt{x^2 + x} - x$

- $y = x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$

- $y = x + \cos(x)$

- $y = 2x - \tan(x)$ for $-\pi/2 < x < \pi/2$

21. Optimization Problems

(1) Find two numbers whose difference is 80 and whose product is a minimum.

(2) What is the minimum vertical distance between the parabolas $y = x^2 + 4$ and $y = 2x - x^2$?

(3) Find the dimensions of a rectangle with area 250 sq. m. whose perimeter is as small as possible.

(4) We have 150 sq. cm. of cardboard. What is the maximum volume of a box with open top and a square base that can be constructed from this cardboard.

(5) Find the point on the line $y = 2x - 3$ that is closest to the origin.

(6) Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

(7) Find the point on the line $y = 3x + 1$ that is closest to the point $(3, 1)$.

(8) A right circular cylinder is inscribed in a cone with height h and has radius r. Find the largest possible volume of such a cylinder.

22. Newton's Method

(1) Explain the Newton-Raphson method. Draw an illustration.

(2) Starting with $x_1 = 2$, find the third approximation x_3 to the root of $x^3 - 3x - 3 = 0$.

(3) Use Newton-Raphson (or simply, Newton's method) to find $\sqrt[7]{3}$ correct to eight decimal places.

(4) Find roots up to 6 decimal places of

- $\sqrt{x+1} = x^2 - x$

- $\sin(x) = x^2 - 2$

23. Antiderivatives

(1) When is function F said to be an antiderivative of f on interval I? Give 5 examples.

(2) What is the most general antiderivative of f on I?

f	General F
$x^n, n \neq -1$	
$\cos(x)$	
$\sin(x)$	
$\sec^2(x)$	
$\sec(x)\tan(x)$	
$\csc^2(x)$	
$\csc(x)\cot(x)$	

(3) Find the most general antiderivative of the function. Check your answer:

- $f(x) = x^3 - 2x^2 + 5x - 7$
- $f(x) = x(3-x)^2$.
- $f(x) = \pi^7$
- $f(x) = \dfrac{4x^5 + 5x^3 - 8x^2 + x - 1}{x^7}$.
- $f(x) = 2\cos(x) + 3\sin(x) - \sec^2(x) + 4\sec(x)\tan(x)$
- $f(x) = 2x^{\frac{4}{3}} - 5x^{\frac{7}{3}}$
- $f(x) = \sqrt[4]{x} + \csc^2(x) - \csc(x)\cot(x)$
- $f(x) = x^5 - 7x + 10$

(4) Given $f(x) = 3x^2 - \sin(x)$, find its antiderivative F when $F(0) = 2$.

(5) Find f:

- $f^{(4)}(x) = x^2 - 3x,\ f(0) = 1,\ f'(0) = 2,\ f''(0) = 3, f'''(0) = 4.$

- $f''(x) = x^2 - 3x + \cos(x)$

- $f^{(3)}(x) = x^5 - 3x^2 + \sin(x)$

- $f''(x) = x^{-\frac{2}{3}}$, with $f(1) = 2$, $f'(-1) = 3$.

- Find a function f such that $f'(x) = x^5$ and the line $y = x$ is tangent to the graph of f.

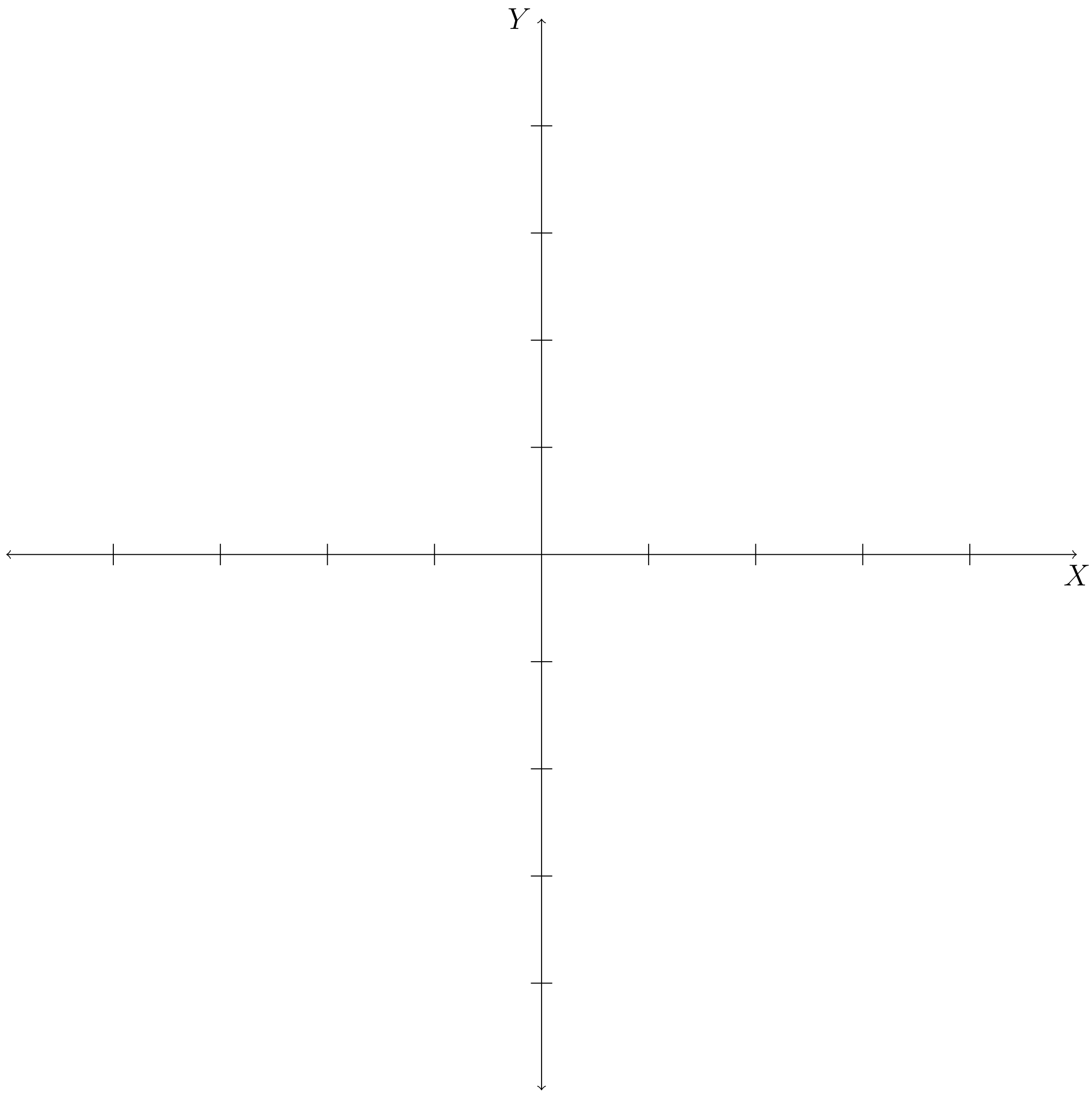
- A particle is moving with velocity $v(t) = 3.5\sqrt[3]{t}$ and $s(0) = 5$ ft. Find the position function $s(t)$.

24. Review Chapter 3

(1) Find the local and absolute extreme values of
- $f(x) = \dfrac{4x-5}{x^2+1}$ on $[-3, 3]$.

- $f(x) = \sin^2(x) + \cos(x)$ on $[0, \pi]$.

(2) In the given coordinate plane, draw the graph of an even and continuous function with domain $[-5, 5]$ such that $f(0) = 2$, $f(1) = 3$, $f(2) = 1$, $f(3) = 0$, $f(4) = -1$, $f(5) = -3$, $f'(0)$ does not exist, $f'(1) = 0$, $f'(2) = -2$, $f'(3) = 0$, $f' > 0$ on $(0, 1)$, $f' \leq 0$ on $(1, 5)$, $f'' < 0$ on $(0, 3)$, and $f'' > 0$ on $(3, 5)$.



(3) Find the point on the circle $x^2 + y^2 = 5$ that is closest to $(3, 4)$.

(4) Sketch the curve $y = \dfrac{1}{(x-1)^2} - \dfrac{1}{(x-3)^2}$.

(5) Sketch the curve $y = \sqrt[3]{x^2 + 4}$.

(6) Find f:

- $f''(x) = 3 - 4x + 5x^2$ with $f(0) = 2$ and $f'(0) = 5$.

- $f'(x) = \dfrac{\sqrt[3]{u} - 3u^2}{u^4}$ with $f(1) = 2$.

25. Areas and Distance

(1) Draw an illustration of four rectangles to estimate the area under the parabola $y = x^2$ from $x = 1$ to $x = 3$ using
- left endpoints;
- right endpoints;
- midpoints;

Guess the actual area.

(2) Draw an illustration of four rectangles to estimate the area under the parabola $y = \sqrt{x}$ from $x = 1$ to $x = 3$ using

- left endpoints;

- right endpoints;

- midpoints;

Guess the actual area.

(3) Speedometer readings for a motorcycle at 15 second intervals are given in the table.

- Estimate the distance travelled by the motorcycle during this time period using the velocities at the beginning of the time intervals.
- Give another estimate using the velocities at the end of the time periods.
- Are your estimates in the previous parts upper or lower estimates? Explain.

t (seconds)	0	15	30	45	60
v (ft/sec)	40	38	34	30	32

(4) State the definiton of the area A of the region under the graph of a continuous functon using limit Riemann sums. Draw an illustration to explain this procedure.

(5) Use the definition from the previous question to find an expression for te area under the graph of f as a limit:

- $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$ for $2 \leq x \leq 6$.
- $f(x) = \sqrt{\cos(x)}$ on $[0, \pi/2]$.

(6) Determine the region whose area is equal to

- $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{6n} \tan\left(\frac{i\pi}{6n}\right)$

- $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \sqrt{2 + \frac{3i}{n}}$

26. The Definite Integral

(1) What is the Definite Integral of a function f from a to b?

(2) The symbol $\int$ was introduced by ________________________ and is called an ________________________. It is an elongated S and is chosen because an integral is a ____________________________________.

(3) In the notation $\int_a^b f(x)dx$,
$f(x)$ is called ____________________________________,
a and b are called ____________________________________,
a is the ____________________________________,
and b is the ____________________________________.

(4) The symbol dx simply indicates that ____
____.

(5) The sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called ____________________________________, named after the German mathematician ____________________________________.

(6) **Theorem:** If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuites on $[a, b]$, then
____;
that is, ________________________ exists.

(7) **Theorem on page 298**

(8) Complete the following formulae and check them for $n = 1, 2, 3, 4, 5$:

- $\sum_{i=1}^{n} 1 =$ ____________________________

- $\sum_{i=1}^{n} i =$ ____________________________

- $\sum_{i=1}^{n} i^2 =$ ____________________________

- $\sum_{i=1}^{n} i^3 =$ ____________________________

(9) Evaluate the Riemann sum for $f(x) = x^3 + 2x^2 - 3x + 5$ taking the sample points to be left end points for $a = 0, b = 2$ and $n = 4$.

(10) Evaluate the Riemann sum for $f(x) = x^3 + 2x^2 - 3x + 5$ taking the sample points to be midpoints for $a = 0, b = 2$ and $n = 4$.

(11) Evaluate $\int_0^2 (x^3 + 2x^2 - 3x + 5)dx$ using limits of Riemann sums.

(12) **Properties of Definite Integrals:** Explain in your own words

- $\int_b^a f(x)dx =$____________________________________.

- $\int_a^a f(x)dx =$____________________________________.

- $\int_a^b cf(x)dx =$____________________________________.

- $\int_a^b [f(x) + g(x)]dx =$____________________________________.

- $\int_a^b [f(x) - g(x)]dx =$____________________________________.

- $\int_a^b c dx =$____________________________________.

Fill in the blanks with the appropriate inequality.

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx$ ____________________ 0.

- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx$ ____________________ $\int_a^b g(x)dx$.

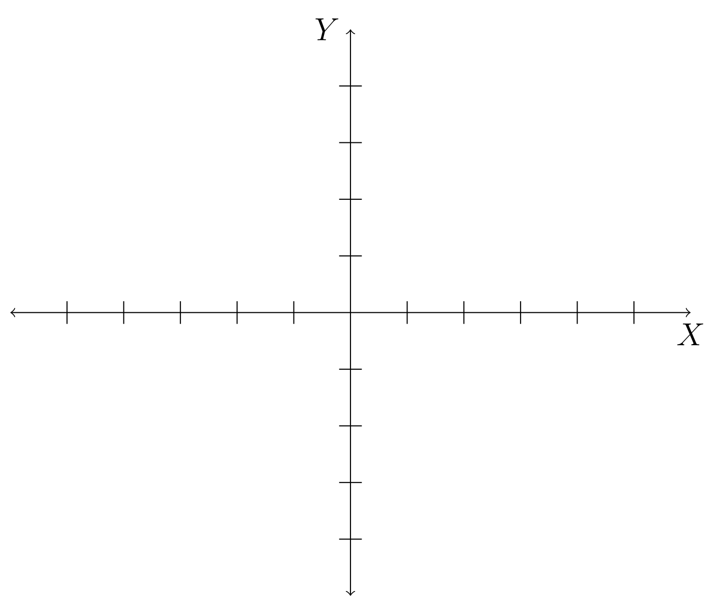
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a)$ __________ $\int_a^b f(x)dx$ __________ $M(b-a)$.

(13) Estimate $\int_1^8 \sqrt[3]{x}dx$.

(14) Find the Riemann sum for $f(x) = \cos(x)$, $0 \leq x \leq 3\pi/2$ with 6 terms using right end points. Use a calculator correct up to 4 decimal places.

(15) Find the Riemann sum for $f(x) = \cos(x)$, $0 \leq x \leq 3\pi/2$ with 6 terms using left end points. Use a calculator correct up to 4 decimal places.

(16) In the given coordinate plane below, draw the graph of a function f on the interval $[-5, 5]$ with $f(-5) = 2$, $f(-3) = 1$, $f(1) = -2$, $f(3) = 3$, and $f(5) = 1$. Estimate $\int_{-5}^{5} f(t)dt$. (Your answer will depend on your graph).



(17) Using $n = 4$ and midpoints, find the approximate value $\int_1^4 \sqrt{x^2+2}\, dx$.

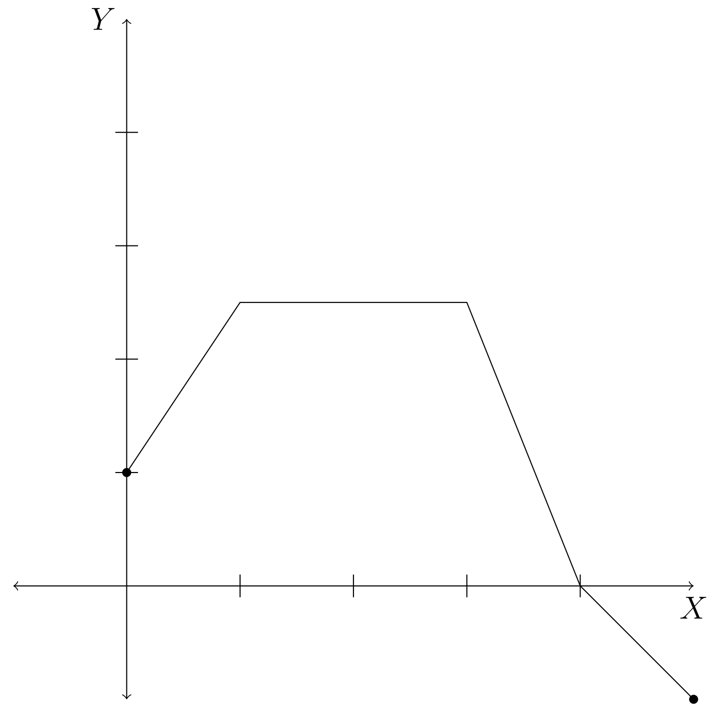
(18) $\lim_{n\to\infty} \sum_{i=1}^{n} (2 + \frac{8i}{n})[5(x_i^*)^7 - 7x_i^*] =$ ________________________.

(19) Evaluate (using limits of Riemann sums): $\int_1^5 (x^2 - 3x + 5)dx$.

(20) Evaluate $\int_{-5}^{5} (2 + \sqrt{25 - x^2})dx$.

27. The Fundamental Theorem of Calculus

(1) Let f be the function whose graph is shown. Let $g(x) = \int_0^x f(t)dt$.
Find $g(1), g(2), g(3), g(4), g(5)$.



(2) State the **Fundamental Theorem of Calculus, Part 1**. Write down the proof of this theorem from your textbook.

(3) Find the derivative of:

- $g(x) = \int_0^x \sqrt{4+t^2}dt$

- $h(x) = \int_0^x \sin\left(\dfrac{\pi t^2}{2}\right) dt$

- $k(x) = \int_0^{x^6} \cos(t^2+5)dt$ (Hint: Do not forget the Chain rule).

(4) State the **Fundamental Theorem of Calculus, Part 2**. Write down the proof of this theorem from your textbook.

(5) State the **Fundamental Theorem of Calculus**.

(6) Find the derivative of the following:

- $f(x) = \int_0^x \sqrt{r^4 + r^2 + 5}\ dr$
- $g(t) = \int_t^1 \left(\dfrac{1}{u^2} + 3u^2\right) du$
- $f(x) = \int_0^{\sqrt{x}} \sqrt{r^4 + r^2 + 5} dr$
- $g(t) = \int_{\sin(t)}^1 \left(\dfrac{1}{u^2} + 3u^2\right) du$

(7) Evaluate the integral:

- $\int_0^3 r^4 + r^2 + 5\ dr$
- $\int_1^8 \left(\dfrac{1}{u^2} + 3u^2\right) du$
- $\int_1^3 \dfrac{r^4 + r^2 + 5}{r^9} dr$
- $\int_0^8 \left(\sqrt[3]{u} + 3u^2\right) du$

- $\int_{-\pi/4}^{\pi/4} \sec(\theta)\tan(\theta)\ d\theta$
- $\int_1^8 \left(\sqrt[3]{\frac{5}{u}}\right) du$

- $\int_{-\pi/2}^{\pi} f(t)dt$ where $f(t) = \begin{cases} \cos(t) & \textit{for } -\pi/2 \leq t \leq \pi/2 \\ \sin(t) & \textit{for } \pi/2 \leq t \leq \pi. \end{cases}$

28. Indefinite Integrals and the Net Change Theorem

(1) What does it mean to say $\int f(x)dx = F(x)$? Present an example.

(2) What is $\int (x^3 + x^2)dx$?

(3) Complete the table:

$\int cf(x)dx$	
$\int [f(x) + g(x)]dx$	
$\int [f(x) - g(x)]dx$	
$\int kdx$	
$\int x^n dx$	
$\int \sin(x)dx$	
$\int \cos(x)dx$	
$\int \sec(x)\tan(x)dx$	
$\int \sec^2(x)dx$	
$\int \csc(x)\cot(x)dx$	
$\int \csc^2(x)dx$	

(4) Find $\int (5x^3 - 3x^2 + 2x + 3\sec^2(x))dx$

(5) Find $\int \frac{\sin(x)}{\cos^2(x)}dx$

(6) Verify by differentiation that the formula is correct: $\int \cos^2(x)dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$.

(7) Find the general indefinite integral:

- $\int(\sqrt{x^5} + \sqrt[3]{x^7})dx$
- $\int(\sqrt{x^5} + \sqrt[3]{x^7})^2dx$
- $\int \sec(\theta)(\sec(\theta) + 3\tan(\theta))d\theta$
- $\int \left(u^4 + 3 + \dfrac{1}{u^4}\right) du$
- $\int \left(\dfrac{\sin(2t)}{\cos(t)}\right) dt$
- $\int \left(\dfrac{x^5 + x^7}{x^3}\right) dx$

(8) Evaluate the following definite integrals:

- $\int_{-2}^{3}(x^5+x^3)^2dx$
- $\int_{-1}^{7}\left(\sqrt{x^5}+x^7\right)dx$
- $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\sec(\theta)(\sec(\theta)+3\tan(\theta))d\theta$
- $\int_{0}^{7}(u^2+3)^2du$
- $\int_{-\pi/2}^{\pi/2}(t^2\sin(t))dt$
- $\int_{-\pi/2}^{\pi}(\sin(t))dt$
- $\int_{-\pi/2}^{\pi}|\sin(t)|dt$
- $\int_{1}^{8}\left(\frac{1}{x^5}+x^7\right)dx$

(9) The acceleration function in km/sec^2 is $a(t)=3t+7$ where time t is in seconds, and $0\le t\le 10$. Let $v(t)$ (in km/sec) and $s(t)$ (in km) be the velocity and position functions respectively with the initial velocity, $v(0)=45\ km/sec$ and the initial position $s(0)=4\ km$. Find the velocity function and the position functions. Then, find the total distance covered.

29. The Substitution Rule

(1) State the Substitution Rule for integration.

(2) Find $\int 4x^3\sqrt{x^4+10}dx$

(3) Find $\int x^3\cos(x^4+10)dx$

(4) Find $\int \sqrt{3x-5}dx$

(5) Explain the Substitution Rule for Definite Integrals.

(6) Find $\int_1^3 \sqrt{3x+5}dx$

(7) Find $\int_2^7 \dfrac{1}{(3x+5)^5}dx$

(8) Find $\int_0^5 x^5\sqrt{1+x^2}dx$

(9) Suppose that f is a continuous function on $[-a, a]$.

- If f is even on $[-a, a]$ then $\int_{-a}^{a} f(x)dx =$
- Find $\int_{-3}^{3}(x^4 + 2x^2 - 1)dx$

- If f is odd on $[-a, a]$ then $\int_{-a}^{a} f(x)dx =$
- Find $\int_{-3}^{3}(x^5 + 2x^3 - x)dx$

(10) Find the integral:

- $\int x^7\sqrt{x^8 + 5}\ dx$
- $\int \dfrac{1}{(3 - 7t)^8}dt$

- $\int \dfrac{\csc^2(1/x)}{x^2}\ dx$
- $\int x^6 \cos(x^7 + 5)dx$

- $\int x^3 \sec(x^4+5)\tan(x^4+5)\ dx$
- $\int (5t-7)^{3.55} dt$
- $\int_0^5 \frac{1}{(1+\sqrt{x})^5}\ dx$
- $\int_{-\pi/2}^{\pi/2} x^8 \sin(x) dt$
- $\int_0^a x\sqrt{a^2-x^2}\ dx$
- $\int_{-a}^{a} \sqrt{a^2-x^2} dx$

30. Review Chapter 4

(1) Let $f(x) = x^2 - 4x$ for $0 \le x \le 2$.

- Using right end points and $n = 4$ find the Riemann sum.
- Using left end points and $n = 4$ find the Riemann sum.
- Using midpoints and $n = 4$ find the Riemann sum.
- Use the definition of definite integral using limit of Riemann sum to find $\int_0^2 (x^2 - 4x)dx$.
- Use the Fundamental theorem to find $\int_0^2 (x^2 - 4x)dx$.

(2) Suppose $\int_2^5 f(x)dx = 12$.

- $\int_5^2 f(x)dx =$ ________________________.
- $\int_2^5 7f(x)dx =$ ________________________.
- If $\int_2^3 f(x)dx = 3$ then $\int_3^5 f(x)dx =$ ________________________.

(3) Evaluate the integral :

- $\int_0^5 (x^5 + 2x^4 - 3x^2 - 7x + 9)dx$
- $\int_1^{10} (2+x)^{12} dx$
- $\int_0^4 (\sqrt[3]{x} + 3\sqrt{x})^2 dx$
- $\int_1^5 \sqrt{2+x^8}\ x^7 dx$
- $\int_0^3 \sin(4t)dt$
- $\int_{-5}^5 \dfrac{\sin(4t)}{1+7x^8} dx$

- $\int_5^{12} \dfrac{3x^2+2}{\sqrt{x^3+2x+9}} dx$
- $\int \csc^{12}(x)\cot(x)dx$
- $\int \sin(\cos(x))\sin(x)dx$
- $\int_0^9 |\sqrt{x}-4|dx$

(4) Find the derivative of:

- $f(x) = \int_0^x \sqrt{t^2+7}dt$
- $f(x) = \int_1^{\sin(x)} \dfrac{1-t^2}{1+t^4} dt$

31. Practice Problems

Below is a collection of problems from all chapters in no particular order.

(1) Evaluate the following limits:

(a) $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} x+1, & x \neq 2 \\ \sqrt{\pi}, & x = 2, \end{cases}$

(b) $\lim_{t\to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2+t} \right)$

(c) $\lim_{x\to\infty} \dfrac{2x^6 + 45x^4 + 5}{3x^6 + 500x^2 + 100x + 2000}$

(2) For what values of c is the given function continuous at $x = 3$?

$$f(x) = \begin{cases} c^2 - x^2, & x \leq 3 \\ \dfrac{x^2 - 5x + 6}{x - 3}, & x > 3 \end{cases}$$

(3) Evaluate the integral: $\displaystyle\int_0^3 e^x \sin(e^x)dx$.

(4) (a) Using the definition of the derivative, find the derivative of $f(x) = x + \dfrac{1}{2x}$.

(b) State the domain of f and f'.

(5) Sketch the graph of a function that satisfies all of the given conditions.

- $f'(x) > 0$, for all $x \neq 1$
- $x = 1$ is a vertical asymptote
- $f'' > 0$, for $x \in (-\infty, 1) \bigcup (3, \infty)$
- $x = 3$ is an inflection point.

(6) Prove that the equation $x^3 + x - 1 = 0$ has exactly one real root. (Hint: Rolle's Theorem).

(7) Find the derivative of the function: $y = \displaystyle\int_{\sin x}^{1} \sqrt{1+t^2}dt$.

(8) A particle moves with velocity function $v(t) = \sin t - \cos t$ and its initial displacement is $s(0) = 0$ m. Find its position function after t seconds.

(9) Let $f(x) = \dfrac{1}{1-x^2}$ with derivatives $f'(x) = \dfrac{2x}{(1-x^2)^2}$, and $f''(x) = \dfrac{6x^2+2}{(1-x^2)^3}$.
For the function f, determine:

(a) The vertical and horizontal asymptotes.
(b) The intervals of increase and decrease.
(c) The local maximum and minimum values.
(d) The intervals of concavity and inflection points.
(e) Use the information from above to sketch f.

(10) Evaluate the integral: $\displaystyle\int_0^3 \left[\frac{d}{dx}\sqrt{4+x^2} \right] dx$.

(11) A particle is moving along a horizontal straight line. Its position, $s(t)$, velocity, $v(t)$, and acceleration, $a(t)$, for $t \geq 0$ are as follows:
$s(t) = t^3 - 6t^2 - 15t$, $v(t) = 3t^2 - 12t - 15$, $a(t) = 6t - 12$.
Determine the time intervals, if any, during which the object is moving left.

(12) Find the number(s) at which f is discontinuous. Explain your answer.

$$f(x) = \begin{cases} x, & x \leq 1 \\ \dfrac{1}{x}, & 1 < x < 3 \\ \sqrt{x-3}, & x \geq 3 \end{cases}$$

(13) State whether the following statements are TRUE or FALSE. If FALSE, provide a counterexample, i.e. give an example where the statement fails. A sketch is sufficient.

(a) A function can have at most two horizontal asymptotes, but infinitely many vertical asymptotes.

(b) If $\lim_{x \to \pi} f(x)g(x)$ exists, then the limit must be $f(\pi)g(\pi)$.

(c) If $f(x) \leq g(x)$ for all $x \neq a$, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

(d) If $f(1) > 0$ and $f(3) < 0$, then there exists a number c between 1 and 3 such that $f(c) = 0$.

(e) If $f'(r)$ exists, then $\lim_{x \to r} f(x) = f(r)$.

(14) A spherical balloon is being deflated at a rate of $5cm^3/sec$. At the instance when the balloon contains $\dfrac{32\pi}{3}cm^3$ of gas, how fast is its radius decreasing? (Recall: $V = \dfrac{4}{3}\pi r^3$)

(15) Find the limit: $\displaystyle\lim_{x \to \infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$

(16) Consider the function $f(x) = x^3 - 12x^2 + 36x = x(x-6)^2$ with derivatives

$$f'(x) = 3(x-2)(x-6) \qquad f''(x) = 6(x-4).$$

Find the intervals of increase and decrease, the local maximum and minimum, the intervals of concavity and inflection points. Use all this information to sketch the graph of f.

(17) If f is a differentiable function, state the definition of $f'(x)$.

(18) Find the slope of the tangent line $\dfrac{dy}{dx}$, to the curve $\cos(xy) = 1 + \sin y$.

(19) A box with a square base and open top must have a volume of $32cm^3$. Find the dimensions of the box that minimize the amount of material used.

(20) Evaluate the integral: $\displaystyle\int_1^2 \frac{1+\sqrt[3]{x}}{\sqrt{x}}dx.$

(21) Let y be a function of x. Find $y'(x)$. Leave your answers unsimplified.

(a) $y = \arcsin(\arctan(2a\pi x))$, $a \in \mathbb{R}$

(b) $y = (\sqrt{x})^x$

(c) $y = \dfrac{e^{\sec x} + \cos e^x}{x}$

(22) Find the derivative of $f(z) = \sqrt{\dfrac{z-1}{z+1}}$. Leave your answer unsimplified.

(23) A right triangle with a hypotenuse of 6 inches is rotated about one of its legs to generate a right circular cone. What is the height of the greatest possible volume of this cone?

(24) Evaluate the integral: $\displaystyle\int \sqrt{x}\sin(1+x^{3/2})dx.$

(25) Find the equation of the *normal line* to the curve $\sin(y) + 2xy^3 + 2x = 4$ at the point (2, 0).

(26) Let $f(x) = x^2e^{-x^2}$. For what value(s) of x does $f(x)$ have a horizontal tangent?

(a) Find the linear approximation $L(x)$ of $f(x) = \cos(x - \frac{\pi}{6})$ at $a = 0$.

(b) Use this to approximate $\cos(\dfrac{-\pi}{3})$.

(27) Evaluate the integral: $\displaystyle\int \frac{1}{(5t+4)^{2.7}}dt$

(28) A particle is moving in a straight horizontal line. Its position, $s(t)$, velocity, $v(t)$ and acceleration functions, $a(t)$ for $t \geq 0$ are: $s(t) = t^3 - 6t^2 + 9t, v(t) = 3t^2 - 12t + 9, a(t) = 6t - 12.$

(a) Determine when the particle is moving to the left.

(b) When is the particle speeding up?

(29) Evaluate the integral: $\displaystyle\int_{\pi/4}^{\pi/3} -\csc^2 x dx.$

(30) Find the derivative of the integral function: $g(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \sec 3t \tan 3t dt$. Leave your answer unsimplified.

(31) Use Newton's Method to make the second approximation x_2 to the solution of $2e^x = ex$. Take $x_1 = 1$.

(32) Find the derivative: $\dfrac{d}{dx}\left(\displaystyle\int_1^{\cos(x)} \sqrt{1-t^2}dt\right).$

(33) Circle the correct answer. Let $h(x) = f(g(\sin 4x))$. Then $f'(x) =$

(a) $f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \cos 4x \cdot 4$

(b) $f'(g'(\sin 4x) \cdot \sin 4x \cdot 4$

(c) $f'(g(\sin 4x)) + f(g'(\sin 4x))$

(d) $\dfrac{f'(g(\sin 4x)) - f(g'(\sin 4x))}{g(\sin 4x)^2}$

(e) None of the above

(34) Circle the correct answer. An experimenter measures the radius of a circle to be 10 cm, with a maximum error of 3/10 cm. The maximum error in his calculation of the AREA is

(a) $dA = 3\pi cm^2$
(b) $dA = 4\pi cm^2$
(c) $dA = 5\pi cm^2$
(d) $dA = 6\pi cm^2$
(e) None of the above

(35) Determine the following limits:

(a) $\lim_{x \to 0^+} (\cos x)^{\frac{1}{x^2}}$
(b) $\lim_{x \to \infty} x \tan(\frac{1}{x})$

(36) Find the derivative of $F(x) = \int_{x^2}^{x^4} \cos(\sqrt{t})dt$.

(37) A rectangle with base on the x-axis has upper vertices on the parabola $y = 4 - \frac{1}{2}x^2$. Find the maximum perimeter.

(38) Circle the correct answer. The absolute maximum of the function $f(x) = x^3 - 3x + 1$ on [0,3] occurs when $x =$

(a) -1
(b) 0
(c) 1
(d) 3
(e) None of the above

(39) Let $f(x) = \dfrac{1}{x}$, $x \in [1, 3]$. Find all numbers c that satisfy the conclusion of the Mean Value Theorem.

(40) Circle the correct answer. For a function $g(x)$, we are given that $g'(x) = \dfrac{x}{x+5}$ and $g''(x) = \dfrac{5}{(x+5)^2}$. The interval(s) where $g(x)$ is simultaneously decreasing AND concave up is/are:

(a) $(0, \infty)$
(b) $(-\infty, -5)$
(c) $(-5, 0)$
(d) $(\infty, -5) \bigcup (-5, 0)$
(e) None of the above

(41) Let $g(x) = \int_{-2}^{x} f(t)dt$ where $f(x) = \begin{cases} x+2, & -2 \le x \le 0 \\ 1, & 0 < x \le 1 \\ 4-3x, & 1 < x \le 2. \end{cases}$

(a) Find an expression for $g(x)$.
(b) Sketch the graphs of f and g.
(c) Where is f continuous? Where is f differentiable? Where is g differentiable?

(42) A man standing on a building is watching a bicycle through a telescope as it approaches the building directly below. If the telescope is 310 ft above ground level and if the bicycle is approaching the building at 10ft/sec, at what rate is the angle, A, of the telescope, changing when the bicycle is 100 ft from the building?

(43) Evaluate the integral:

(a) $\int_{-1}^{1} \dfrac{r}{(1+r^2)^4} dr$
(b) $\int \cos^4 x \sin x dx$
(c) $\int \dfrac{\sec^2 x}{\sqrt{1+\tan x}} dx$
(d) $\int_{-a}^{0} y^2 \left(1 - \dfrac{y^3}{a^2}\right)^{-2} dy$

(44) Starting with $x_1 = 2$, find the second approximation x_2 to the root of the equation $x^3 - 2x - 5 = 0$, using Newton's method. You may leave your answer unsimplified.

(45) A long rectangular sheet of metal, 16 inches wide, is to be turned up at both sides to make a horizontal gutter with vertical sides. How many inches should be turned up at each side for maximum carrying capacity?

(46) If $2x \le g(x) \le x^4 - x^2 + 2$ for all x, find $\lim_{x \to 1} g(x)$.

(47) Using that $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$, evaluate the integral $\int_{1}^{3} (4-2x)dx$ by evaluating the limit of Riemann Sums.

(48) Find $f(t)$ given that $f'(t) = t + \dfrac{1}{t^3}$, and $f(1) = 6$.

(49) Find the derivative of the given function: $y = \sqrt{x + \sin x}$.

(50) Use Implicit Differentiation to find the slope of the tangent line to the curve

$$\sin(x + y) = 2x - 2y$$

at the point (π, π).

(51) Find the limit: $\displaystyle\lim_{t \to 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t}$.

(52) If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x, find $\displaystyle\lim_{x \to 1} g(x)$.

(53) Determine f given that $f''(x) = \sin x$, $f'(0) = -2$ and $f(0) = 1$.

(54) Evaluate the indefinite integral:

(a) $\displaystyle\int \tan x \sec^2 x dx$

(b) $\displaystyle\int \frac{3x^2}{(x^3+1)^2} dx$

(55) Explain why the function $f(x)$ is discontinuous at $x = 1$.

$$f(x) = \begin{cases} 1 - x^2, & x < 1 \\ 1/x. & x \geq 1 \end{cases}$$