CSI 35 LECTURE NOTES (Ojakian)

Topic 1: Proofs

OUTLINE

(References: Wells 1 - 4, 15, 27, 28, 30, 80-84, 86; Rosen: 1.6, 1.7, 1.8, 5.5)

1. Mathematical Proofs

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1. Mathematical Proof

   (a) An argument that goes from the assumptions to the conclusion in a deductive manner.

   (b) Example: Consider summing two evens. What can you say? Can you prove it? Experiment with numbers.

   (c) The steps in a proof can be thought of as putting Inference Rules into action.

   (d) We start with focus on Propositional Logic proofs.

2. Inference Rules

   (a) Forms of reasoning that are always valid: If the premises are true then the conclusion is true.

   (b) Key Rules: See Inference Rules handout.

   (c) Putting the rules together into a proof

      i. Example: Section 1.6.4 (page 77): Example 6

      ii. Example (they try): Section 1.6.4 (page 77-78): Example 7

      iii. Example (quantifiers). Section 1.6.7 (page 80): Example 12.

      iv. Example (they try). Section 1.6.7 (page 80 - 81): Example 13.

   (d) Fallacies

      i. Fallacy of affirming the conclusion. Example: Section 1.6.6 (page 79): Example 10.

   (e) Propositional Logic Exercises. Section 1.6 (pages 82ff):

      i. Propositional Logic Inference Rules: 1 - 4, 7, 8

      ii. Propositional Logic Proofs: 5, 6

      iii. Drawing your own conclusions: 9, 10

   (f) Quantifier Logic Exercises. Section 1.6 (pages 82ff):

      i. Analyzing proofs: 13, 14

      ii. Proof Checking: 15, 16

      iii. Fallacies: 17, 18, 23, 24

      iv. Proof Checking: 19, 20

      v. Quantifiers proofs: 25 - 35

3. Structured Proofs

   (a) Consider summing two evens. What can you say? Can you prove it? Experiment with numbers.

   (b) Conjecture, counter-example, and proof

   (c) I will use the expression "**Structured Proof**" to refer to a proof which is broken down into simple steps, in which the steps are successively numbered (i.e. 1, 2, 3, etc) and for each step a justifications is written down; the justification can be 1) by definition, or 2) by given, or 3) by a known fact, or 4) by indicating how earlier steps logically imply this step (include the numbers of the steps you use).

       In (3) when I say "known fact" I do *not* mean facts you think are simple about the new definition (ex. you can't just assume things about divisibility). I mean facts from *earlier courses* (ex. distributivity, basic algebra, facts about integers, etc)

   (d)

       PROBLEM 1. *Give a structured proof for the sum of two evens.*

       PROBLEM 2. *Prove that the sum of an odd integer and an even integer is odd.*

4. Divisibility Proofs

   (a) Definition: x divides y. How define? (define so $x \neq 0$).

   (b) Prove or disprove the following.

       i. 103 divides itself (can use "Universal Instantiation").
       ii. Every integer divides itself (exception?).
       iii. Every integer divides 0 (exception?).
       iv. If $a|b$ then $a \leq b$

5. Using Cases in Proofs

   Break up the statement you are trying to prove into a number of easier-to-prove statements

   (a) Section 4.1: 9, 11, 12 (cases on a odd or even) and related 44, 45

   (b) Prove the 100 is not the cube of an integer.

   (c) Prove the triangle inequality: $|x + y| \leq |x| + |y|$ (for x, y integers).

6. Proofs on program correctness (Section 5.5)

   (a) One kind of question: Does the program always terminate?

   (b) Second kind of question: Given some initial assertion, verify some final assertion.

   PROBLEM 3. *Consider the following program:*

```
y = 5
if y <= 5:
    z = x + 2*y
else:
    z = x - 2*y
```

   Suppose the initial assertion is that $x = 3$*. Then prove the final assertion that* $z = 13$.

   PROBLEM 4. *Consider the following program:*

```
x = 0
for k in range(1, n+1):
    x = x + 2*k
```

   Suppose the initial assertion is that $n = 100$*. Then prove the final assertion that* $x = 10100$.

   PROBLEM 5. *Consider the following program:*

```
while x < 10000:
    if x % 7 == 0:
        x = x + 1
    else:
        x = x + 3
```

   Prove that whatever integer the initial value of x *is, the program terminates.*

   (c) **Exercises. Section 5.5: 1 - 4**

7. Proof by Contradiction

   To this point proofs have been Direct. Now Indirect or by contradiction.

   (a) Prove that if n is not prime, then at least one of its factors is $\leq n$.

   (b) Section 4.1: exercise 10

   PROBLEM 6. *Prove that for all positive integers* n*, if* n^2 *is even then so is* n.

   PROBLEM 7. *Prove that among 100 consecutive days, any 51 of these days must contain 2 consecutive days.*

8. Proving Equivalences

   (a) Must prove two implications.

   PROBLEM 8. *Prove that for all positive integers* n*,* n^2 *is even if and only if* n *is even.*

   (Note: part of this proof already done)

   (b) The use of Lemmas:

      i. Lemma: The product of two evens is even.

      ii. Lemma: The product of two odds is odd