CSI 35 LECTURE NOTES (Ojakian)

Topic 10: More on Graphs

OUTLINE

(References: Wells: 157, 159, Finan: Ch. 7, Rosen: 10.4, 10.5, 10.8)

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1. <u>Eulerian and Hamiltonian</u>

   Eulerian vs. Hamiltonian like: Street-cleaning versus salesperson.

   (a) Initial examples

   PROBLEM 1. *Wells exercise 157.4.2 and 157.4.3 (which Eulerian? which Hamiltonian?)*

   PROBLEM 2. *Wells exercise 157.2.1 (Eulerian)*

   PROBLEM 3. *Wells exercise 157.3.3 (Hamiltonian)*

   (b) Concept of "Necessary and Sufficient Conditions"

      i. Example: Integer divisible by 3 and 4.

   (c) Eulerian: Necessary and Sufficient Conditions?

      i. Try finding some necessary ones? some sufficient ones? both necessary and sufficient??

      ii. Statement of Euler's Theorem

   (d) Hamiltonian: Necessary and Sufficient Conditions?

      i. Try finding some necessary ones? some sufficient ones? both necessary and sufficient??

      ii. Statement of Ore's Theorem

   (e) Applications

      i. Gray codes and the hypercube

   (f) Proof of Euler's Theorem (via "iterations")

   PROBLEM 4. *For an Eulerian graph, find an Eulerian Circuit using the method from the proof of Euler's Theorem. Do it so that in each iteration, a circuit is chosen that does not repeat vertices (and even better, try to make it the shortest circuit option at that point).*

2. <u>Coloring</u>

   One Motivation: Schedule student exam times so there is no conflict.

   PROBLEM 5. *Find the chromatic number of K_n and C_n and the wheel graph W_n (which has a center vertex and n other vertices).*

   PROBLEM 6. *Use Sage Math to find a scheduling (i.e. coloring) for BCC courses.*

   PROBLEM 7. *Wells 159.2.2 (coloring bipartite)*

   PROBLEM 8. *4 coloring problem and Wells 159.2.7 - Can you find such a place?*

3. Connectivity

How reliable is the network?

(a) Define vertex connectivity (κ) and edge connectivity (λ).

i. Examples: Cycles, Trees, Complete Graphs

ii. Relationship between them?

(b) How many paths between two vertices: vertex disjoint and not. Connection to κ.

i. Matrix product (just for square matrices)

ii. Matrix product theorem

(c) Problems

PROBLEM 9. *For a graph, use Sage to find its vertex connectivity and edge connectivity*

PROBLEM 10. *For a small graph, finds its adjacency matrix, and various powers. Compare this to the vertex disjoint paths the correspond to vertex connectivity.*

4. Matching

Students: Read Rosen page 660 in advance!

One Motivation: Match workers to jobs?

PROBLEM 11. *Create problem from students in class, along with jobs to perform. Is there a complete matching from students to jobs?*

(a) Examples.

(b) Statement Hall's Marriage Theorem

PROBLEM 12. *Consider the graph on the board (with 3 vertices on the left). Does it satisfy the condition in Hall's Theorem? If not add, some edges so that it does satisfy the condition.*

(c) Proof of Hall's Marriage Theorem!

PROBLEM 13. *Follow the notation of the proof of Hall's Theorem.*

i. Draw a bipartite graphs, with 4 vertices in W_1, which falls into case 1.

ii. Choose an edge $\{a, b\}$, where $a \in W_1$ and $b \in W_2$.

iii. Remove edge $\{a, b\}$, along with vertices a and b. Verify Hall's condition holds on this smaller graph, and find a complete matching on the smaller graph.

iv. Find a complete matching on the original graph, by combining the matching from the last step with the edge $\{a, b\}$.

PROBLEM 14. *Follow the notation of the proof of Hall's Theorem.*

i. Draw a bipartite graphs, with 4 vertices in W_1, which falls into case 2.

ii. Choose a subset $W_1' \subseteq W_1$ such that $|W_1'| = |N(W_1')|$. Find a complete matching from W_1', then remove all these vertices from the graph.

iii. Verify that what remains satisfies the condition in Hall's Theorem. And obtain a complete matching on this remaining part.

iv. Find a complete matching on the original graph, by combining the matchings from the last two steps.