MTH 05. Basic Concepts of Mathematics I

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To my parents and teachers

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Suggested Syllabus

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• Simplifying expressions, cont. • Exponents and roots	1.3.4, 1.4.7	p. 37, 38 # 41-52 Finish Classroom Exercises p. 44-45; p. 49 # 1-7
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Topic(s)	Text Section(s)	**HOMEWORK**
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Topic(s)	Text Section(s)	**HOMEWORK**
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COURSE GRADE: To pass MTH 05, you must have a class average of at least 74% (or C), and you must achieve a score of at least 60 on the CUNY Common Final Exam. Your class average will be calculated using the following weighting:

$$.35(\text{CUNY Common Final Exam Score}) + .65(\text{average of 4 class tests}).$$

Contents

Chapter 1

Real Numbers

1.1 Introduction

The study of real numbers is an ambitious project. We will learn just enough about the real numbers as our course requires.

We first introduce the number systems as they were historically developed. We would like to inform the reader that not all mathematicians will agree with the terminology presented here.

The set of **natural numbers** is

$$\mathbb{N} = \{1, 2, 3, \cdots\}.$$

The set of natural numbers is **closed** under addition. That is, given any two natural numbers, their sum is also a natural number. Addition is a **commutative** operation. That is, $a+b = b+a$ for any natural numbers a, b. The set $\mathbb{N}$ is also closed under multiplication, and multiplication is also commutative. That is, $a \cdot b = b \cdot a$ for any natural numbers a, b. Further, it contains the multiplicative identity 1. That is, $1 \cdot a = a = a \cdot 1$ for any natural number a. But $\mathbb{N}$ does not contain the **additive identity**.
The set of **whole numbers** is

$$\mathbb{W} = \{0, 1, 2, 3, \cdots\}.$$

Notice that $\mathbb{W}$ contains just one more element than $\mathbb{N}$, the number 0. The number 0 is the additive identity. That is, $a + 0 = a = 0 + a$ for any natural number a. Since every natural number is also a whole number, we say that $\mathbb{N}$ is a **subset** of $\mathbb{W}$, written mathematically as $\mathbb{N} \subset \mathbb{W}$. Thinking in non-technical terms, we could say, " $\mathbb{N}$ is contained in $\mathbb{W}$."

Notice that while $\mathbb{W}$ is closed under addition and multiplication, it is **not closed** under subtraction. For example, $3 - 7$ is not a whole number. This brings us to the next number system. The set of **integers** is

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

The set $\mathbb{Z}$ is closed under addition, multiplication, and subtraction, and has the additive identity. Further, $\mathbb{W}$ is a subset of $\mathbb{Z}$. So we have $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z}$.

Notice that $\mathbb{Z}$ is not closed under division. For example, $7 \div 3$ is not an integer. Therefore, a new number system is needed which would contain $\mathbb{Z}$ and be closed under addition, subtraction, multiplication, and division. The set of **rational numbers** is denoted by $\mathbb{Q}$. It is difficult to list all the numbers of $\mathbb{Q}$. We therefore use **set-builder notation**.

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

such that

The set of

Here, $\in$ is read as "elements of," and $\neq$ is read as "not equal to.". Thus, $\mathbb{Q}$ is equal to the set of all fractions $\frac{a}{b}$ such that a, b are integers, and b is not equal to 0.

The set $\mathbb{Q}$ is closed under addition, subtraction, multiplication, and division by nonzero elements, and contains 0 (the additive identity) and 1 (the multiplicative identity). Addition and multiplication are commutative and associative. Moreover, multiplication satisfies the **distributive law** over addition. That is,

$$a \cdot (b + c) = a \cdot b + a \cdot c \textit{ for all rational numbers } a, b, c.$$

For an integer n, we can view n as a rational number as $\frac{n}{1}$. For example, $5 \in \mathbb{Z}$ can be viewed as $\frac{5}{1} \in \mathbb{Q}$. Therefore, $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q}$. Here are some more examples of rational numbers.

$-8 = \frac{-8}{1} = \frac{8}{-1} = -\frac{8}{1}$	$0 = \frac{0}{1}$
$2.3 = \frac{23}{10}$	$-4.5 = -\frac{45}{10} = \frac{-45}{10} = \frac{45}{-10}$
$3.456 = \frac{3456}{1000}$	$0.12 = \frac{12}{100}$
$5\frac{1}{2} = \frac{11}{2}$	$-7\frac{1}{3} = -\frac{22}{3}$
$0.5555\cdots = 0.\overline{5} = \frac{5}{9}$	$0.646464\cdots = 0.\overline{64} = \frac{64}{99}$

The rational numbers can be arranged in a line, but they leave infinitely many holes. These holes get filled by **irrational numbers**. That is, an irrational number cannot be written in the form of a fraction of two integers. Examples of irrational numbers are

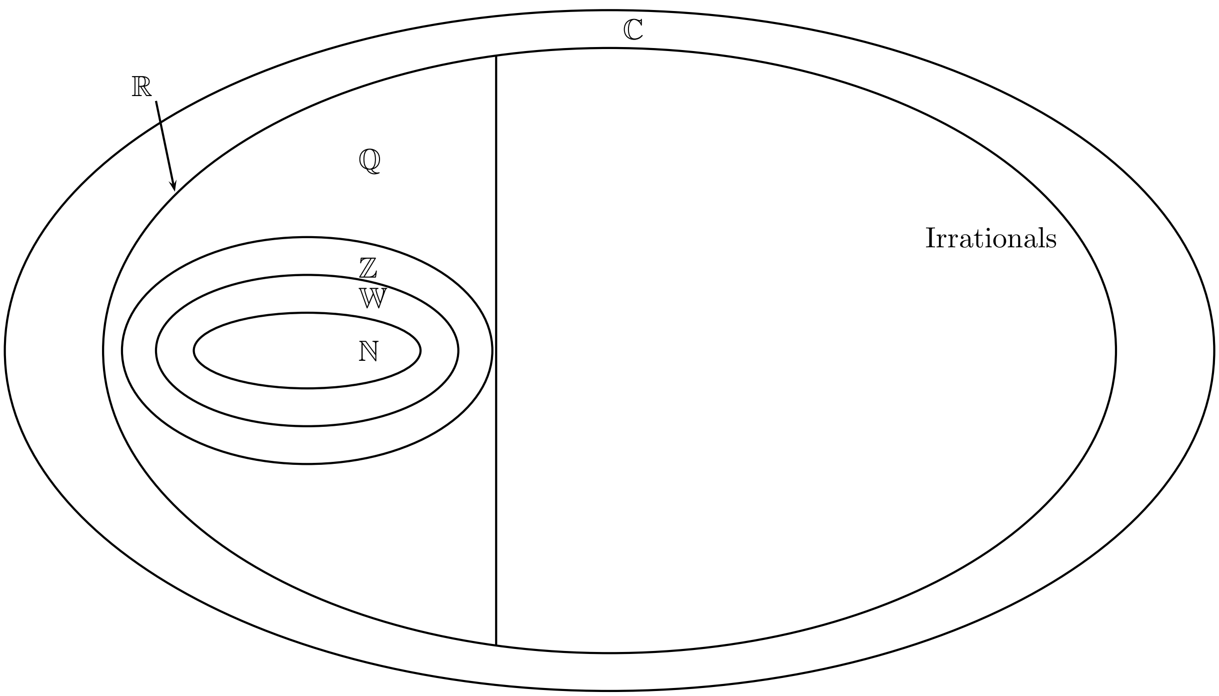
$$\sqrt{2}, \sqrt{3}, \pi \quad (pi), e \quad (\textit{the Euler number}).$$

Irrational numbers are extremely useful, and we will encounter them throughout our course. Numbers which are rational or irrational, are called the **real numbers**. The set of real numbers is denoted by $\mathbb{R}$. There is a bigger number system containing $\mathbb{R}$ on which addition and multiplication are commutative and associative, which is closed under subtraction and division by non-zero numbers, and which contains 0 and 1. That number system is the set of **complex numbers** denoted by $\mathbb{C}$. Let $i = \sqrt{-1}$. It turns out that every element of $\mathbb{C}$ can be written in the form of $a + bi$ where a, b are real numbers. That is,

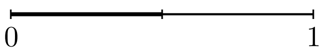
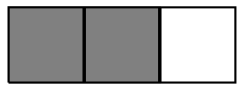
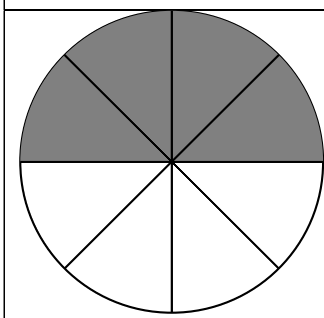
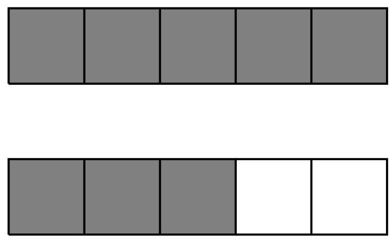
$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$

We will see more on complex numbers towards the end of this course. So we will focus only on the real numbers.

The following figure is a Venn diagram representation of the number systems introduced so far.



First we understand the rational numbers. The following figures help us understand the meaning of fractions.

Figure	Value	Figure	Value
	$\frac{1}{2}$		$\frac{2}{3}$
	$\frac{4}{8} = \frac{1}{2}$		$\frac{8}{5} = 1\frac{3}{5}$

Notice that $\frac{4}{8} = \frac{1}{2}$. This helps us understand how to reduce fractions to their lowest terms:

$$\frac{4}{8} = \frac{4 \times 1}{4 \times 2} = \frac{\overset{1}{\cancel{4}} \times 1}{\overset{1}{\cancel{4}} \times 2} = \frac{1}{2}.$$

Here is another example of reducing a fraction to its lowest term done in several steps:

$$\frac{180}{200} = \frac{10 \times 18}{10 \times 20} = \frac{\overset{1}{\cancel{10}} \times 18}{\overset{1}{\cancel{10}} \times 20} = \frac{18}{20} = \frac{2 \times 9}{2 \times 10} = \frac{\overset{1}{\cancel{2}} \times 9}{\overset{1}{\cancel{2}} \times 10} = \frac{9}{10}.$$

This process can be reversed. That is,

$$\frac{1}{2} = \frac{4 \times 1}{4 \times 2} = \frac{4}{8}, \quad \text{and} \quad \frac{9}{10} = \frac{20 \times 9}{20 \times 10} = \frac{180}{200}.$$

Here are more examples

$\frac{2}{5} = \frac{7 \times 2}{7 \times 5} = \frac{14}{35}.$	$\frac{2}{5} = \frac{2 \times 2}{2 \times 5} = \frac{4}{10}.$	$\frac{2}{5} = \frac{8 \times 2}{8 \times 5} = \frac{16}{40}.$
$-\frac{3}{7} = -\frac{4 \times 3}{4 \times 7} = -\frac{12}{28}.$	$-\frac{3}{7} = -\frac{5 \times 3}{5 \times 7} = -\frac{15}{35}.$	$-\frac{3}{7} = -\frac{8 \times 3}{8 \times 7} = -\frac{24}{56}.$

Classroom Exercises : Fill in the blanks:

1. $\dfrac{2}{3} = \dfrac{\qquad\qquad}{18}$.
2. $\dfrac{7}{11} = \dfrac{56}{\qquad\qquad}$.
3. Reduce the following fractions to their lowest terms:
   - $\dfrac{25}{75}$
   - $\dfrac{32}{36}$
   - $-\dfrac{100}{250}$

Notice in the figure drawn on the previous page, $\dfrac{8}{5} = 1\dfrac{3}{5}$. This number written in the form of a fraction, $\dfrac{8}{5}$, and in the form of a **mixed number**, $1\dfrac{3}{5}$. A fraction in which the numerator is greater than or equal to the denominator is called an **improper fraction**. We can convert an improper fraction to a mixed number.

Example 1 :

$\dfrac{12}{7} = 1\dfrac{5}{7}$. To see this, consider the following division:

$$\begin{array}{rl} & 1 \leftarrow \text{quotient} \\ \hline 7\,\big) & 12 \\ & -07 \\ \hline & 5 \leftarrow \text{remainder} \end{array}$$

That is, $12 = 7 \times 1 + 5$.

So when we divide 12 by 7 we get 1 and five-sevenths more.

That is, $\dfrac{12}{7} = 1\dfrac{5}{7}$.

Example 2 :

$\dfrac{85}{6} = 14\dfrac{1}{6}$. To see this, consider the following division:

$$\begin{array}{rl} & 14 \leftarrow \text{quotient} \\ \hline 6\,\big) & 85 \\ & -6 \\ \hline & 25 \\ & -24 \\ \hline & 01 \leftarrow \text{remainder} \end{array}$$

That is, $85 = 6 \times 14 + 1$.

So when we divide 85 by 6 we get 14 and one-sixth more.

That is, $\dfrac{85}{6} = 14\dfrac{1}{6}$.

Example 3:
$-\frac{41}{7} = -5\frac{6}{7}$. We go through the same process as in the previous case, ignoring the negative sign. When 41 is divided by 7, we get a quotient of 5, and a remainder of 6. Thus, $\frac{41}{7} = 5\frac{6}{7}$. Now introduce the sign back to get $-\frac{41}{7} = -5\frac{6}{7}$.

Classroom Exercises : Convert the following improper fractions to mixed numbers:
- $\frac{35}{6}$
- $\frac{100}{8}$
- $-\frac{230}{4}$

Going in the reverse direction, given a mixed number we can convert it into an improper fraction.

Example 1:

$$1\frac{8}{11} = 1 \textit{ and } \frac{8}{11} = 1 + \frac{8}{11} = \frac{11}{11} + \frac{8}{11} = \frac{19}{11}.$$

Thus, we simply write,

$$1\frac{8}{11} = \frac{1 \times 11 + 8}{11} = \textit{(recall that multiplication takes precedence over addition)}\ \frac{11+8}{11} = \frac{19}{11}.$$

Example 2:

$$3\frac{4}{13} = \frac{3 \times 13 + 4}{13} = \frac{39+4}{13} = \frac{43}{13}.$$

Example 3:
When a negative mixed number is given, keep the negative sign out of your calculations, but keep it throughout.

$$-7\frac{2}{15} = -\frac{7 \times 15 + 2}{15} = -\frac{105+2}{15} = -\frac{107}{15}.$$

Classroom Exercises : Convert the following mixed numbers to improper fractions:
- $3\frac{5}{6}$
- $6\frac{3}{8}$
- $-7\frac{3}{4}$

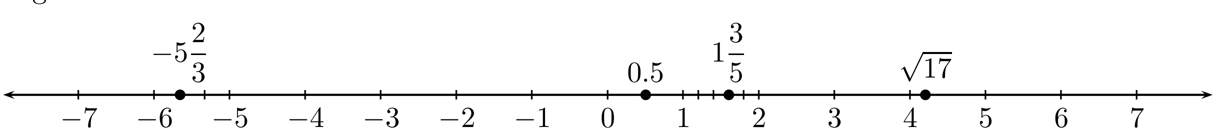
Any two real numbers can be compared using the following symbols of inequality:

Symbol	Meaning
$<$	Less than
$\leq$	Less than or equal to
$>$	Greater than
$\geq$	Greater than or equal to

The following table gives examples of inequalities which are true or false:

Mathematical statement	Translation to English	Truth value
$5 < 10$	5 is less than 10	True
$5 \leq 10$	5 is less than or equal to 10	True
$-12 > -23$	Negative 12 is greater than Negative -23	True
$-12 \geq -23$	Negative 12 is greater than or equal to negative -23	True
$4 \leq 4$	4 is less than or equal to 4	True
$4 \geq 4$	4 is greater than or equal to 4	True
$4 < 4$	4 is less than 4	False
$5 < -3$	5 is less than -3	False

Real numbers greater than 0 are called positive, and the real numbers less than 0 are called negative. The real numbers are arranged on a number line in an increasing order when read from left to right. Thus, all the positive numbers are written to the right of 0, and all the negative numbers are written to the left of 0.



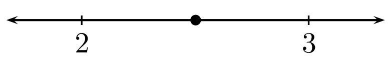
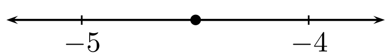
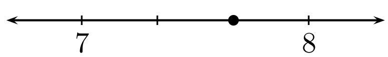
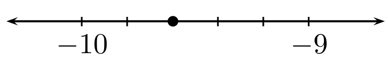
Notice that the placement of $\sqrt{17}$ has to be estimated. Since 17 is between 16 and 25, we note that $\sqrt{17}$ has to be between $\sqrt{16}$ and $\sqrt{25}$. Therefore, $\sqrt{17}$ is between 4 and 5. Furthermore, since 17 is closer to 16 than to 25, we place $\sqrt{17}$ closer to 4 than to 5. Without a calculator, we can think of $\sqrt{17}$ as 4.something.

Classroom Exercises :

1. Estimate the values of:
   - $\sqrt{28}$ • $\sqrt{2}$ • $-\sqrt{37}$

2. Plot the following numbers on the number line:
   - 1.2 • -2.3 • $\frac{3}{7}$ • $-\frac{2}{3}$ • $3\frac{3}{4}$ • $-3\frac{2}{5}$

The following figure explains how we plot fractions and mixed numbers on the real number line.

Location of the point	Value of the point
	2.5 is the midpoint of the line segment joining 2 and 3.
	-4.5 is the midpoint of the line segment joining -5 and -4.
	$7\frac{2}{3}$ is two-thirds of the distance from 7 towards 8.
	$-9\frac{3}{5}$ is the three-fifths of the distance from -9 towards -10.

Once we arrange all the numbers on the number line, we can introduce the concept of the **absolute value** of a real number. The absolute value of a real number is its distance from 0 on the real number line. We therefore have,

$$\begin{array}{ll} |5| = 5 & \text{because the distance of 5 from 0 is 5;} \\ |-7| = 7 & \text{because the distance of } -7 \text{ from 0 is 7;} \\ |0| = 0 & \text{because the distance of 0 from 0 is 0.} \end{array}$$

Classroom Exercises:

1. Fill in the blanks using $<, >, \leq, \geq, =$ whichever apply (in case a question has more than one answer, then write down every answer that applies):

   (a) 5____________8.

   (b) -42____________ -47.

   (c) -3____________7.

(d) $18\underline{\hspace{5cm}}10.$

(e) $5\underline{\hspace{5cm}}5.$

(f) $-2\underline{\hspace{5cm}}-7.$

2. Plot the following numbers on the number line:

   (a) 0.9

   (b) -2.4

   (c) $\frac{3}{4}$

   (d) $-\frac{3}{5}$

   (e) $5\frac{2}{7}$

   (f) $-4\frac{2}{3}$

3. Find the absolute values:

   (a) $|203|$

   (b) $|-203|$

   (c) $|\sqrt{7}|$

   (d) $-|\sqrt{7}|$

   (e) $|-\sqrt{7}|$

   (f) $|0|$

1.1.1 Homework Exercises

1. Reduce the following fractions to their lowest terms:

   (a) $\frac{12}{18}$

   (b) $\frac{34}{60}$

   (c) $\frac{250}{400}$

   (d) $\frac{125}{765}$

   (e) $\frac{30}{85}$

2. Fill in the blanks:
   (a) $\dfrac{1}{4} = \dfrac{\qquad}{12} = \dfrac{5}{\qquad} = \dfrac{\qquad}{28} = \dfrac{9}{\qquad}$
   (b) $\dfrac{2}{5} = \dfrac{\qquad}{10} = \dfrac{8}{\qquad} = \dfrac{20}{\qquad} = \dfrac{\qquad}{75}$
3. Convert the following fractions to mixed numbers:
   (a) $\dfrac{12}{5}$
   (b) $-\dfrac{30}{7}$
   (c) $\dfrac{53}{9}$
   (d) $-\dfrac{24}{11}$
   (e) $\dfrac{180}{25}$
4. Convert the following mixed numbers to fractions:
   (a) $3\dfrac{1}{8}$
   (b) $-7\dfrac{8}{11}$
   (c) $9\dfrac{4}{7}$
   (d) $-8\dfrac{3}{14}$
   (e) $-9\dfrac{2}{7}$
5. Estimate the values of the following:
   (a) $\sqrt{2}$
   (b) $\sqrt{5}$
   (c) $\sqrt{10}$
   (d) $\sqrt{19}$
   (e) $\sqrt{50}$

6. Fill in the blanks using $<, >, \leq, \geq, =$ whichever apply (in case a question has more than one answer, then write down every answer that applies):

   (a) 4______________7.

   (b) -24______________-27.

   (c) -4______________7.

   (d) 14______________7.

   (e) 4______________4.

   (f) -4______________-7.

7. Plot the following numbers on the number line:

   (a) 0.7

   (b) -3.5

   (c) $\frac{2}{5}$

   (d) $-\frac{4}{7}$

   (e) $7\frac{2}{3}$

   (f) $-8\frac{1}{5}$

8. Find the absolute values:

   (a) $|102|$

   (b) $|-23|$

   (c) $|\sqrt{5}|$

   (d) $-|\sqrt{5}|$

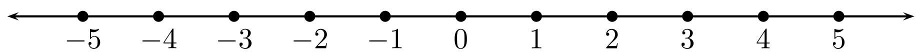
   (e) $|-\sqrt{5}|$

   (f) $|0|$

1.2 Addition and Subtraction of Real Numbers

1.2.1 Addition

To **add** two numbers is to find their combined value. We denote addition by the symbol $+$
Recall: Real numbers are placed on a number line. The negative numbers are placed to the left of zero, and the positive numbers are placed to the right of zero.



To add a positive number is to move to the right. To add a negative number is to move to the left.
For instance, to understand $5+4$, we start at 5 on the number line, and move 4 units to the right to reach 9. Is this the same as $4+5$? That is, would you get the same answer had you started at 4 and moved 5 units to the right?
Likewise, to understand $5+(-4)$, we start at 5 on the number line, and move 4 units to the left to reach 1. Is this the same as $(-4)+5$? That is, would you get the same answer had you started at -4 and moved 5 units to the right?
Classroom Exercises: Use the number line (you will have to extend the number line in your mind) to find the following sums:

(a) $(-4)+8$

(b) $28+(-7)$

(c) $(-8)+(-9)$

(d) $12+(-19)$

(e) $(-13)+0$

(f) $0+15$

The number 0 is called the **additive identity.** This is because, $a+0=0+a=a$ for any real number a.
Another important way of understanding addition of signed numbers is by using the notions of *profit* and *loss*. For instance, the combined value of \$ 5 and \$ 6 is \$ 11. That is,

$$5+6=11.$$

On the other hand, if I make a profit of \$ 10, and a loss of \$ 3, then combining my profit and loss gives me a *profit* of only \$ 7. That is,

$$10+(-3)=7.$$

Likewise, if I make a profit of \$ 11, and a loss of \$ 15, then combining my profit and loss gives me a *loss* of \$ 4. Therefore,

$$11+(-15)=-4.$$

This way of viewing signed numbers as *profit* (for positive numbers) and *loss* (for negative numbers) helps us simplify complicated additions.

For instance,

$(-3) + 10 + (-8) + 12 + (-9) + 20$	
$= 10 + 12 + 20 + (-3) + (-8) + (-9)$	*get the profits together, and then the losses together;*
$= 42 + (-20)$	*combine your profits, and combine your losses;*
$= 22.$	*the final balance.*

Here is another example:

$23 + (-8) + 11 + (-7) + (-9) + (-12) + 1 + (-18) + 2$	
$= 23 + 11 + 1 + 2 + (-8) + (-7) + (-9) + (-18) + (-12)$	*get the profits together, and then the losses together;*
$= 37 + (-54)$	*combine your profits, and combine your losses;*
$= (-17).$	*the final balance.*

Classroom Exercises:

(a) $(-12) + 4 + (-8) + (-7) + 8$

(b) $9 + (-7) + 18 + (-9)$

By now you have probably discovered the following rules of addition:

- To add two numbers of the same sign, add their absolute values and keep the sign.
- To add two numbers of opposite signs, subtract the smaller absolute value from the larger absolute value and keep the sign of the number with the larger absolute value.
- **Commutativity for addition:** $a + b = b + a$ for any real numbers a and b.

We also have the **Associativity for addition:**

$$(a + b) + c = a + (b + c) \textit{ for any real numbers } a, b \textit{ and } c.$$

This rule means that, for the purposes of addition, we can group our numbers in whichever order we want to.
Before we proceed further, recall the addition and subtraction of numbers in decimal form.

Classroom Exercises:

$$\begin{array}{r} 2.359 \\ +1.423 \\ \hline \end{array} \qquad \begin{array}{r} 897.123 \\ +98.12 \\ \hline \end{array} \qquad \begin{array}{r} 63.7201 \\ -23.783 \\ \hline \end{array} \qquad \begin{array}{r} 125.35 \\ -67.7895 \\ \hline \end{array}$$

Now we can simplify the following:

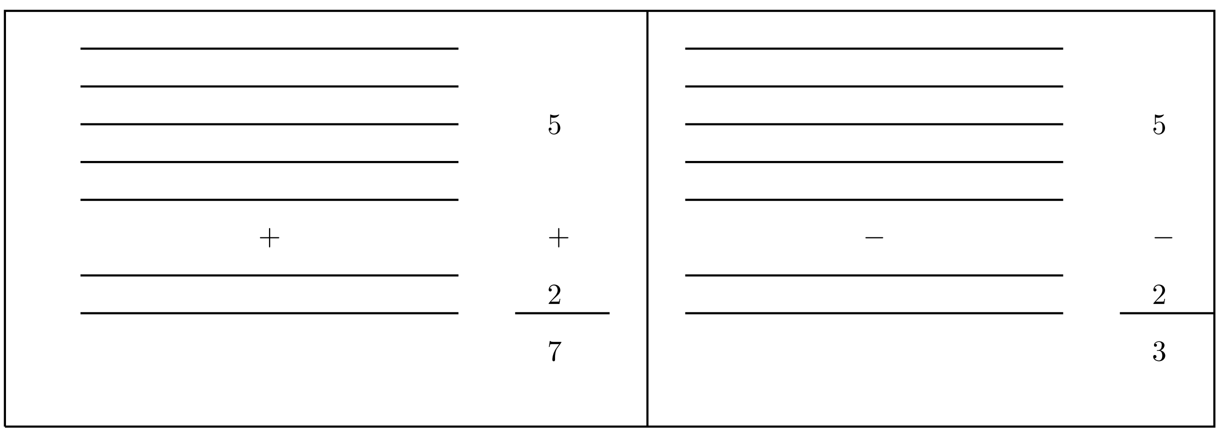
$$
\begin{aligned}
&32.879 + (-59.23) + (-23.789) + 24.599 + 32.788 && \\
=&32.879 + 24.599 + 32.788 + (-59.23) + (-23.789) && \textit{group the positive and negative numbers ;} \\
=&90.266 + (-83.019) && \textit{add the positive numbers,} \\
& && \textit{and add the negative numbers;} \\
=&7.247 && \textit{the final answer.}
\end{aligned}
$$

Classroom Exercises: Find the following sums

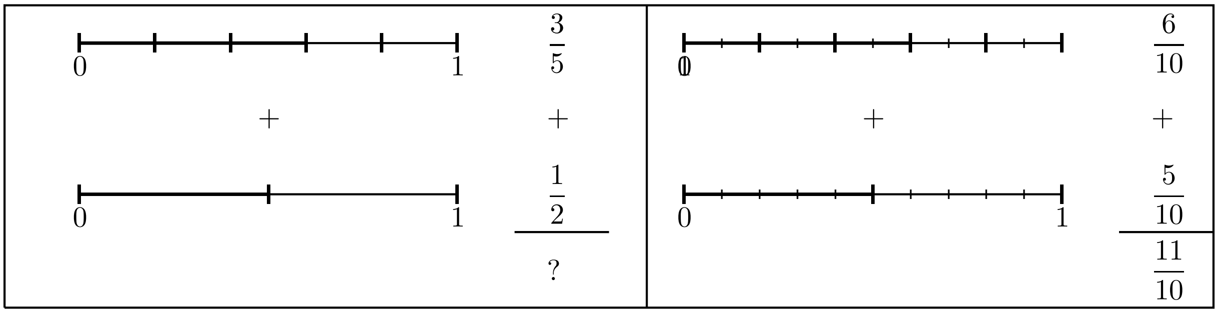
(a) $(-89.234) + (-72.1234) + 48.88 + (-20.331) + 52.123$

(b) $157.339 + (-98.56) + 72.114 + (-48.231)$

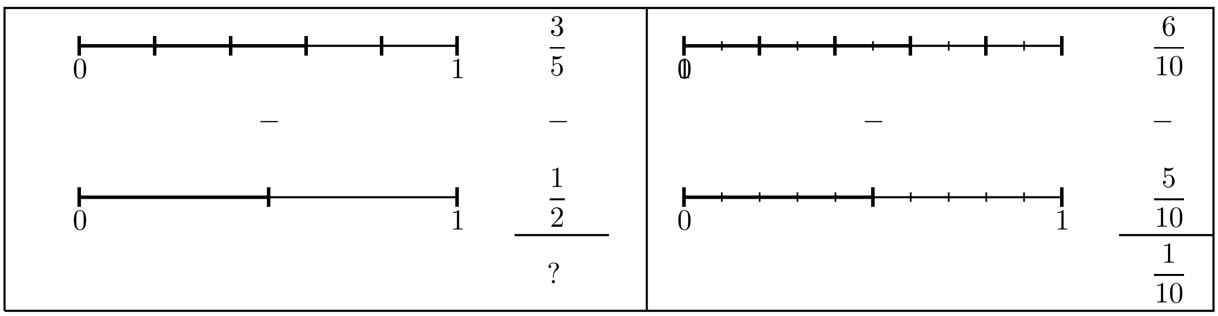
Let us now try to understand the addition of fractions. First, a preliminary: When we add 5 units and 2 units, we get 7 units. When we subtract 2 units from 5 units, we get 3 units. Units can be thought of as line segments of the same length. We see this pictorially here:



Now, consider the following picture: We want to add $\frac{3}{5}$ and $\frac{1}{2}$. As they represent different lengths, we cannot directly add them. But, we see $\frac{3}{5} = \frac{6}{10}$ and $\frac{1}{2} = \frac{5}{10}$. Now, we can add $\frac{6}{10}$ and $\frac{5}{10}$ to get $\frac{11}{10}$.



In the same way, notice the following subtraction:



Remark : To add or subtract fractions, we need *common denominators*. We prefer using the *Least Common Denominator (LCD)* which you must have learned in your previous mathematics classes.

Let us try a few problems without drawing pictures.

1.

$$\begin{aligned}
&\frac{2}{7}+\frac{3}{5} && \textit{The least common denominator is } 35;\\
=&\frac{2\times 5}{7\times 5}+\frac{3\times 7}{5\times 7}\\
=&\frac{10}{35}+\frac{21}{35}\\
=&\frac{10+21}{35}=\frac{31}{35}. && \textit{The final answer.}
\end{aligned}$$

2.

$$\begin{aligned}
&\left(-\frac{3}{8}\right)+\frac{7}{12} && \textit{The least common denominator is } 24;\\
=&\left(-\frac{3\times 3}{8\times 3}\right)+\frac{7\times 2}{12\times 2}\\
=&\left(-\frac{9}{24}\right)+\frac{14}{24}\\
=&\frac{(-9)+14}{24}=\frac{5}{24}. && \textit{The final answer.}
\end{aligned}$$

3.

$$\begin{aligned}
&\left(-\frac{7}{30}\right)+\frac{8}{45}+\left(-\frac{11}{60}\right) && \textit{The least common denominator is } 180;\\
=&\left(-\frac{7\times 6}{30\times 6}\right)+\frac{8\times 4}{45\times 4}+\left(-\frac{11\times 3}{60\times 3}\right)\\
=&\left(-\frac{42}{180}\right)+\frac{32}{180}+\left(-\frac{33}{180}\right)\\
=&\frac{(-42)+32+(-33)}{180}\\
=&\frac{32+(-42)+(-33)}{180} && \textit{Grouping the positives and the negatives};\\
=&\frac{32+(-75)}{180}=\frac{-43}{180}=-\frac{43}{180} && \textit{The final answer.}
\end{aligned}$$

Classroom Exercises: Evaluate

(a) $\frac{2}{7}+\frac{3}{5}$

(b) $\left(-\frac{2}{3}\right)+\frac{3}{5}+\left(-\frac{4}{15}\right)$

1.2.2 Subtraction

Once you have mastered addition of signed numbers, subtraction is straightforward. First, a definition:
The **opposite** of a real number a is $-a$. Note, $a+(-a)=0$. The opposite of a is also called **the additive inverse** of a.
The opposite of 4 is -4. Note, $4+(-4)=0$.
The opposite of (-6) is 6. Note, $(-6)+6=0$.

Subtracting a number is the same as **adding its opposite**.

For instance,

$$
\begin{aligned}
10 - 4 &= 10 + (-4) = 6 \\
32 - (-8) &= 32 + 8 = 40 \\
(-3) - 6 &= (-3) + (-6) = \underline{\hspace{2cm}} \quad \textit{(Fill in the blank.)}
\end{aligned}
$$

Let us see why this should be true. To understand $10 - 4$ on the number line, we start at 10 and move 4 units to the left, thus obtaining 6. This is the same procedure we used to calculate $10 + (-4)$.

Likewise, to understand $32-(-8)$, we start at 32 and move **left** by -8 units. Now what does it mean to move left by -8 units? It means, move right by 8 units. That is why $32-(-8) = 32+8$.

Classroom Exercises:

(a) $(-4) - 8 = (-4) + \underline{\hspace{1cm}} = \underline{\hspace{1cm}}$

(b) $28 - (-7) = 28 + \underline{\hspace{1cm}} = \underline{\hspace{1cm}}$

(c) $(-8) - (-9) = (-8) + \underline{\hspace{1cm}} = \underline{\hspace{1cm}}$

(d) $12 - (-19) = 12 + \underline{\hspace{1cm}} = \underline{\hspace{1cm}}$

Here is a complicated exercise involving subtractions, which we convert to an addition problem.

$$
\begin{aligned}
&(-3) - 10 + (-8) + 12 - (-9) + 20 \\
=&(-3) + (-10) + (-8) + 12 + 9 + 20 && \textit{Changing the subtractions to additions} \\
=&(-21) + 41 && \textit{Combine the negative numbers, and the positive numbers.} \\
=&20. && \textit{The final answer.}
\end{aligned}
$$

Here is another example:

$$
\begin{aligned}
&23 - (-8) + 11 + (-7) + (-9) + (-12) - 1 - (-18) + 2 \\
=&23 + 8 + 11 + (-7) + (-9) + (-12) + (-1) + 18 + 2 \\
=&23 + 8 + 11 + 18 + 2 + (-7) + (-9) + (-12) + (-1) && \textit{Group the positive and negative numbers.} \\
=&62 + (-29) = 33 && \textit{The final answer.}
\end{aligned}
$$

Classroom Exercises:

(a) $(-12) - 4 - (-8) + (-7) + 8$

(b) $9 + (-7) + 18 - (-9)$

(c) $(-89.234) - (-72.1234) + 48.88 + (-20.331) - 52.123$

(d) $157.339 - (-98.56) + 72.114 - (-48.231)$

(e) $\frac{5}{6}-\frac{7}{12}$

(f) $\left(-\frac{1}{3}\right)+\frac{8}{5}-\left(-\frac{14}{15}\right)$

(g) $-\left(-\frac{3}{5}\right)+\frac{2}{15}-\left(-\frac{2}{3}\right)$

(h) $\frac{8}{9}+\left(-\frac{5}{6}\right)-\left(-\frac{1}{2}\right)$

An important note: When we say, "Subtract a from b," we mean $b-a$. When we say, "The difference of a and b," we mean $|a-b|$.

1. Subtract 8 from 36. Answer: $36-8=28$.
2. Subtract -8 from 64. Answer: $64-(-8)=64+8=72$.
3. Subtract 8 from 6. Answer: $6-8=-2$.
4. Subtract 23 from -6. Answer: $-6-23=-29$.
5. Subtract -12 from -35. Answer: $-35-(-12)=-35+12=-23$.
6. The difference of 12 and 4 is $|12-4|=|8|=8$.
7. The difference of -12 and 7 is $|-12-7|=|-19|=19$.
8. The difference of 12 and -20 is $|12-(-20)|=|12+20|=|32|=32$.
9. The difference of -12 and -28 is $|-12-(-28)|=|-12+28|=|16|=16$.

Classroom Exercises:

(i) Subtract 4 from 16.

(ii) Subtract -4 from 20.

(iii) Subtract 4 from -12.

(iv) Subtract -4 from -14.

(v) Find the difference of 4 and 16.

(vi) Find the difference of -4 and 20.

(vii) Find the difference of 4 and -12.

(viii) Find the difference of -4 and -14.

1.2.3 Homework Exercises

Add

1. $78 + (-45)$
2. $(-34) + 48$
3. $83 + 76$
4. $(-24) + (-78)$
5. $(-34.579) + (-26.126)$
6. $(-71.223) + 56.34$
7. $59.865 + (-39.234)$
8. $\left(-\frac{2}{3}\right) + \frac{4}{15}$
9. $\left(-\frac{9}{8}\right) + \left(-\frac{4}{5}\right)$
10. $\frac{7}{12} + \left(-\frac{5}{8}\right)$
11. $18 + (-5) + (-8) + 22 + (-7) + (-8)$
12. $(-13) + 12 + (-20) + (-5) + 4 + 8$
13. $(-10) + (-3) + (-5) + (-6)$
14. $13 + (-7) + (-9) + 14 + 2$
15. $28 + (-5) + (10) + 2 + (-15)$
16. $25.987 + (-18.321) + (-11.345) + 59$
17. $(-52.139) + 80.776 + (-567.98) + (-35.62) + 78.9121$
18. $(-123.58) + (-890.56) + 235.45 + 324.78$
19. $56.345 + (-87.221) + 56 + (-345.12) + (-24.78)$
20. $(-23.678) + (-61.054) + 20.12 + 3.78 + (-3.45)$

21. $\left(-\frac{2}{11}\right)+\frac{3}{4}+\left(-\frac{9}{22}\right)$

22. $\frac{7}{12}+\left(-\frac{3}{5}\right)+\frac{5}{6}$

23. $\left(-\frac{2}{3}\right)+\frac{7}{4}+\left(-\frac{9}{2}\right)+\frac{5}{3}+\left(-\frac{3}{4}\right)+\frac{5}{6}$

24. $\frac{5}{8}+\left(-\frac{13}{6}\right)+\left(-\frac{11}{12}\right)+\frac{5}{3}+\left(-\frac{5}{4}\right)+\frac{7}{6}$

25. $\left(-\frac{9}{13}\right)+\left(-\frac{8}{39}\right)+\frac{7}{3}$

Simplify (combination of both add and subtract):

26. $78-(-45)$

27. $(-34)-48$

28. $83-76$

29. $(-24)-(-78)$

30. $(-34.579)-(-26.126)$

31. $(-71.223)+56.34$

32. $59.865-(-39.234)$

33. $\left(-\frac{2}{9}\right)-\frac{4}{5}$

34. $\left(-\frac{9}{8}\right)-\left(-\frac{5}{6}\right)$

35. $\frac{17}{12}-\left(-\frac{15}{8}\right)$

36. $18+(-5)+(-8)-22-(-7)+(-8)$

37. $(-13)-12+(-20)-(-5)-4+8$

38. $(-10)+(-3)-(-5)-(-6)$

39. $13+(-7)-(-9)+14-2$

40. $28 + (-5) - (10) + 2 - (-15)$

41. $25.987 + (-18.321) - (-11.345) - 59$

42. $(-52.139) - 80.776 + (-567.98) - (-35.62) + 78.9121$

43. $(-123.58) - (-890.56) - 235.45 + 324.78$

44. $56.345 + (-87.221) + 56 - (-345.12) - (-24.78)$

45. $(-23.678) + (-61.054) - 20.12 + 3.78 - (-3.45)$

46. $\left(-\frac{2}{11}\right) - \frac{3}{4} + \left(-\frac{9}{22}\right)$

47. $\frac{7}{12} + \left(-\frac{3}{5}\right) - \frac{5}{6}$

48. $\left(-\frac{2}{3}\right) + \frac{7}{4} - \left(-\frac{9}{2}\right) + \frac{5}{3} + \left(-\frac{3}{4}\right) + \frac{5}{6}$

49. $\frac{5}{8} - \left(-\frac{13}{6}\right) + \left(-\frac{11}{12}\right) + \frac{5}{3} + \left(-\frac{5}{4}\right) + \frac{7}{6}$

50. $\left(-\frac{9}{13}\right) - \left(-\frac{8}{39}\right) - \frac{7}{3}$

Answer the following:

51. Subtract 12 from 53.

52. Subtract 12 from -53.

53. Subtract -12 from 53.

54. Subtract -12 from -53.

55. Find the difference of 56 and 89.

56. Find the difference of 56 and -89.

57. Find the difference of -56 and 89.

58. Find the difference of -56 and -89.

1.3 Multiplication and Division of Real Numbers

1.3.1 Multiplication

Multiplying is a quick way of adding. For instance, the phrase "three times four" stands for $4+4+4$. Instead of writing

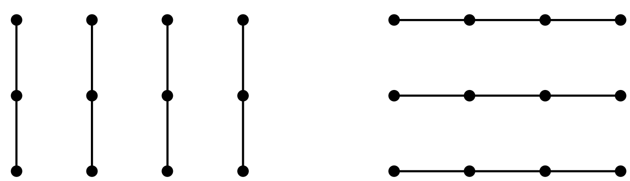
$$5+5+5+5+5+5+5+5+5+5+5+5 \qquad \textit{twelve times five, or five written twelve times;}$$

we simply write

$$12 \times 5.$$

Therefore, the multiplication-table helps us to find the sum when five is written twelve times by just saying $12 \times 5 = 60$.

A clarification: Sometimes we get confused by a phrase of the kind "three times four." Is it $3+3+3+3$ or $4+4+4$? When dealing with the phrase "three times four," ask yourself the question, "How many times?" That tells you, "three times." In other words, the number four has to be written three times. Fortunately for us, "three times four" equals "four times three."



Four times three. Three times four.

Notice, $5 \times (-8) =$ five times $(-8) = (-8) + (-8) + (-8) + (-8) + (-8) = -40$. We thus get $+ \times - = -$. To understand $- \times +$ and $- \times -$ we will need some abstract discussions of opposites (the additive inverses). To avoid any distraction from our current lesson, let us agree to the following rules:

$$\begin{aligned} + \times + &= + \\ + \times - &= - \\ - \times + &= - \\ - \times - &= + \end{aligned}$$

When numbers are multiplied, the result is called their **product.**

Classroom Exercises:

(a) $5 \times 6 = 30$.

(b) $10 \times (-3) = -30$.

(c) $(-11)15 = (-11) \times 15 = -165$.

(d) $(-9)(-12) = (-9) \times (-12) = 108$.

(e) $(-4)8 = (-4) \times 8 =$

(f) $8(-7) = 8 \times (-7) =$

(g) $(-8)(-9) = (-8) \times (-9) =$

(h) $12(-9) = 12 \times (-9) =$

(i) $(-13) \times 0 =$

(j) $0 \times 15 =$

Properties of multiplication:

Multiplication is Commutative : $a \times b = b \times a$ for any real numbers a and b. This tells us that $10 \times 2 = 2 \times 10$. Ordering of numbers while multiplying does not matter.

Multiplication is Associative : $(a \times b) \times c = a \times (b \times c)$ for any real numbers a, b and c. For instance, compare the two calculations:

$$\begin{aligned} &(10 \times 2) \times 3 \\ =&20 \times 3 \\ =&60. \end{aligned} \qquad\qquad \begin{aligned} &10 \times (2 \times 3) \\ =&10 \times 6 \\ =&60. \end{aligned}$$

This important property allows us to multiply several numbers without worrying about the order of multiplication. For instance,

$$\begin{aligned} &(-12) \times 3 \times (-5) \times 4 \\ =&(-36) \times (-5) \times 4 \\ =&180 \times 4 \\ =&720. \end{aligned} \qquad\qquad \begin{aligned} &(-12) \times 3 \times (-5) \times 4 \\ =&(-12) \times (-15) \times 4 \\ =&(-12) \times (-60) \\ =&720. \end{aligned}$$

Multiplication is Distributive over Addition : $a \times (b + c) = a \times b + a \times c$ for any real numbers a, b and c. For example, compare the two calculations:

$$\begin{array}{ll} \quad 10 \times (2+3) & \qquad \quad 10 \times 2 + 10 \times 3 \\ = 10 \times 5 & \qquad = 20 + 30 \\ = 50. & \qquad = 50. \end{array}$$

Using these properties we now simplify expressions involving multiplications. Try and simplify using other orderings of the numbers and multiplications, and see whether we agree.

- Here is an example:

$$\begin{array}{ll} (-2) \times 3 \times (-5) \times 9 \times (-7) & \\ = -2 \times 3 \times 5 \times 9 \times 7 & \textit{We are using } - \times - \times - = - \\ = \underbrace{-2 \times 3} \times \underbrace{5 \times 9} \times 7 & \textit{Multiply two numbers at a time;} \\ = \underbrace{-6 \quad \times \quad 45} \times 7 & \\ = \underbrace{-270 \times 7} & \\ = -1890. & \textit{The final answer.} \end{array}$$

- Here is another example:

$$\begin{array}{ll} 3 \times (-9) \times 4 \times (-3) \times (-5) \times (-2) & \\ = 3 \times 9 \times 4 \times 3 \times 5 \times 2 & \textit{We are using } - \times - \times - \times - = + \\ = \underbrace{3 \times 9} \times \underbrace{4 \times 3} \times \underbrace{5 \times 2} & \textit{Multiply two numbers at a time;} \\ = \underbrace{27 \quad \times \quad 12} \times 10 & \\ = \underbrace{324 \times 10} & \\ = 3240. & \textit{The final answer.} \end{array}$$

Classroom Exercises: Find the products:

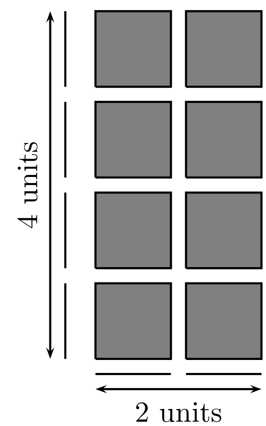
1. $(-3) \times (-10) \times (-5) \times (-7)$

2. $2 \times (-10) \times (-5) \times (-7)$

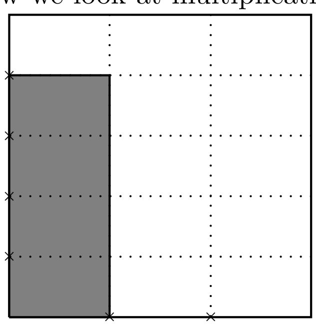
We next want to understand multiplication of fractions.

Here is a geometric way of looking at multiplication, using the fact that the area of a rectangle is the product of its length and its width.

2 units $\times$ 4 units $=$ 8 square units.



Now we look at multiplication of fractions geometrically.



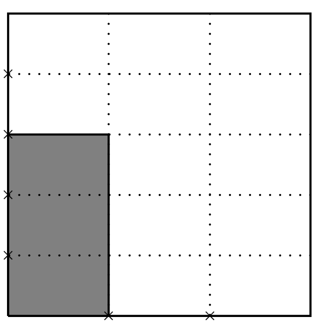
The adjoining square represents 1 square unit.

The shaded region represents $\frac{4}{15}$ of the area.

The length of the shaded region is $\frac{1}{3}$, while its width is $\frac{4}{5}$.

Therefore, $\frac{1}{3} \times \frac{4}{5} = \frac{4}{15}$.

Here is another situation:



Here, the length of the shaded region is $\frac{1}{3}$, while its width is $\frac{3}{5}$.

Therefore, $\frac{1}{3} \times \frac{3}{5} = \frac{3}{15}$.

Note that five times the shaded region is equal to the entire square.

That is, $\frac{3}{15} = \frac{1}{5}$.

We thus conclude:

- $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ for any whole numbers a, b, c, d with $b, d \neq 0$. That is, to multiply two fractions, we multiply the numerators to obtain the numerator of the product, and we multiply the denominators to obtain the denominator of the product.

- $\dfrac{a \times c}{b \times c} = \dfrac{a}{b}$ for any whole numbers a, b, c with $c \neq 0$. That is, a common factor of the numerator and the denominator can be *cancelled.*

Remark : For multiplication of fractions, we *do not* need common denominators.

Example 1 :

$$\begin{aligned} &\frac{2}{7} \times \frac{4}{3} \\ =&\frac{2 \times 4}{7 \times 3} \qquad \textit{There are no common factors in the numerator and the denominator.} \\ =&\frac{8}{21}. \end{aligned}$$

Example 2 :

$$\begin{aligned} &\frac{1}{2} \times \frac{4}{7} = \frac{1 \times 4}{2 \times 7} \quad (2 \textit{ is a common factor in the numerator and the denominator}) \\ =&\frac{1 \times \cancelto{2}{4}}{\cancelto{1}{2} \times 7} = \frac{2}{7}. \end{aligned}$$

Example 3 :

$$\begin{aligned} &\frac{15}{13} \times \left(-\frac{39}{25}\right) = -\frac{15}{13} \times \frac{39}{25} \\ =&-\frac{15 \times 39}{13 \times 25} \quad (5 \textit{ and } 13 \textit{ are common factors in the numerator and the denominator}) \\ =&-\frac{\cancelto{3}{15} \times \cancelto{3}{39}}{\cancelto{1}{13} \times \cancelto{5}{25}} = -\frac{9}{5}. \end{aligned}$$

Example 4 :

$$\begin{aligned} &\left(-\frac{3}{5}\right) \times \frac{15}{7} \times \left(-\frac{14}{3}\right) \times \left(-\frac{2}{9}\right) \\ =&-\frac{3}{5} \times \frac{15}{7} \times \frac{14}{3} \times \frac{2}{9} \qquad \textit{We are using } - \times - \times - = - \\ =&-\frac{3 \times 15 \times 14 \times 2}{5 \times 7 \times 3 \times 9} \qquad \textit{There are several common factors in the numerator and the denominator.} \\ =&-\frac{\cancelto{1}{3} \times \cancelto{3}{15} \times \cancelto{2}{14} \times 2}{\cancelto{1}{5} \times \cancelto{1}{7} \times \cancelto{1}{3} \times 9} = -\frac{\cancelto{1}{3} \times 2 \times 2}{\cancelto{3}{9}} = -\frac{4}{3}. \end{aligned}$$

Classroom Exercises:

(a) $\frac{4}{5} \times \frac{15}{8}$

(b) $\left(-\frac{24}{25}\right) \times \frac{5}{12} \times \left(-\frac{20}{9}\right)$

Multiplication of numbers written in decimal form can be addressed using fractions. Recall that a number written in decimal form can be written in the form of a fraction. Examples:

$$23.45 = \frac{2345}{100}, \quad 0.4578 = \frac{4578}{10000}, \quad 2345.1 = \frac{23451}{10}, \ \textit{and}\ 5 = \frac{5}{1}.$$

We can now multiply:

Example 1: Note that we do not cancel any factor of the denominators here. You may cancel factors and eventually divide to get the same answer in decimal form but this would be more work.

$$2.3 \times 0.5 = \frac{23}{10} \times \frac{5}{10} = \frac{23 \times 5}{10 \times 10} = \frac{115}{100} = 1.15$$

Example 2:

$$\begin{aligned} &(-1.36) \times (-8.811) \\ =&1.36 \times 8.811 \quad \textit{We are using } - \times - = + \\ =&\frac{136}{100} \times \frac{8811}{1000} = \frac{1198296}{100000} = 11.98296 \end{aligned}$$

Classroom Exercises: Find the products:

(a) $9.9 \times 1.2 \times 0.3$

(b) $0.9 \times (-1.3) \times (-1.1)$

1.3.2 Division

Once you master multiplication of fractions, division becomes a straightforward exercise. First, a few definitions:
The **multiplicative identity** is 1. What this means is, $1 \times a = a$ and $a \times 1 = a$ for any real number a. For any *nonzero* real number a, its **multiplicative inverse** (or reciprocal) is $\frac{1}{a}$. Notice,

$$a \times \frac{1}{a} = \frac{a}{1} \times \frac{1}{a} = \frac{a \times 1}{1 \times a} = \frac{\cancelto{1}{a} \times 1}{1 \times \cancelto{1}{a}} = \frac{1}{1} = 1.$$

Number	Reciprocal
5	$\frac{1}{5}$
$\frac{1}{8}$	$\frac{8}{1} = 8$
$\frac{4}{5}$	$\frac{5}{4}$
$-\frac{8}{17}$	$-\frac{17}{8}$

We write the phrase, "a divided by b" in short form as $a \div b$. When we have several $\times$ or $\div$ in an expression, we read from **left to right**. That is, $24 \div 2 \div 3 = 12 \div 3 = 4$. Note,

$$a \div b = \frac{a}{b} \textit{ which is same as } \frac{a}{1} \times \frac{1}{b}$$

That is, **division by a nonzero number, is multiplication by its reciprocal.** For example,

$$10 \div 3 = \frac{10}{3}. \textit{ On the other hand, } \frac{10}{1} \times \frac{1}{3} = \frac{10 \times 1}{1 \times 3} = \frac{10}{3}.$$

More examples:

$$12 \div 8 = \frac{12}{1} \div \frac{8}{1} = \frac{12}{1} \times \frac{1}{8} = \frac{12 \times 1}{1 \times 8} = \frac{\cancelto{3}{12} \times 1}{1 \times \cancelto{2}{8}} = \frac{3}{2}$$

$$50 \div 15 = \frac{50}{1} \div \frac{15}{1} = \frac{50}{1} \times \frac{1}{15} = \frac{50 \times 1}{1 \times 15} = \frac{\cancelto{10}{50} \times 1}{1 \times \cancelto{3}{15}} = \frac{10}{3}$$

Therefore, these division problems can be converted to multiplication problems. Do keep in mind that this method is unnecessary for the purposes of simpler problems. As long as your method is mathematically sound, it does not matter which approach you use.

Example 1 : The expression $12 \div 3 \div 2$ can be simplified (always go from left to right) to $4 \div 2 = 2$. This is a much simpler procedure as compared to the following:

$$
\begin{aligned}
&12 \div 3 \div 2 && = \frac{12}{1} \div \frac{3}{1} \div \frac{2}{1} \\
&= \frac{12}{1} \times \frac{1}{3} \times \frac{1}{2} && = \frac{12 \times 1 \times 1}{1 \times 3 \times 2} \\
&= \frac{12}{6} = \frac{\cancelto{2}{12}}{\cancelto{1}{6}} = 2
\end{aligned}
$$

Example 2 : The following evaluation does need all the steps shown.

$$
\begin{aligned}
&\frac{4}{9} \div \frac{5}{12} \times \left(-\frac{20}{27}\right) \div \frac{10}{3} \\
&= \frac{4}{9} \times \frac{12}{5} \times \left(-\frac{20}{27}\right) \times \frac{3}{10} && \textit{Divisions are changed to multiplications by reciprocals;} \\
&= -\frac{4}{9} \times \frac{12}{5} \times \frac{20}{27} \times \frac{3}{10} && \textit{There is one } - \\
&= -\frac{4 \times 12 \times 20 \times 3}{9 \times 5 \times 27 \times 10} \\
&= -\frac{4 \times \cancelto{4}{12} \times \cancelto{2}{20} \times \cancelto{1}{3}}{\cancelto{3}{9} \times 5 \times \cancelto{9}{27} \times \cancelto{1}{10}} && \textit{Cancelling common factors} \\
&= -\frac{32}{135} && \textit{The final answer.}
\end{aligned}
$$

Example 3 : The following evaluation shows us how to divide numbers written in decimal form.

$$
\begin{aligned}
&1.395 \times (-0.2) \div 0.3 \\
&= \frac{1395}{1000} \times \left(-\frac{2}{10}\right) \div \frac{3}{10} && \textit{Converting decimal numbers to fractions;} \\
&= \frac{1395}{1000} \times \left(-\frac{2}{10}\right) \times \frac{10}{3} && \textit{Divisions are changed to multiplications} \\
& && \textit{by reciprocals;} \\
&= -\frac{1395}{1000} \times \frac{2}{10} \times \frac{10}{3} && \textit{There is one } - \\
&= -\frac{1395 \times 2 \times 10}{1000 \times 10 \times 3} = -\frac{\cancelto{465}{1395} \times 2 \times \cancelto{1}{10}}{1000 \times \cancelto{1}{10} \times \cancelto{1}{3}} \\
&= -\frac{930}{1000} = -0.093. && \textit{Cancel out the common factors } 3, 10
\end{aligned}
$$

Example 4 : Yet another example of a simplification process.

$$\begin{aligned}
&2.5 \div 1.2 \div (-3.24) \times (-0.75) \\
&= \frac{25}{10} \div \frac{12}{10} \div \left(-\frac{324}{100}\right) \times \left(-\frac{75}{100}\right) && \textit{Converting decimal numbers to fractions;} \\
&= \frac{25}{10} \times \frac{10}{12} \times \left(-\frac{100}{324}\right) \times \left(-\frac{75}{100}\right) && \textit{Divisions are changed to multiplications by reciprocals;} \\
&= \frac{25}{10} \times \frac{10}{12} \times \frac{100}{324} \times \frac{75}{100} && \textit{Use } - \times - = + \\
&= \frac{25 \times 10 \times 100 \times 75}{10 \times 12 \times 324 \times 100} \\
&= \frac{25 \times \cancelto{1}{10} \times \cancelto{1}{100} \times \cancelto{25}{75}}{\cancelto{1}{10} \times \cancelto{4}{12} \times 324 \times \cancelto{1}{100}} = \frac{625}{1296} && \textit{The final answer.}
\end{aligned}$$

Classroom Exercises: Simplify the following:

(a) $40 \div 5 \times 2$

(b) $40 \div 5 \div 2$

(c) $42 \div 7 \times 6$

(d) $\dfrac{3}{14} \div \left(-\dfrac{6}{7}\right) \div 5$

(e) $7.5 \div 0.25 \times (-1.2)$

An important note: You probably have noticed that throughout this section we have described division by a **nonzero** number. The reason is quite simple. **Division by** 0 **is undefined.** That is,

$$a \div 0 \textit{ or } \frac{a}{0} \textit{ is undefined for any number } a.$$

Thus, $\frac{1}{0}$ is undefined. So is $\frac{0}{0}$ undefined. This should not be confused with $\frac{0}{1}$ which is 0. **Zero divided by any nonzero number is zero.** Thus,

$$0 \div a = \frac{0}{a} = 0 \textit{ whenever } a \textit{ is nonzero.}$$

Classroom Exercises: Simplify:

(a) $0 \div 3$

(b) $3 \div 0$

(c) $0 \div 0$

(d) $3 \times 0 \div 5$

(e) $3 \times 5 \div 0$

(f) $2 \div 4 \times 0$

(g) $2 \div 0 \times 4$

1.3.3 Simplifying expressions involving $\times, \div, +, -$

Recall that multiplication is used to quickly calculate when a number is added several times. This means that

$$5 \times 4 + 2 = 4 + 4 + 4 + 4 + 4 + 2 = 20 + 2 = 2.$$

That is, when we encounter $5 \times 4 + 2$, we multiply first. This motivates the following ordering: **To simplify an expression involving $\times, \div, +, -$, first perform $\times, \div$ and then perform $+, -$.**

Example 1 : Simplify $3 + 4 \times 8$

$$\begin{aligned} & 3 + 4 \times 8 \\ & = 3 + 32 && \textit{Multiplication first} \\ & = 35 && \textit{Final answer.} \end{aligned}$$

Example 2 : Simplify $5 - 7 \times 3$

$$\begin{aligned} & 5 - 7 \times 3 \\ & = 5 - 21 && \textit{Multiplication first} \\ & = 5 + (-21) = -16 && \textit{Final answer.} \end{aligned}$$

Example 3 : Simplify $12 - 8 \div 5$

$$\begin{aligned} & 12 - 8 \div 5 \\ & = 12 - \frac{8}{5} && \textit{Division first} \\ & = \frac{12}{1} - \frac{8}{5} && \textit{Least common denominator is } 5 \\ & = \frac{12 \times 5}{1 \times 5} - \frac{8}{5} = \frac{60 - 8}{5} = \frac{52}{5}. && \textit{The final answer.} \end{aligned}$$

Example 4 : Simplify $(-7)+5\times\frac{2}{3}\div\frac{4}{5}+6$

$$
\begin{aligned}
&(-7)+5\times\frac{2}{3}\div\frac{4}{5}+6 && \textit{Address multiplications and divisions first.}\\
&=(-7)+\frac{5}{1}\times\frac{2}{3}\times\frac{5}{4}+6 && \textit{Division is multiplication by the reciprocal.}\\
&=(-7)+\frac{5\times 2\times 5}{1\times 3\times 4}+6\\
&=(-7)+\frac{5\times \cancelto{1}{2}\times 5}{1\times 3\times \cancelto{2}{4}}+6\\
&=(-7)+\frac{25}{6}+6\\
&=\frac{(-7)\times 6}{6}+\frac{25}{6}+\frac{6\times 6}{6} && \textit{The least common denominator is } 6\\
&=\frac{-42}{6}+\frac{25}{6}+\frac{36}{6}=\frac{19}{6} && \textit{The final answer.}
\end{aligned}
$$

Example 5 : Simplify $20+\frac{1}{3}\times\frac{9}{5}\div\frac{6}{25}-5$

$$
\begin{aligned}
&20+\frac{1}{3}\times\frac{9}{5}\div\frac{6}{25}-5 && \textit{Address multiplications and divisions first.}\\
&=20+\frac{1}{3}\times\frac{9}{5}\times\frac{25}{6}-5 && \textit{Division is multiplication by the reciprocal.}\\
&=20+\frac{1\times 9\times 25}{3\times 5\times 6}-5\\
&=20+\frac{1\times \cancelto{3}{9}\times \cancelto{5}{25}}{\cancelto{1}{3}\times \cancelto{1}{5}\times 6}-5 && \textit{Simplifying the fractions}\\
&=20+\frac{\cancelto{5}{15}}{\cancelto{2}{6}}-5=20+\frac{5}{2}-5\\
&=\frac{20\times 2}{2}+\frac{5}{2}-\frac{5\times 2}{2} && \textit{The least common denominator is } 2\\
&=\frac{40}{2}+\frac{5}{2}-\frac{10}{2}=\frac{35}{2} && \textit{The final answer.}
\end{aligned}
$$

There are times when multiplcation or division can be done only after some other simplifications have been done. For instance,

Example 6 : Simplify $\dfrac{3+(-4)}{(-3)-(-9)}$. Notice that the division can be performed only after the numerator and denominators are first simplified. So, a possible simplification is

$$\frac{3+(-4)}{(-3)-(-9)} = \frac{-1}{-3+9} = \frac{-1}{6} = -\frac{1}{6}$$

Example 7 : Simplify $\dfrac{3\times 4-11}{(-2)\times 3\times(-4)+2}$ Again, we first simplify the numerator and the denominator. Within the numerator and the denominator, multiplications have to be performed first. Thus,

$$\frac{3\times 4-11}{(-2)\times 3\times(-4)+2} = \frac{12-11}{24+2} = \frac{1}{26}.$$

Example 8 : Simplify $\dfrac{3\div 4-11}{(-2)\div 3\div(-4)+2}$. First simplify the numertor and the denominator.

$$\frac{\frac{3}{1}\times\frac{1}{4}-11}{\frac{(-2)}{1}\times\frac{1}{3}\times\frac{1}{(-4)}+2} = \frac{\frac{3\times 1}{1\times 4}-11}{\frac{(-2)\times 1\times 1}{1\times 3\times(-4)}+2} \qquad \textit{Division is multiplication by the reciprocal}$$

$$= \frac{\frac{3}{4}-11}{\frac{(-2)}{(-12)}+2} = \frac{\frac{3}{4}-11}{\frac{\cancelto{1}{(-2)}}{\cancelto{6}{(-12)}}+2} \qquad \textit{Simplify fraction}$$

$$= \frac{\frac{3}{4}-11}{\frac{1}{6}+2}$$

$$= \frac{\frac{3}{4}-\frac{11\times 4}{4}}{\frac{1}{6}+\frac{2\times 6}{6}} = \frac{\frac{3}{4}-\frac{44}{4}}{\frac{1}{6}+\frac{12}{6}} \qquad \textit{Common denominators}$$

$$= \frac{\frac{3-44}{4}}{\frac{1+12}{6}} = \frac{\frac{-41}{4}}{\frac{13}{6}}$$

$$= \frac{-41}{4}\div\frac{13}{6}$$

$$= \frac{-41}{4}\times\frac{6}{13} = \frac{-41\times 6}{4\times 13}$$

$$= \frac{-41\times\cancelto{3}{6}}{\cancelto{2}{4}\times 13} = \frac{-123}{26} = -\frac{123}{26} \qquad \textit{The final answer.}$$

Classroom Exercises: Simplify:

(a) $3 \times 4 - 7$

(b) $8 \div 5 + 12$

(c) $(-9) \div \frac{3}{4} \times \frac{2}{5} + 7$

(d) $12 \div 7 \times 2$

(e) $15 \times 2 + 3 \div 7 - 8$

(f) $\dfrac{\frac{1}{2} + \frac{3}{5} - \frac{2}{3}}{\frac{2}{3}}$

(g) $(-8) - 12 \div 5 \times 8$

(h) $\dfrac{(-2) + 5 \times 3 \times 7 - 8}{7 - 3 \times (-5) + 8}$.

(i) $\dfrac{3 \div 4 \times 5 - 5 \div 8 \div 3}{7 \times 2 \div 5 + 2 \div 3 \times 4}$

(j) $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}$

You may skip certain steps **only after you have achieved mastery over these calculations.** Until then, try to write as many steps as you need. For instance, consider the following simplification.

Simplication of $(-8) - 12 \div 5 \times 8 + 20 \times \dfrac{3}{4} \div 6 - (-12)$

$$
\begin{aligned}
&(-8) - 12 \div 5 \times 8 + 20 \times \frac{3}{4} \div 6 - (-12) \\
&= (-8) - \frac{12}{1} \div \frac{5}{1} \times \frac{8}{1} + \frac{20}{1} \times \frac{3}{4} \div \frac{6}{1} - (-12) && \textit{Address multiplications and divisions first} \\
&= (-8) - \frac{12}{1} \times \frac{1}{5} \times \frac{8}{1} + \frac{20}{1} \times \frac{3}{4} \times \frac{1}{6} - (-12) && \textit{Division is multiplication by the reciprocal.} \\
&= (-8) - \frac{12 \times 1 \times 8}{1 \times 5 \times 1} + \frac{20 \times 3 \times 1}{1 \times 4 \times 6} - (-12) \\
&= (-8) - \frac{96}{5} + \frac{\cancelto{5}{20} \times \cancelto{1}{3} \times 1}{1 \times \cancelto{1}{4} \times \cancelto{2}{6}} - (-12) && \textit{Simplifying the fractions} \\
&= (-8) - \frac{96}{5} + \frac{5}{2} - (-12) \\
&= (-8) - \frac{96}{5} + \frac{5}{2} + 12 && \textit{Note that } -(-12) = +12 \\
&= \frac{(-8) \times 10}{10} - \frac{96 \times 2}{5 \times 2} + \frac{5 \times 5}{2 \times 5} + \frac{12 \times 10}{10} && \textit{The least common denominator is } 10 \\
&= \frac{-80}{10} - \frac{192}{10} + \frac{25}{10} + \frac{120}{10} \\
&= \frac{-80 - 192 + 25 + 120}{10} \\
&= \frac{-272 + 145}{10} = \frac{-127}{10}. && \textit{The final answer.}
\end{aligned}
$$

1.3.4 Homework Exercises

Multiply

1. $(-4) \times 5$
2. $3 \times (-7)$
3. $(-7) \times 12$
4. $(-12) \times (-11)$
5. 0×2507
6. $(-3566) \times 0$
7. $3 \times (-7) \times 8$
8. $(-6) \times (-5) \times 9$

9. $5 \times (-8) \times 12 \times (-2)$
10. $(-4) \times (-5) \times 3 \times 9 \times (-7)$
11. $\frac{3}{5} \times \frac{2}{11}$
12. $\frac{3}{5} \times \frac{25}{27}$
13. $\left(-\frac{7}{9}\right) \times \frac{3}{28}$
14. $\left(-\frac{24}{28}\right) \times \left(-\frac{7}{30}\right)$
15. $\left(-\frac{4}{15}\right) \times \frac{4}{7} \times \left(-\frac{7}{2}\right)$
16. $\frac{21}{20} \times \left(-\frac{50}{49}\right) \times \left(-\frac{8}{9}\right)$
17. $\frac{12}{13} \times \frac{14}{15} \times \left(-\frac{25}{36}\right)$
18. $\left(-\frac{18}{25}\right) \times \left(-\frac{50}{9}\right) \times \left(-\frac{3}{5}\right)$
19. $\left(-\frac{23}{15}\right) \times \frac{12}{5} \times \frac{15}{8}$
20. $\left(-\frac{7}{9}\right) \times \left(-\frac{10}{7}\right) \times \left(-\frac{36}{49}\right)$

Simplify (multiplications and divisions)

21. $(-4) \div 5$
22. $3 \div (-7)$
23. $(-7) \div 13$
24. $(-12) \div (-11)$
25. $0 \div 2507$
26. $(-3566) \div 0$

27. $3 \times (-7) \div 8$

28. $(-6) \div (-5) \times 9$

29. $5 \times (-8) \div 12 \div (-2)$

30. $(-4) \div (-5) \times 3$

31. $\dfrac{3}{5} \div \dfrac{2}{11}$

32. $\dfrac{13}{25} \div \dfrac{39}{50}$

33. $\left(-\dfrac{7}{9}\right) \div \dfrac{1}{7}$

34. $\left(-\dfrac{24}{35}\right) \div \left(-\dfrac{12}{35}\right)$

35. $\left(-\dfrac{4}{15}\right) \times \dfrac{12}{35} \div \left(-\dfrac{7}{2}\right)$

36. $\dfrac{21}{20} \div \left(-\dfrac{50}{49}\right) \div \left(-\dfrac{8}{9}\right)$

37. $\dfrac{12}{13} \times \dfrac{14}{15} \div \left(-\dfrac{25}{36}\right)$

38. $\left(-\dfrac{18}{25}\right) \div \left(-\dfrac{50}{9}\right) \div \left(-\dfrac{3}{5}\right)$

39. $\left(-\dfrac{23}{15}\right) \times \dfrac{12}{5} \div \dfrac{15}{8}$

40. $\left(-\dfrac{8}{9}\right) \times \left(-\dfrac{10}{7}\right) \div \left(-\dfrac{36}{49}\right) \div \dfrac{7}{39}$

Simplify (multiplications, divisions, additions, and subtractions)

41. $(-6) \div 3 \times 9 - 5$

42. $5 \times (-8) - 7 + 10 \div 2 \div (-5)$

43. $(-4) \div 3 + (-5) + 3 \div 9$

44. $\dfrac{1}{5} \div \dfrac{2}{3} + \dfrac{5}{7}$

45. $\frac{13}{25} \times \frac{5}{3} - \frac{13}{15}$

46. $(-10) \div 8 + 7$

47. $\frac{5}{6} - \frac{2}{9} \times \frac{3}{5} + 2$

48. $12 + 3 \div 2 - 2 \times 4$

49. $9 - \frac{5}{4} \times 8 - 4$

50. $2 \times \frac{3}{5} + \frac{7}{2} \div 3 \times 2 - 2$

51. $\frac{4 + 5 \div 7}{-3 \times 8}$

52. $\frac{\frac{1}{2} - \frac{3}{5} + \frac{7}{10}}{5 + (-2) \times 4}$

53. $\frac{\frac{1}{2} + \frac{5}{6} - \frac{7}{3}}{3 + \frac{1}{2}}$

54. $1 + \frac{1}{1 + \frac{1}{2}}$

55. $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}$

1.4 Simple exponents, roots, and absolute values

1.4.1 Exponents and Radicals

Exponents are used to describe several self-multiplications. Instead of writing $2 \times 2 \times 2 \times 2 \times 2$ we write 2^5. For any natural number n, and any real number a, we write

$a^n = a \times a \times \cdots \times a$; *here, a appears n times. Further, $a^0 = 1$ for $a \neq 0$, and 0^0 is undefined.*

We say that a is the **base** and n is the **exponent.** Some examples:

- 2^4: Here, the base is 2 and the exponent is 4. The value of $2^4 = 2 \times 2 \times 2 \times 2 = 16$.
- 3^2: Here, the base is 3 and the exponent is 2. The value of $3^2 = 3 \times 3 = 9$.
- 4^0: Here, the base is 4 and the exponent is 0. The value of $4^0 = 1$.
- $(-5)^2$: Here, the base is -5 and the exponent is 2. The value of $(-5)^2 = (-5) \times (-5) = 25$.
- -5^2: Here, the base is 5 and the exponent is 2. The value of $-5^2 = -5 \times 5 = -25$.
- $\left(\dfrac{2}{5}\right)^3$: Here, the base is $\left(\dfrac{2}{5}\right)$ and the exponent is 3.

  The value of $\left(\dfrac{2}{5}\right)^3 = \left(\dfrac{2}{5}\right) \times \left(\dfrac{2}{5}\right) \times \left(\dfrac{2}{5}\right) = \dfrac{8}{125}$.

The **inverse** operation of taking exponents is the operation of extracting a root. That is,

$\sqrt{9} = 3$ *because* $3^2 = 9$.

$\sqrt{25} = 5$ *because* $5^2 = 25$.

$\sqrt{1} = 1$ *because* $1^2 = 1$.

$\sqrt{0} = 0$ *because* $0^2 = 0$.

$\sqrt[3]{8} = 2$ *because* $2^3 = 8$.

$\sqrt[3]{27} = 3$ *because* $3^3 = 27$.

$\sqrt[3]{64} = 4$ *because* $4^3 = 64$.

$\sqrt[3]{1} = 1$ *because* $1^3 = 1$.

$\sqrt[3]{0} = 0$ *because* $0^3 = 0$.

$\sqrt[3]{-8} = -2$ *because* $(-2)^3 = -8$.

$\sqrt[3]{-27} = -3$ *because* $(-3)^3 = -27$.

The symbol $\sqrt{}$ is called the **radical.** In an expression $\sqrt{a}$, the number a is called the **radicand.**

a^2 *is read "a -squared;"* $\sqrt{a}$ *is read "square-root of* a*"*

a^3 *is read "a -cubed;"* $\sqrt[3]{a}$ *is read "cube-root of* a*"*

Properties of exponents and radicals:

- $a^n \times a^m = a^{n+m}$. For example, $3^2 \times 3^4 = (3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3^6$.
- $(a^n)^m = a^{nm}$. For example, $(5^2)^3 = (5^2) \times (5^2) \times (5^2) = (5 \times 5) \times (5 \times 5) \times (5 \times 5) = 5^6$.
- $(a \times b)^n = a^n \times b^n$. For example, $(4 \times 5)^2 = 4 \times 5 \times 4 \times 5 = 4 \times 4 \times 5 \times 5 = 4^2 \times 5^2$.
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ for $b \neq 0$. For example, $\left(\frac{4}{5}\right)^2 = \frac{4}{5} \times \frac{4}{5} = \frac{4 \times 4}{5 \times 5} = \frac{4^2}{5^2}$.
- $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$ for $a, b \geq 0$. For example, $\sqrt{4 \times 9} = \sqrt{36} = 6 = 2 \times 3 = \sqrt{4} \times \sqrt{9}$.
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ for $a, b \geq 0, b \neq 0$. For example, $\sqrt{\frac{4}{9}} = \frac{2}{3} = \frac{\sqrt{4}}{\sqrt{9}}$.
- $\sqrt[3]{a \times b} = \sqrt[3]{a} \times \sqrt[3]{b}$. For example, $\sqrt[3]{8 \times 27} = \sqrt{216} = 6 = 2 \times 3 = \sqrt[3]{8} \times \sqrt[3]{27}$.
- $\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$ for $b \neq 0$. For example, $\sqrt[3]{\frac{8}{27}} = \frac{2}{3} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$.

Remark :

- $(a+b)^2 \neq a^2 + b^2$. Check with an example. Let $a = 1$ and $b = 2$. Then

$(a+b)^2$	$a^2 + b^2$
$= (1+2)^2$	$= 1^2 + 2^2$
$= 3^2 = 9.$	$= 1 + 4 = 5.$

  Therefore, $(1+2)^2 \neq 1^2 + 2^2$.

- $(a-b)^2 \neq a^2 - b^2$. Check with an example. Let $a = 1$ and $b = 2$. Then

$(a-b)^2$	$a^2 - b^2$
$= (1-2)^2$	$= 1^2 - 2^2$
$= (-1)^2 = 1.$	$= 1 - 4 = -3.$

  Therefore, $(1-2)^2 \neq 1^2 - 2^2$.

- $\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$. Check with an example. Let $a=1$ and $b=4$. Then

$$\begin{array}{ll} \sqrt{a+b} & \sqrt{a}+\sqrt{b} \\ =\sqrt{1+4} & =\sqrt{1}+\sqrt{4} \\ =\sqrt{5}=2.236067\cdots & =1+2=3. \end{array}$$

  Therefore, $\sqrt{1+2} \neq \sqrt{1}+\sqrt{2}$.

- You can check that $\sqrt{a-b} \neq \sqrt{a}-\sqrt{b}$
- You can check that $\sqrt[3]{a+b} \neq \sqrt[3]{a}+\sqrt[3]{b}$
- You can check that $\sqrt[3]{a-b} \neq \sqrt[3]{a}-\sqrt[3]{b}$

1.4.2 Simplifying expressions involving exponents and radicals

Now we are ready to simplify expressions involving exponents and radicals.
Example 1: Simplify $-(-3)^4$.

$$-(-3)^4 = -(-3)(-3)(-3)(-3) = -81 \text{ \textit{There are five negatives.}}$$

Example 2: Simplify $(-1)^5$.

$$(-1)^5 = (-1)(-1)(-1)(-1)(-1) = -1.$$

Example 3: Simplify $(-1)^{98}$. Note, $(-1)^{98} = \underbrace{(-1)(-1)\cdots(-1)}_{98 \textit{ times}} = +1$.

Example 4: Simplify $-\sqrt{49}$. Note, $-\sqrt{49} = -7$.
Example 5: Simplify $\sqrt{98}$. Note, $\sqrt{98} = \sqrt{49\times 2} = \sqrt{49}\sqrt{2} = 7\sqrt{2}$.
Example 6: Simplify $\sqrt{180}$

$$\sqrt{180} = \sqrt{4\times 45} = \sqrt{4\times 9\times 5} = \sqrt{4}\sqrt{9}\sqrt{5} = 2\times 3\times\sqrt{5} = 6\sqrt{5}.$$

Example 7: Simplify $\sqrt{(-3)^2}$.

$$\sqrt{(-3)^2} = \sqrt{9} = 3.$$

Example 8: Simplify $-\left(\sqrt{(-5)^2}\right)^3$. Start simplifying from the innermost parentheses.

$$-\left(\sqrt{(-5)^2}\right)^3 = -\left(\sqrt{25}\right)^3 = -(5)^3 = -125.$$

Example 9: Simplify $\left(\sqrt{5}\right)^2$.

$$\left(\sqrt{5}\right)^2 = 5. \quad \textit{This is the meaning of } \sqrt{5}.$$

Yet another way of understanding the above problem is

$$\left(\sqrt{5}\right)^2 = \sqrt{5}\sqrt{5} = \sqrt{25} = 5.$$

Example 10: Simplify $\left(\sqrt[3]{\frac{64}{27}}\right)^2$. Note, $\left(\sqrt[3]{\frac{64}{27}}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$.

Classroom Exercises : Find the value of

(a) $(-1)^{101}$

(b) $-(-2)^6$

(c) $(-3)^4$

(d) $\left(-\sqrt{81}\right)^3$

(e) $\sqrt{(-3)^2}$

(f) $\sqrt{300}$

(g) $\left(\sqrt{7}\right)^2$

1.4.3 Simplifying expressions involving exponents, radicals, and $\times, \div$

Suppose we want to simplify the expression, $2^3 \times 5$. This is equal to $2 \times 2 \times 2 \times 5 = 8 \times 5 = 40$. In other words, when we encounter multiplication and exponents in the same expression, the exponents get performed first. This motivates the following rule:

To simplify an expression involving exponents, radicals, multiplications, and divisions, first perform exponents and radicals, and then perform multiplications and divisions.

Example 1: Simplify $\sqrt{9} \times (4)^3 \div \sqrt[3]{125}$.

$$\sqrt{9} \times (4)^3 \div \sqrt[3]{125} = 3 \times 64 \div 5 = \frac{192}{5}.$$

Example 2: Simplify $\sqrt{72} \div \sqrt{144} \times (-2)^3$

$$\sqrt{72} \div \sqrt{144} \times (-2)^3 = \sqrt{36 \times 2} \div 12 \times (-8) = 6\sqrt{2} \div 12 \times (-8)$$

$$= \frac{6\sqrt{2}}{12} \times (-8) = \frac{-48\sqrt{2}}{12} = -\frac{\cancelto{4}{48}\sqrt{2}}{\cancelto{1}{12}} = -4\sqrt{2}.$$

Example 3: Simplify $\frac{1}{\sqrt{5}}$. (That is, **rationalize the denominator.** This means, the denominator should be free of the radical symbol.)

$$\frac{1}{\sqrt{5}} = \frac{1 \times \sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{\sqrt{5}}{5}; \textit{ recall that } \sqrt{5}\sqrt{5} = 5.$$

Example 4: Simplify $\sqrt[3]{-343} \div \sqrt{49} \times \sqrt{\frac{25}{12}}$.

$$\begin{aligned}
&\sqrt[3]{-343} \div \sqrt{49} \times \sqrt{\frac{25}{12}} \\
=&(-7) \div 7 \times \frac{\sqrt{25}}{\sqrt{12}} && \textit{Simplify radicals} \\
=&(-7) \div 7 \times \frac{5}{\sqrt{4 \times 3}} \\
=&\frac{(-7)}{1} \times \frac{1}{7} \times \frac{5}{2\sqrt{3}} && \textit{Multiplication and division now} \\
=&\frac{-\cancel{7}^{1} \times 5}{\cancel{7}^{1} \times 2\sqrt{3}} \\
=&-\frac{5}{2\sqrt{3}} \\
=&-\frac{5\sqrt{3}}{2\sqrt{3}\sqrt{3}} && \textit{Rationalize the denominator} \\
=&-\frac{5\sqrt{3}}{2 \times 3} = -\frac{5\sqrt{3}}{6}. && \textit{The final answer.}
\end{aligned}$$

As a rule we always rationalize the denominator to obtain the final answer.
Example 5: Simplify $\sqrt{169} \times \sqrt[3]{8} \div 3^3 \div \sqrt{20}$.

$$\begin{aligned}
\sqrt{169} \times \sqrt[3]{8} \div 3^3 \div \sqrt{20} =&13 \times 2 \div 27 \div \sqrt{4 \times 5} && \textit{Simplify exponents and radicals} \\
=&13 \times 2 \div 27 \div 2\sqrt{5} \\
=&\frac{13}{1} \times \frac{2}{1} \times \frac{1}{27} \times \frac{1}{2\sqrt{5}} && \textit{Division is multiplication by reciprocal} \\
=&\frac{13 \times \cancel{2}^{1}}{27 \times \cancel{2}^{1}\sqrt{5}} = \frac{13\sqrt{5}}{27\sqrt{5}\sqrt{5}} && \textit{Rationalize the denominator} \\
=&\frac{13\sqrt{5}}{27 \times 5} = \frac{13\sqrt{5}}{135}. && \textit{The final answer.}
\end{aligned}$$

Many a time, one has to simplify the radicand (in a radical expression) or the base (in an exponential expression) first.

Example 6: Simplify $\sqrt{36 \times 100 \div 49 \div 81}$.

$$\sqrt{36 \times 100 \div 49 \div 81} = \sqrt{\frac{36}{1} \times \frac{100}{1} \times \frac{1}{49} \times \frac{1}{81}} = \sqrt{\frac{36 \times 100}{49 \times 81}} = \frac{\overset{2}{\cancel{6}} \times 10}{7 \times \underset{3}{\cancel{9}}} = \frac{20}{21}.$$

Example 7: Simplify $(90 \div 9 \div 12 \div 2)^2$

$$(90 \div 9 \div 12 \div 2)^2 = \left(\frac{90}{1} \times \frac{1}{9} \times \frac{1}{12} \times \frac{1}{2}\right)^2 = \left(\frac{\overset{10}{\cancel{90}} \times 1 \times 1 \times 1}{1 \times \underset{1}{\cancel{9}} \times 12 \times 2}\right)^2 = \left(\frac{\overset{5}{\cancel{10}}}{12 \times \underset{1}{\cancel{2}}}\right)^2 = \frac{25}{144}.$$

Classroom Exercises : Simplify the following (Do not forget to rationalize the denominator whenever necessary) :

1. $(a)\ \sqrt{8}\quad (b)\ \sqrt{18}\quad (c)\ \sqrt{32}\quad (d)\ \sqrt{50}$
2. $(a)\ \sqrt{12}\quad (b)\ \sqrt{27}\quad (c)\ \sqrt{48}\quad (d)\ \sqrt{75}$
3. $(a)\ \sqrt{20}\quad (b)\ \sqrt{45}\quad (c)\ \sqrt{80}\quad (d)\ \sqrt{125}$
4. $(a)\ \sqrt{24}\quad (b)\ \sqrt{54}\quad (c)\ \sqrt{28}\quad (d)\ \sqrt{63}$
5. $(a)\ \sqrt{\frac{1}{8}}\quad (b)\ \sqrt{\frac{1}{18}}\quad (c)\ \sqrt{\frac{1}{32}}\quad (d)\ \sqrt{\frac{1}{50}}$
6. $(a)\ \sqrt{\frac{1}{12}}\quad (b)\ \sqrt{\frac{1}{27}}\quad (c)\ \sqrt{\frac{1}{48}}\quad (d)\ \sqrt{\frac{1}{75}}$
7. $(a)\ \sqrt{\frac{1}{20}}\quad (b)\ \sqrt{\frac{1}{45}}\quad (c)\ \sqrt{\frac{1}{80}}\quad (d)\ \sqrt{\frac{1}{125}}$
8. $(a)\ \sqrt{\frac{1}{24}}\quad (b)\ \sqrt{\frac{1}{54}}\quad (c)\ \sqrt{\frac{1}{28}}\quad (d)\ \sqrt{\frac{1}{63}}$
9. $\sqrt{8} + \sqrt{18} - 3\sqrt{32} - \sqrt{50}$
10. $\sqrt{12} - 2\sqrt{27} + 4\sqrt{48} + \sqrt{75}$
11. $\sqrt{20} - 5\sqrt{45} + 8\sqrt{80} + \sqrt{125}$
12. $\sqrt{24} + 2\sqrt{54} - 3\sqrt{28} + \sqrt{63}$

13. $\sqrt{121} \div \sqrt[3]{125} \times \left(\sqrt{5}\right)^2$

14. $-\sqrt{289} \div \sqrt{150}$

15. $\sqrt{\dfrac{400}{45}}$

16. $(1 \div 3 \div 2 \times 5)^2$

17. $(4^2 \div 2^4)^0$

1.4.4 Simplifying expressions involving exponents, radicals, and $\times, \div, +, -$

Recall, to simplify an expression involving $\times, \div, +, -$, we first perform $\times, \div$ (left to right) and then perform $+, -$ (left to right).
Therefore, to simplify an expression involving exponents, radicals, $\times, \div, +$, and $-$ we perform the operations in the following order:

1. Exponents and Radicals are computed first.

2. Multiplications and Divisions are computed second (left to right).

3. Additions and Subtractions come at the end (left to right).

Example 1: Simplify $5^2 \div \sqrt{36} + \sqrt[3]{27} \times 2^3$

$$\begin{aligned}
& 5^2 \div \sqrt{36} + \sqrt[3]{27} \times 2^3 \\
=& 25 \div 6 + 3 \times 8 && \textit{Exponents and radicals first} \\
=& \frac{25}{6} + 3 \times 8 && \textit{Multiplications and divisions next (left to right)} \\
=& \frac{25}{6} + 24 \\
=& \frac{25}{6} + \frac{24 \times 6}{6} && \textit{The least common denominator is } 6 \\
=& \frac{25}{6} + \frac{144}{6} = \frac{25 + 144}{6} = \frac{169}{6}.
\end{aligned}$$

Example 2: Simplify $4^2 + \sqrt{80} \times 3 - \sqrt{125}$.

$$
\begin{aligned}
& 4^2 + \sqrt{80} \times 3 - \sqrt{125} \\
=& 16 + \sqrt{16 \times 5} \times 3 - \sqrt{25 \times 5} && \textit{Simplify exponents and radicals first} \\
=& 16 + 4 \times \sqrt{5} \times 3 - 5\sqrt{5} \\
=& 16 + 4 \times 3 \times \sqrt{5} - 5\sqrt{5} && \textit{Multiplication is commutative} \\
=& 16 + 12\sqrt{5} - 5\sqrt{5} \\
=& 16 + 7\sqrt{5} && \textit{Additions and subtractions last.}
\end{aligned}
$$

Example 3: Simplify $\sqrt{(-3)^2 + (-9)^2}$.

$$\sqrt{(-3)^2 + (-9)^2} = \sqrt{9 + 81} = \sqrt{90} = \sqrt{9 \times 10} = \sqrt{9}\sqrt{10} = 3\sqrt{10}.$$

Notice in this case that $\sqrt{9 + 81} \neq \sqrt{9} + \sqrt{81}$.
Classroom Exercises : Simplify

(a) $\sqrt{12} + \sqrt{75} - \sqrt{27}$

(b) $3\sqrt{8} - 4\sqrt{50} + 2\sqrt{18}$

(c) $\dfrac{3^2}{\sqrt{5}} - \dfrac{4\sqrt[3]{27}}{\sqrt{5}} \div \dfrac{\sqrt{20}}{12}$

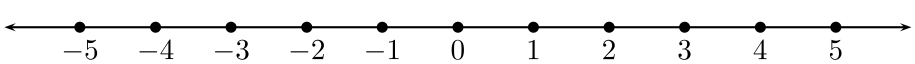
(d) $\dfrac{\sqrt{6 \times 4}}{5} + \dfrac{\sqrt{2}}{5} \div \dfrac{\sqrt{3}}{4}$

(e) $\dfrac{\sqrt{8}}{3} + \dfrac{\sqrt{18}}{4} - \dfrac{5\sqrt{2}}{3}$

(f) $\sqrt{5^2 + 4^2}$

1.4.5 Absolute value and simplifications involving absolute values

The **absolute value** of a real number is its distance from 0 on the real number line. Absolute value of number a is denoted by $|a|$.



- The distance between 4 and 0 on the number line is 4. Hence, $|4| = 4$.
- The distance between -4 and 0 on the number line is 4. Hence, $|-4| = 4$.

- The distance between 0 and 0 on the number line is 0. Hence, $|0| = 0$.

The absolute value is closely related to exponents and radicals. Pay careful attention to the numbers in the following three examples.

- $\sqrt{4^2} = \sqrt{16} = 4$.
- $\sqrt{(-4)^2} = \sqrt{16} = 4$.
- $\sqrt{0^2} = \sqrt{0} = 0$.

In other words, $\sqrt{a^2} = |a|$ *for any real number* a.
Classroom Exercises :

(a) $\sqrt{5^2} =$

(b) $\sqrt{(-5)^2} =$

(c) $\sqrt{7^2} =$

(d) $\sqrt{(-7)^2} =$

(e) $-\sqrt{7^2} =$

(f) $-3\sqrt{2^2} =$

Mind you, $\sqrt[3]{a^3} \neq |a|$. For instance, $\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$ which is not the same as $|-2|$.
Properties of the absolute value:

- $|a \times b| = |a| \times |b|$ for any real numbers a, b. For example, $|(-2) \times 3| = |-6| = 6 = |-2| \times |3|$.
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for any real numbers a, b with $b \neq 0$. For example, $\left|\frac{-2}{3}\right| = \left|-\frac{2}{3}\right| = \frac{2}{3} = \frac{|-2|}{|3|}$.

Remark:

- $|a + b| \neq |a| + |b|$. Check for $a = (-2)$ and $b = 1$.
  $|(-2) + 1| = |-1| = 1$. Whereas, $|-2| + |1| = 2 + 1 = 3$.
- $|a - b| \neq |a| - |b|$. Check for $a = (-2)$ and $b = 1$.
  $|(-2) - 1| = |-3| = 3$. Whereas, $|-2| - |1| = 2 - 1 = 1$.

Classroom Exercises : Find the values of

(a) $\left|\frac{-9}{10}\right|$ (b) $|2 \times (-7)|$ (c) $\sqrt{\left(\frac{-9}{10}\right)^2}$ (d) $\sqrt{(2 \times (-7))^2}$

1.4.6 Simplifying expressions involving absolute values

When we want to simplify an expression involving exponents, radicals, $|\ \ |, \times, \div, +, -$, the absolute value takes the same priority as exponents and radicals. That is,

1. Exponents, Radicals, and Absolute Values are computed first.
2. Multiplications and Divisions are computed second.
3. Additions and Subtractions come at the end.

As always, if exponents, radicals, and absolute values themselves have some complicated expressions within them, then the inner complications have to be simplified. In other words, exponents, radicals, and absolute values can themselves be **grouping** mechanisms.
Example 1: Simplify $|-10| \times 3^2 - \sqrt{9}$.

$$\begin{aligned} & |-10| \times 3^2 - \sqrt{9} && \textit{Absolute value, exponent and radical first} \\ =& 10 \times 9 - 3 && \textit{Multiplication next} \\ =& 90 - 3 = 87. && \textit{Subtraction last.} \end{aligned}$$

Example 2: Simplify $2^3 \div |-5| \times \sqrt{49} + 8$.

$$\begin{aligned} & 2^3 \div |-5| \times \sqrt{49} + 8 && \textit{Exponent, radical, absolute value first} \\ =& 8 \div 5 \times 7 + 8 && \textit{Multiplication and division next} \\ =& \frac{8}{5} \times 7 + 8 = \frac{8}{5} \times \frac{7}{1} + 8 && \textit{Addition last; the least common denominator is } 5 \\ =& \frac{56}{5} + \frac{40}{5} = \frac{96}{5}. \end{aligned}$$

Example 3: Simplify $\left| \dfrac{\sqrt[3]{125} \times |-2| + 12^2 \div |-8|}{-|-3+5|^2} \right| + 4$. In this case, the outermost absolute value is to be performed after the expression inside is simplified.

$$\begin{aligned} & \left| \frac{\sqrt[3]{125} \times |-2| + 12^2 \div |-8|}{-|-3+5|^2} \right| + 4 && \textit{The inner exponents, radicals and absolute values} \\ & && \textit{are to be simplifed first} \\ =& \left| \frac{5 \times 2 + 144 \div 8}{-|2|^2} \right| + 4 \\ =& \left| \frac{10+18}{-4} \right| + 4 && \textit{Simplify multiplications and divisions inside} \\ =& \left| \frac{28}{-4} \right| + 4 = \left| -\frac{28}{4} \right| + 4 = \frac{28}{4} + 4 = 7 + 4 = 11. \end{aligned}$$

Classroom Exercises:

(a) $|-4|$

(b) $-|4|$

(c) $-|-4|$

(d) $|3-8\times 2|$

(e) $|(-1)^{81}|$

(f) $|(-12)\div 6\times 2|$

(g) $|(-1)^3+3\div 5-4|$

(h) $(-2)^3+3\div|2-7|+1$

(i) $\left|\dfrac{\sqrt[3]{27}-(-2)^4+3}{8\times(-2)}\right|$

1.4.7 Homework Exercises

Do not forget to rationalize the denominator whenever necessary.
Simplify (exponents and radicals):

1. $(-1)^{2001}$
2. $-(-5)^3$
3. $-(-(-2)^4)$
4. $-\sqrt{1000}$
5. $\sqrt{128}$
6. $\sqrt{243}$
7. $\sqrt{150}$
8. $\left(\sqrt{15}\right)^2$
9. $\sqrt[3]{\dfrac{1}{64}}$
10. $\left(\sqrt[3]{7}\right)^3$

Simplify (exponents, radicals, multiplications, and divisions)

11. $\sqrt{196} \times \sqrt[3]{1000}$
12. $\sqrt{\dfrac{1}{128}}$
13. $\sqrt[3]{64} \div \sqrt{12}$
14. $\sqrt{12} \div \sqrt{98} \times \sqrt[3]{8}$
15. $\sqrt{30} \times \sqrt{2} \times \sqrt{3}$
16. $\sqrt[3]{8^2 \times 5^2 \div 20^2}$
17. $6^3 \div \sqrt{108} \div \sqrt{162}$
18. $\sqrt{1 \div 8 \div 18 \div 25}$
19. $(80 \div \sqrt{400} \times \sqrt[3]{7})^3$
20. $\left(\sqrt{(-5)^2 \div (-4)^2}\right)^3$

Simplify (exponents, radicals, multiplications, divisions, additions, and subtractions)

21. $\sqrt{3} + \sqrt{48} - \sqrt{75}$
22. $\sqrt{\dfrac{9}{2}} - \sqrt[3]{5 \div 5} \div \dfrac{\sqrt{50}}{3^2}$
23. $\left(\dfrac{3}{2}\right)^3 + \sqrt{\dfrac{25}{64}} \div \sqrt[3]{\dfrac{1}{10}}$
24. $\sqrt{24} - 4\sqrt{6} + \sqrt{150}$
25. $\sqrt{8} - \sqrt{18} + \sqrt{50}$
26. $\dfrac{5^3 \times 2^3 - \sqrt{8}\sqrt{72}}{\sqrt{25} + \sqrt{3}\sqrt{75}}$
27. $\sqrt{\dfrac{\sqrt[3]{125} + 2^2 5^3 - 1}{(-4)^2\sqrt{9} + (-2)^3}}$

Simplify (absolute values, exponents, radicals, multiplications, divisions, additions, and subtractions)

28. $\left|(-1)^{2009}\right|$

29. $\left|(-7)^3\right|$

30. $\left|-(-(-3)^4)\right|$

31. $\left|-\sqrt{24}\right|$

32. $\sqrt{|-12|} \div \sqrt{|-98|} \times |\sqrt[3]{-8}|$

33. $3 - 5 + |7 - 12| \times |(-2)^3 - 12| \div 3$

34. $|3 - 5 + 7 \div 2 \times 3|$

35. $|(-2)^{2011}|^0$

36. $\left|-\sqrt{150} + \sqrt{2} \times \sqrt{3}\right|$

37. $\left|2^3 \times 5^2 - 20^2 \div 3\right|$

38. $-80 \div \sqrt{|-400|} \times \left|\sqrt[3]{-2})^3\right|$

39. $\left|-\sqrt{|25| \div (-4)^2}\right|^3$

40. $\left|\dfrac{2^3 - 5^0 - \sqrt{8}\sqrt{72}}{\sqrt{25} + \sqrt{3}\sqrt{75}}\right|$

Chapter 2

Evaluating Algebraic Expressions and Functions

2.1 Order of operations

2.1.1 Parentheses and Order of operations

In an algebraic expression, the **parentheses** are used as a grouping mechanism so as to overrule the precedence in the order of operations. For example,

- In the expression $2 \times (3 + 4)$, the number 2 is multiplied to $(3 + 4)$. In other words, the number 2 sees the $(3 + 4)$ as just one number. Without the parentheses, the expression would be $2 \times 3 + 4$, in which case 2 is multiplied to 3 alone.

  The parentheses allow us to overrule the precedence in the order of operations. For instance, when you encounter a multiplication and an addition in an expression, you are supposed to perform the multiplication first. But the parentheses in $2 \times (3+4)$ requires us to add 3 and 4 first to get a 7, and then multiply by 2. That is, $2 \times (3 + 4) = 2 \times 7 = 14$.

  Compare this to $2 \times 3 + 4 = 6 + 4 = 10$.

- Consider $3 + 5^2$. Here, the exponent 2 applies to 5 alone. As per the order of operations, exponent gets performed first. So, $3 + 5^2 = 3 + 25 = 28$.

  Compare this with $(3 + 5)^2$. The parentheses suggest that $(3 + 5)$ should be thought of as one group, and the exponent applies to **all of** $(3 + 5)$. Therefore, to simplify the expression $(3 + 5)^2$, we let the parentheses overrule the order of operations, and add first. Thus, $(3 + 5)^2 = 8^2 = 64$.

Therefore, to simplify an expression involving parentheses, exponents, radicals, absolute values $\times, \div, +$, or $-$ we perform the operations in the following order:

1. **Parentheses** have to be simplified first. If there are several nested parentheses, then start from the innnermost parentheses and work outwards.
2. **Exponents, Radicals, and Absolute values** are computed next.
3. **Multiplications and Divisions** are computed next. If there are several multiplications and divisions, then you may go from **left to right**, or convert the divisions into multiplications by reciprocals and then proceed.
4. **Additions and Subtractions** come at the end. If there are several additions and subtractions, then you may go from **left to right**, or combine all the additions together, and all the subtractions together and then proceed.

In algebraic expressions, we may encounter not just parentheses, ( ), but also brackets, [ ], braces, { }, fraction bars, as in $\frac{2 \times 3 + 5}{4 + 7}$, and radicals, as in $\sqrt{3 \times 12 + 64}$. Brackets, braces, fraction bars and radicals perform the same function as parentheses.
Let us work on some examples.
Example 1: Simplify $(7 - 2) \times 5 + 6^2$.

$$\begin{aligned}
(7-2) \times 5 + 6^2 =& 5 \times 5 + 6^2 && \textit{Parentheses first}\\
=& 5 \times 5 + 36 && \textit{Exponent next}\\
=& 25 + 36 = 61. && \textit{Multiplication next}
\end{aligned}$$

Example 2: Simplify $(4 + 5)[3 - 2\{4 - (5 - (-2))\}]$.

$$\begin{aligned}
(4+5)[3-2\{4-(5-(-2))\}] =& (9)[3-2\{4-(5+2)\}] && \textit{Parentheses first, starting from the}\\
=& (9) \times [3 - 2 \times \{4 - 7\}] && \textit{innermost ones;}\\
=& (9) \times [3 - 2 \times \{-3\}]\\
=& (9) \times [3 - \{-6\}]\\
=& 9 \times [3 + 6] = 9 \times 9 = 81.
\end{aligned}$$

Example 3: Simplify $2 - \sqrt{|3[2 + 5(3 - 4)^3]|}$.

$$\begin{aligned}
& 2 - \sqrt{|3[2+5(3-4)^3]|} && \textit{Parentheses first, starting from the innermost ones}\\
=& 2 - \sqrt{|3[2+5(-1)^3]|} && \textit{Within the brackets, exponent first}\\
=& 2 - \sqrt{|3[2+5 \times (-1)]|} && \textit{Within the brackets, multiplication first}\\
=& 2 - \sqrt{|3[2+(-5)]|} && \textit{Simplify the brackets}\\
=& 2 - \sqrt{|3 \times [-3]|} && \textit{Simplify the radicand}\\
=& 2 - \sqrt{|-9|}\\
=& 2 - \sqrt{9} = 2 - 3 = -1.
\end{aligned}$$

Example 4: Simplify $(5+2)^3$. Note, $(5+2)^3 = 7^3 = 343$.
Many students make the mistake of writing $(5+2)^3 = 5^3 + 2^3 = 125 + 8 = 133$. This is incorrect because, **parentheses have to be computed first.** Compare Example 4 with the next example.
Example 5: Simplify $5^3 + 3 \times 5^2 \times 2 + 3 \times 5 \times 2^2 + 2^3$.

$$\begin{aligned}
&5^3 + 3 \times 5^2 \times 2 + 3 \times 5 \times 2^2 + 2^3 && \textit{Exponents first} \\
=&125 + 3 \times 25 \times 2 + 3 \times 5 \times 4 + 8 && \textit{Multiplications next} \\
=&125 + 150 + 60 + 8 && \textit{Additions last} \\
=&343.
\end{aligned}$$

Notice that the answers to Examples 4 and 5 are the same. Indeed, we have a formula

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

That is, $(a+b)^3 \neq a^3 + b^3$.
Classroom Exercises : Simplify

(a) $(10 + (-3))^2$

(b) $10^2 + 2 \times 10 \times (-3) + (-3)^2$

(c) $\left(\frac{2}{3} - \frac{4}{5}\right)\left(\frac{2}{3} + \frac{4}{5}\right)$

(d) $\left(\frac{2}{3}\right)^2 - \left(\frac{4}{5}\right)^2$

(e) $3[4^2 \div 2^3 + \sqrt{25}(|3 \times (-7) + 4| + 5)]$

2.1.2 Homework Exercises

Simplify

1. $(5+6)^2$
2. $5^2 + 2 \times 5 \times 6 + 6^2$
3. $(7-6)^2$
4. $7^2 - 2 \times 7 \times 6 + 6^2$
5. $(10 + (-12))^3$

6. $10^3 + 3 \times 10^2 \times (-12)$

7. $\frac{3}{5} + \{3 - 7(5 + (1 + 2))\} \div 2$

8. $\frac{2}{\sqrt{|(-3)^2 - 5^3|}}$

9. $\frac{1}{6}(7 + 8)8 - \left(\frac{2}{3}\right)^2$

10. $\left(5 + \frac{\sqrt{289} + 7}{7^2 - 23^{58}}\right)^0$ (Do not work too hard here).

2.2 Evaluating algebraic expressions and functions

Algebra entails working with expressions involving variables. For example, the area of a rectangle is given by the formula $A = l \times w$ where A stands for the area, l stands for the length, and w stands for the width of the rectangle. Once you have such a formula, you can find the area of a rectangle of any length or width by **evaluating** the formula $A = l \times w$ at the given values of l and w.

In this lesson we practise evaluating algebraic expressions.

Example 1: Evaluate the expression $x^3 + 2x + 4$ for $x = 2$.

Given such a task, the very first step is to place parentheses around the value of the variable. This is particularly important if the value of the variable is negative.

In this problem, simply write $x = (2)$. Now proceed.

$$\begin{aligned}
&x^3 + 2x + 4 && \textit{Replace every } x \textit{ by } (2)\\
=&(2)^3 + 2(2) + 4 && \textit{Now we follow the order of operations.}\\
=&8 + 2 \times 2 + 4 && \textit{Exponents first. Also, } 2(2) = 2 \times 2\\
=&8 + 4 + 4 && \textit{Multiplication next}\\
=&16.
\end{aligned}$$

Example 2: Evaluate the expression $-x^2 + 4$ for $x = -5$.

First write parentheses around the value for x. That is, $x = (-5)$. Now proceed.

$$\begin{aligned}
&- x^2 + 4 && \textit{Replace } x \textit{ by } (-5),\\
=&- (-5)^2 + 4 && \textit{Exponent first; remember, the square applies to } (-5)\\
=&- (25) + 4\\
=&- 25 + 4 = -21.
\end{aligned}$$

Example 3: Evaluate the expression $x + y$ for $x = -2$ and $y = 3$.
First set $x = (-2)$ and $y = (3)$. Now proceed.

$$x + y = (-2) + (3) = 1.$$

Example 4: Evaluate the expression $\sqrt{x^2 + y^2}$ for $x = -2$ and $y = 3$.
First set $x = (-2)$ and $y = (3)$. Now proceed.

$$\begin{aligned} &\sqrt{x^2 + y^2} && \textit{Replace } x \textit{ by } (-2) \textit{ and } y \textit{ by } (3) \\ =&\sqrt{(-2)^2 + (3)^2} && \textit{Simplify inside the } \sqrt{\ } \\ =&\sqrt{4 + 9} = \sqrt{13}. \end{aligned}$$

Note that examples 3 and 4 show that $\sqrt{x^2 + y^2} \neq x + y$.
Example 5: Evaluate the expression $\dfrac{a^2 + 3b - c}{4abc}$ for $a = 3$, $b = -4$, and $c = -5$.
First write parentheses around the values for a and b. That is, $a = (3)$, $b = (-4)$, and $c = (-5)$.
Now proceed.

$$\begin{aligned} &\frac{a^2 + 3b - c}{4abc} && \textit{Replace } a \textit{ by } (3), \quad b \textit{ by } (-4), \quad c \textit{ by } (-5) \\ =&\frac{(3)^2 + 3(-4) - (-5)}{4(3)(-4)(-5)} && \textit{Simplify numerator and denominator separately} \\ =&\frac{9 + 3(-4) - (-5)}{4(3)(-4)(-5)} && \textit{Exponent first.} \\ =&\frac{9 + (-12) - (-5)}{240} && \textit{Multiplication next} \\ =&\frac{9 - 12 + 5}{240} = \frac{\cancelto{1}{2}}{\cancelto{120}{240}} = \frac{1}{120}. \end{aligned}$$

Note that the notation of a function can also be used to evaluate algebraic expressions. We will not go into the details of defining a function in this course. But we will quickly present an example.
Example 6: Given function $f(x) = x^2 + 3x - 1$ evaluate $f(-2)$.
We are being asked to substitute $x = (-2)$ for the expression $f(x)$. That is

$$f(-2) = (-2)^2 + 3(-2) - 1 = 4 - 6 - 1 = -3.$$

Classroom Exercises: Evaluate the expressions or functions for the given values of the variables.

(a) $\dfrac{3x+5}{4x-7}$ for $x=-2$.

(b) $\left|4a^2-5b+c\right|$ for $a=-3, b=2, c=5$.

(c) $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ for $a=2, b=-3, c=-5$.

(d) $\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ for $a=-4, b=9, c=8$.

(e) $f(3)$ where $f(x)=x^3-2$.

(f) $f(-3)$ where $f(x)=-x^2+7$.

(g) $f(0)$ where $f(x)=2x^8-7x^2+\dfrac{1}{3}x+2$.

(h) $f(1)$ where $f(x)=2x^2-3x+4$.

We now use this knowledge to evaluate various formulae. Let us recall some of the important formulae we encounter in basic algebra and geometry.

1. **Simple Interest** Let P denote the principal (or the amount deposited) for t years, at an annual rate of interest r. Then, the simple interest earned at the end of t years is given by the formula

$$I = Prt$$

Calculate the interest earned at the end of 5 years if \$ 1,000 is deposited at an annual rate of interest of $r=0.02$.

Solution : First write $P=(1000)$, $r=(0.02)$, and $t=(5)$. Now, the interest earned is

$$I=(1000)(0.02)(5)=1000\times\frac{2}{100}\times 5=100.$$

That is, the interest earned is \$ 100.

2. **Slope of a line** Given any two points (x_1, y_1) and (x_2, y_2) on a line, the slope of the line is denoted by m and is given by the formula

$$m=\frac{y_2-y_1}{x_2-x_1}$$

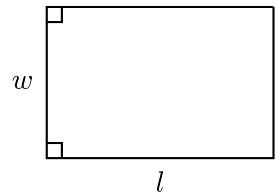
Find the slope of a line that passes through points $(-3, 4)$ and $(7, -5)$.

Solution : The two points are $(x_1, y_1)=(-3, 4)$ and $(x_2, y_2)=(7, -5)$. First write $x_1=(-3)$, $y_1=(4)$, $x_2=(7)$, and $y_2=(-5)$. Now,

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{(-5)-(4)}{(7)-(-3)}=\frac{-9}{10}=-\frac{9}{10}.$$

3. **Area of a rectangle** $A=lw$ is the formula for the area of a rectangle with length l and width w. Find the area when $l=\dfrac{3}{4}$ cm. and $w=\dfrac{8}{9}$ cm.

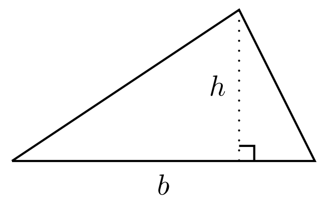
Solution : First write $l = \left(\frac{3}{4}\right)$ and $w = \left(\frac{8}{9}\right)$.



Now, $A = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right) = \frac{\cancel{3}^{1} \times \cancel{8}^{2}}{\cancel{4}^{1} \times \cancel{9}^{3}} = \frac{2}{3}$ sq. cm.

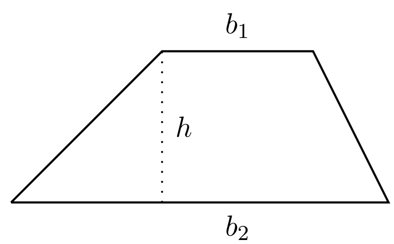
4. **Area of a triangle** The area of a triangle, A, with base b and height h is given by the formula $A = \frac{1}{2}bh$. Find the area of a triangle with $b = \frac{4}{5}$ ft. and $h = \frac{7}{8}$ ft.

   Solution : First $b = \left(\frac{4}{5}\right)$ and $h = \left(\frac{7}{8}\right)$

   $$A = \frac{1}{2}\left(\frac{4}{5}\right)\left(\frac{7}{8}\right) = \frac{1 \times \cancel{4}^{1} \times 7}{2 \times 5 \times \cancel{8}^{2}} = \frac{7}{20} \text{ sq. ft.}$$

5. **Area of a trapezoid** The area of a trapezoid, A, with bases b_1, b_2 and height h is given by the formula $A = \frac{1}{2}(b_1 + b_2)h$. Find the area of a trapezoid with $b_1 = \frac{3}{4}$, $b_2 = \frac{2}{5}$ and $h = \frac{1}{3}$.

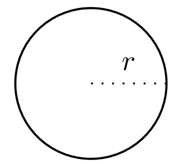
   Solution : First, set $b_1 = \left(\frac{3}{4}\right)$, $b_2 = \left(\frac{2}{5}\right)$ and $h = \left(\frac{1}{3}\right)$.

   $$A = \frac{1}{2}\left(\left(\frac{3}{4}\right) + \left(\frac{2}{5}\right)\right)\left(\frac{1}{3}\right).$$

   Note that the inner parentheses are not necessary in this case, so we have

   $$\begin{aligned} A =& \frac{1}{2}\left(\frac{3}{4} + \frac{2}{5}\right)\left(\frac{1}{3}\right) && \textit{Simplify parentheses first} \\ =& \frac{1}{2}\left(\frac{3 \times 5}{4 \times 5} + \frac{2 \times 4}{5 \times 4}\right)\left(\frac{1}{3}\right) && \textit{The least common denominator is } 20 \\ =& \frac{1}{2}\left(\frac{15 + 8}{20}\right)\left(\frac{1}{3}\right) \\ =& \frac{1}{2}\left(\frac{23}{20}\right)\left(\frac{1}{3}\right) = \frac{1 \times 23 \times 1}{2 \times 20 \times 3} = \frac{23}{120} \text{ sq. units.} \end{aligned}$$

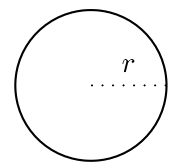
6. **Area of a circle** The area of a circle, A, with radius r is given by the formula $A = \pi r^2$. Find the area of a circle with $r = \sqrt{21}$ cm. (Keep π in your answer).

   Solution : First set $r = \left(\sqrt{21}\right)$.

   $A = \pi\left(\sqrt{21}\right)^2 = \pi(21) = 21\pi$ sq. cm.

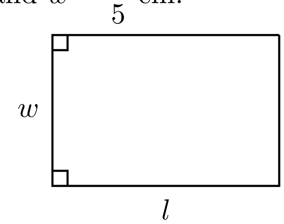
7. **Circumference of a circle** The circumference of the circle is the length of its boundary. It is denoted by C. For a circle wirh radius r, its circumference is given by the formula $C = 2\pi r$. Find the circumference of a circle with $r = \dfrac{2}{3}$ ft.

   Solution : First set $r = \left(\dfrac{2}{3}\right)$.

   $C = 2\pi\left(\dfrac{2}{3}\right) = \dfrac{2\pi}{1}\left(\dfrac{2}{3}\right) = \dfrac{2\pi \times 2}{3} = \dfrac{2 \times 2\pi}{3} = \dfrac{4\pi}{3}$ ft.

8. **Perimeter of a rectangle** The perimeter of a rectangle is the length of its boundary. It is denoted by P. The perimeter of a rectangle with length l and width w is given by the formula $P = 2l + 2w$. Find the perimeter of a rectangle with $l = \dfrac{2}{3}$ cm. and $w = \dfrac{4}{5}$ cm.

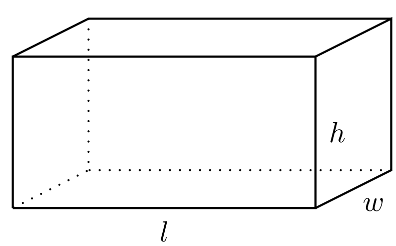
   Solution : First set $l = \left(\dfrac{2}{3}\right)$ and $w = \left(\dfrac{4}{5}\right)$.

   $P = 2\left(\dfrac{2}{3}\right) + 2\left(\dfrac{4}{5}\right) = \dfrac{2}{1}\left(\dfrac{2}{3}\right) + \dfrac{2}{1}\left(\dfrac{4}{5}\right) = \dfrac{4}{3} + \dfrac{8}{5}$

   $= \dfrac{4 \times 5}{3 \times 5} + \dfrac{8 \times 3}{5 \times 3} = \dfrac{20}{15} + \dfrac{24}{15} = \dfrac{44}{15}$ cm.

9. **Volume of a rectangular box** The volume of a rectangular box, V, with length l, width w, and height h is given by the formula $V = lwh$. Find the volume of a rectangular box with $l = 0.5$ cm., $w = 0.3$ cm., and $h = 1.7$ cm.

   Solution : First set $l = (0.5)$, $w = (0.3)$, and $h = (1.7)$.

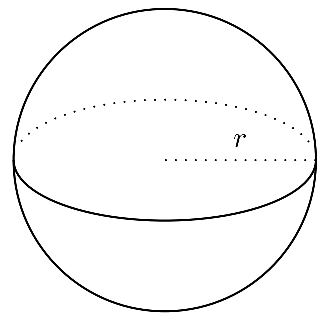
   $V = (0.5)(0.3)(1.7) = \left(\dfrac{5}{10}\right)\left(\dfrac{3}{10}\right)\left(\dfrac{17}{10}\right)$

   $= \dfrac{5 \times 3 \times 17}{1000} = \dfrac{255}{1000} = 0.255$ cubic cm.

10. **Volume of a sphere** The volume of a sphere, V, with radius r is given by the formula $V = \frac{4}{3}\pi r^3$. Find the volume of a sphere with $r = \frac{2}{3}$ cm.

Solution : First set $r = \left(\frac{2}{3}\right)$.

$V = \frac{4}{3}\pi\left(\frac{2}{3}\right)^3 = \frac{4}{3} \times \frac{\pi}{1} \times \frac{8}{27} = \frac{4 \times \pi \times 8}{3 \times 1 \times 27} = \frac{32\pi}{81}$ cubic cm.



Classroom Exercises :

(a) Evaluate $x^3 + 3x^2 + 3x + 1$ for $x = -2$.

(b) Find the slope of a line passing through points $(-2, 3)$ and $(-5, 0)$.

(c) Find the area of a circle with radius $\frac{1}{\sqrt{3}}$ cm.

(d) Find the area of a trapezoid with $b_1 = \frac{2}{3}, b_2 = \frac{4}{5}$ and $h = \frac{1}{4}$.

(e) Find the simple interest when \$ 4,000 is invested at the rate of 5% for 4 years.

2.2.1 Homework Exercises

1. Evaluate $\frac{a^2 - 2ab + b^2}{(a-b)^2}$ for $a = -3$ and $b = -5$.

2. Evaluate $\left((a+b)(a-b) - (a^2 - b^2)\right)^3$ for $a = 4$ and $b = -7$.

3. Evaluate $\sqrt{a^2 + b^2}$ for $a = -9$ and $b = 5$.

4. Evaluate $\sqrt[3]{|a-b| + a^2b^3}$ when $a = 2$ and $b = 5$.

5. Evaluate $(a+b)^3 - a^3 - b^3$ for $a = -2$ and $b = -4$.

6. Find $f(2)$ when $f(x) = 3x^2 + 2x - 5$.

7. Find $f(-2)$ when $f(x) = -3x^2 + 5$.

8. Find $f(0)$ when $f(x) = 2x^{10} - 11x^9 + 77$.

9. Find $f(1)$ when $f(x) = 4x^3 + 5x^2 - 6x + 7$.

10. Find the simple interest when $P = \$12,000$, $r = 0.04$, and $t = 3$ years.

11. Find the slope of a line passing through the points $(-10, 5)$ and $(12, 5)$.

12. Find the area of a rectangle with $l = \sqrt{12}$ ft. and $w = \sqrt{50}$ ft.

13. Find the area of a triangle with $b = \dfrac{3}{2}$ cm. and $h = \dfrac{12}{13}$ cm.

14. Find the area of a trapezoid with $b_1 = \dfrac{3}{13}$ in., $b_2 = \dfrac{1}{26}$ in., and $h = \dfrac{7}{2}$ in.

15. Find the area of a circle with $r = 0.7$ cm.

16. Find the circumference of a circle with $r = 2.8$ cm.

17. Find the perimeter of a rectangle with $l = \dfrac{12}{5}$ ft. and $w = \dfrac{2}{15}$ ft.

18. Find the volume of a rectangular box with $l = \sqrt{8}$ cm., $w = \sqrt{75}$ cm., and $h = 10$ cm.

19. Find the volume of a sphere with $r = \dfrac{3}{5}$ ft.

Chapter 3

Linear Equations

3.1 Solving Linear Equations

A **linear equation** is an equation in which each term is either a constant, or a product of a constant and a variable raised to the first power.
For example, here are a few linear equations:

$3x + 5 = 2$	*Linear equation in one variable, x.*
$3x + 5y - 3 = 2x + 7z$	*Linear equation in three variables, x, y, and z.*
$ax + b = c$	*Linear equation in one variable, x, with* **constants** *a, b, and c;*
	that is, a, b, c are to be thought of some fixed numbers.

Here are a few **nonlinear** equations:

$3x^2 + 5 = 2$	*The variable x has exponent 2.*
$3xy - 3 = 2x + 7z$	*Here, the term $3xy$ has degree 2.*
$3\sqrt{x} + 6 = 2x$	*The term $3\sqrt{x}$ can be written as $3x^{\frac{1}{2}}$. The exponent of x is therefore $\frac{1}{2}$.*
$\dfrac{4}{x} - 7\sqrt{y} = 2$	*The term $\dfrac{4}{x}$ can be written as $4x^{-1}$, which is not a linear term.*
	Also, the term $-7\sqrt{y}$ can be written as $-7y^{\frac{1}{2}}$, which is not a linear term.

A **solution** of a linear equation is that number (or numbers) which when substituted for the variable(s) gives a true statement.

For example, here are a few examples of solutions to linear equations:

- $x = 7$ is a solution of $-2x + 6 = -8$. Here is how we check (LHS stands for Left hand side, and RHS stands for Right hand side):

$$\begin{aligned} LHS &= -2(7) + 6 && \textit{Substitute (7) for } x; \\ &= -14 + 6 && \textit{Simplify} \\ &= -8 = RHS. && \textit{As LHS = RHS, we have a true statement.} \end{aligned}$$

- $x = -3$ is a solution of $3x + 9 = 2(x - 1) + 8$. Here is how we check:

$$\begin{aligned} LHS &= -3(-3) + 9 && \textit{Substitute } (-3) \textit{ for } x; \\ &= -9 + 9 = 0. && \textit{Simplify;} \\ RHS &= 2((-3) - 1) + 8 && \textit{Substitute } (-3) \textit{ for } x; \\ &= 2(-4) + 8 && \textit{Simplify parentheses first;} \\ &= -8 + 8 = 0. && \textit{As LHS = RHS, we have a true statement.} \end{aligned}$$

- $x = 2, y = -3$ is a solution of $-3x + 5y = -4(x - y) - 1$.

$$\begin{aligned} LHS &= -3(2) + 5(-3) && \textit{Substitute values for } x, y; \\ &= -6 - 15 = -21. && \textit{Simplify;} \\ RHS &= -4((2) - (-3)) - 1 && \textit{Substitute values for } x, y; \\ &= -4(2 + 3) - 1 && \textit{Simplify parentheses first;} \\ &= -4(5) - 1 = -20 - 1 = -21. && \textit{As LHS = RHS, we have a true statement.} \end{aligned}$$

Here are a few examples of numbers which are **not solutions** of the given equations.

- $x = -5$ is not a solution of $2x + 7 = 3x + 6$. Here is how we check:

$$\begin{aligned} LHS &= 2(-5) + 7 && \textit{Substitute the given value for } x; \\ &= -10 + 7 = -3 && \textit{Simplify;} \\ RHS &= 3(-5) + 6 && \textit{Substitute the given value for } x; \\ &= -15 + 6 = -9. && \textit{As } LHS \neq RHS, \textit{ we have a } \textbf{false} \textit{ statement.} \end{aligned}$$

- The pair $x = 2, y = 3$ is not a solution of $2x + 8y = 9$.

$$\begin{aligned} LHS &= 2(2) + 8(3) && \textit{Substitute;} \\ &= 4 + 24 = 28 \neq 9. && \textit{As } LHS \neq RHS, \textit{ we have a } \textbf{false} \textit{ statement.} \end{aligned}$$

Classroom Exercises : Check whether the given numbers are solutions to the given equations.

(a) $x = -5$. Equation is: $-2x + 11 = 3(x - 2)$.

(b) $x = 4$. Equation is: $2x + 3 = 7x$.

(c) $x = 3, y = -4$. Equation is: $3x + y = -x - y$

(d) $x = 4, y = -2$. Equation is: $2y - x = -3x - 2y$

To **solve an equation** is to find soltuion(s) of the given equation. The rest of this section is devoted to solving linear equations in one variable, if such a solution exists.
Remark: (1) Some equations may not have any solutions. Here are examples of equations with no solutions:

- $x + 2 = x + 1$. Note that no matter what value we substitute for x, there is no way that $x + 2$ could be equal to $x + 1$. This equation therefore has no solutions.

- $2y - 3 = 5 + 2y$. You can check for yourself that no such number y could exist.

(2) Some equations may have every real number as a solution. Here are some examples:

- $3(x + 3) = 3x + 9$. Every real number substituted in this equation gives a true statement.

- $2 - 3x = -3x + 2$.

(3) Two equations are said to be **equivalent** if they have the same set of solutions. For instance, the equations $3x = 6$ and $3x + 7 = 13$ are equivalent equations. Indeed, note that $x = 2$ is the only solution for both of these equations. So, while solving an equation, we go through various steps, each resulting in an equivalent equation.

3.1.1 Addition property

The **inverse** of addition is subtraction. Likewise, the **inverse** of subtraction is addition. Therefore, to undo addition, we subtract. And, to undo subtraction, we add. Keeping this in mind, we can solve certain linear equations. Think of a linear equation as a **balance.**

Addition property for an equation

When the same number is added to or subtracted from the two sides of an equation, the result is an equivalent equation.

We now solve some equations.

- $$\begin{array}{rl} x+8=-9 & \textit{To isolate } x, \textit{ undo } +8; \\ \underline{-8=-8} & \textit{Subtract 8 from both sides.} \\ x=-17 & \textit{The solution.} \end{array}$$

- $$\begin{array}{rl} x-11=2 & \textit{To isolate } x, \textit{ undo } -11; \\ \underline{+11=+11} & \textit{Add 11 to both sides.} \\ x=13 & \textit{The solution.} \end{array}$$

- $$\begin{array}{rl} 6=x-9 & \textit{To isolate } x, \textit{ undo } -9; \\ \underline{+9= \quad +9;} & \textit{Add 9 to both sides.} \\ 15=x & \textit{The solution.} \end{array}$$

- $$\begin{array}{rl} -12=10+x & \textit{To isolate } x, \textit{ undo 10 which is } +10; \\ \underline{-10=-10} & \textit{Subtract 10 from both sides.} \\ -22=x & \textit{The solution.} \end{array}$$

- $$\begin{array}{ll} x+\frac{8}{3}=-\frac{9}{2} & \textit{To isolate } x, \textit{ undo } +\frac{8}{3}; \\ \underline{-\frac{8}{3}=-\frac{8}{3}} & \textit{Subtract } \frac{8}{3} \textit{ from both sides.} \\ x \quad =-\frac{9}{2}-\frac{8}{3} & \\ x \quad =-\frac{9\times 3}{2\times 3}-\frac{8\times 2}{3\times 2} & \textit{The least common denominator is } 6; \\ x \quad =\frac{-27-16}{6}=-\frac{43}{6} & \textit{The solution.} \end{array}$$

-

$$\begin{aligned}
-\frac{5}{3} &= \frac{9}{4} + x && \textit{To isolate } x, \textit{ undo } +\frac{9}{4};\\
-\frac{9}{4} &= -\frac{9}{4} && \textit{Subtract } \frac{9}{4} \textit{ from both sides.}\\
\hline
-\frac{5}{3} - \frac{9}{4} &= x\\
-\frac{5\times 4}{3\times 4} - \frac{9\times 3}{4\times 3} &= x && \textit{The least common denominator is } 12;\\
\frac{-20-27}{12} &= x\\
-\frac{47}{12} &= x && \textit{The solution.}
\end{aligned}$$

-

$$\begin{aligned}
x - \sqrt{11} &= 3\sqrt{11} && \textit{To isolate } x, \textit{ undo } -\sqrt{11};\\
+\sqrt{11} &= +\sqrt{11} && \textit{Add } \sqrt{11} \textit{ to both sides.}\\
\hline
x &= 4\sqrt{11} && \textit{The solution.}
\end{aligned}$$

-

$$\begin{aligned}
-\frac{2}{\sqrt{10}} &= \frac{3}{\sqrt{10}} + x && \textit{To isolate } x, \textit{ undo } +\frac{3}{\sqrt{10}};\\
-\frac{3}{\sqrt{10}} &= -\frac{3}{\sqrt{10}} && \textit{Subtract } \frac{3}{\sqrt{10}} \textit{ from both sides.}\\
-\frac{5}{\sqrt{10}} &= +x\\
-\frac{5\times\sqrt{10}}{\sqrt{10}\times\sqrt{10}} &= +x && \textit{Rationalize the denominator.}\\
-\frac{\cancelto{1}{5}\sqrt{10}}{\cancelto{2}{10}} &= x\\
-\frac{\sqrt{10}}{2} &= x && \textit{The solution.}
\end{aligned}$$

Classroom Exercises : Solve the following equations:

(a) $x + 22 = -12$

(b) $-\frac{1}{3} = x - \frac{2}{5}$

(c) $10 = 43 + x$

(d) $x + 5 = 5$

(e) $x + \sqrt{3} = 4$

(f) $2\pi = 3\pi + x$

(g) $x - \frac{2}{\sqrt{5}} = \frac{7}{\sqrt{5}}$

3.1.2 Multiplication property

The **inverse** of multiplication is division. Likewise, the **inverse** of division is multiplication. Therefore, to undo multiplication, we divide. And, to undo division, we multiply. Keeping this in mind, we can solve certain linear equations. Thinking of a linear equation as a **balance,** we have the

Multiplication property for an equation
When the same nonzero number is multiplied to the two sides of an equation, the result is an equivalent equation.
When the two sides of an equation are divided by the same nonzero number, the result is an equivalent equation.

We now solve some equations.

- $$\begin{aligned} 6x &= 9 && \textit{To isolate } x \textit{ undo multiplication by } 6; \\ \frac{6x}{6} &= \frac{9}{6} && \textit{Divide both sides by } 6; \\ \frac{\cancelto{1}{6}x}{\cancelto{1}{6}} &= \frac{\cancelto{3}{9}}{\cancelto{2}{6}} && \textit{Simplify;} \\ x &= \frac{3}{2} && \textit{The solution.} \end{aligned}$$

- $$\begin{aligned}
\frac{x}{4} &= 12 && \textit{To isolate } x \textit{ undo division by } 4;\\
\frac{x}{4}\times 4 &= 12\times 4 && \textit{Multiply both sides by } 4;\\
\frac{x\times \cancelto{1}{4}}{\cancelto{1}{4}} &= 12 &&\\
x &= 48 && \textit{The solution.}
\end{aligned}$$

- $$\begin{aligned}
\frac{2}{3} &= \frac{4x}{5} && \textit{To isolate } x \textit{ undo multiplication by } 4 \textit{ and division by } 5;\\
\frac{2}{3}\times\frac{5}{4} &= \frac{4x}{5}\times\frac{5}{4} && \textit{Multiply both sides by } \frac{5}{4};\\
\frac{2\times 5}{3\times 4} &= x &&\\
\frac{\cancelto{1}{2}\times 5}{3\times \cancelto{2}{4}} &= x &&\\
\frac{5}{6} &= x && \textit{The solution.}
\end{aligned}$$

- $$\begin{aligned}
\sqrt{3}x &= 5 && \textit{To isolate } x \textit{ undo multiplication by } \sqrt{3};\\
\frac{\sqrt{3}x}{\sqrt{3}} &= \frac{5}{\sqrt{3}} && \textit{Divide both sides by } \sqrt{3};\\
\frac{\cancelto{1}{\sqrt{3}}x}{\cancelto{1}{\sqrt{3}}} &= \frac{5}{\sqrt{3}} && \textit{Simplify;}\\
x &= \frac{5\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}} && \textit{Rationalize the denominator;}\\
x &= \frac{5\sqrt{3}}{3} && \textit{The solution.}
\end{aligned}$$

•

$$\begin{aligned}
-\frac{3}{7}x &= \frac{2}{3} && \textit{To isolate } x \textit{ undo multiplication by } -3 \textit{ and division by } 7;\\
-\frac{3}{7} \times \frac{x}{1} &= \frac{2}{3} && \textit{Rewrite the problem;}\\
\frac{-3x}{7} \times \frac{7}{-3} &= \frac{2}{3} \times \frac{7}{-3} && \textit{Multiply both sides by } \frac{7}{-3};\\
\frac{-3x \times 7}{7 \times -3} &= \frac{2 \times 7}{3 \times -3} && \\
\frac{\cancel{-3}^{1}x \times \cancel{7}^{1}}{\cancel{7}^{1} \times \cancel{-3}^{1}} &= \frac{14}{-9} && \textit{Simplify;}\\
x &= -\frac{14}{9} && \textit{The solution.}
\end{aligned}$$

- Do not let such a problem trouble you.

$$\begin{aligned}
-x &= 4 && \textit{Realize that } -x = -1 \times x;\\
\frac{-1 \times x}{-1} &= \frac{4}{-1} && \textit{Divide both sides by } -1;\\
x &= -4 && \textit{The solution.}
\end{aligned}$$

Classroom Exercises : Solve the following equations:

(a) $3x = 0$ (What is $0 \div 3$?)

(b) $\frac{x}{8} = \frac{2}{3}$

(c) $-x = 5$

(d) $-\frac{7}{6} = \frac{3}{4}x$

(e) $\sqrt{5} = 2x$

(f) $\frac{\sqrt{3}x}{3} = \frac{2}{5}$ (Do not forget to rationalize the denominator)

(g) $\frac{2x}{7} = \frac{2}{7}$

(h) $-\frac{2x}{7} = \frac{3}{8}$

3.1.3 Combining rules

Now we are ready to combine the rules for addition and multiplication. To isolate x from an expression such as $3x+5$, we first need to isolate $3x$. That is, we use the addition/subtraction property before the multiplication/division property for isolating the variable.

- $$\begin{aligned}
3x + 5 &= 12 && \textit{First undo } +5 \textit{ and then undo } \times 3;\\
-5 &= -5 && \textit{Subtract } 5 \textit{ from both sides;}\\
3x &= 7 &&\\
\frac{3x}{3} &= \frac{7}{3} && \textit{Divide both sides by } 3 \textit{ and note that } \frac{\cancel{3}^{1}x}{\cancel{3}_{1}} = x;\\
x &= \frac{7}{3} && \textit{The solution.}
\end{aligned}$$

- $$\begin{aligned}
-\frac{3x}{4} + 7 &= 24 && \textit{First undo } +7 \textit{ and then undo } \times -\frac{3}{4};\\
-7 &= -7 && \textit{Subtract } 7 \textit{ from both sides;}\\
-\frac{3x}{4} &= 14 &&\\
\frac{4}{-3} \times \frac{-3x}{4} &= \frac{4}{-3} \times 14 && \textit{Multiply both sides by } \frac{4}{-3};\\
x &= -\frac{16}{3} && \textit{The solution.}
\end{aligned}$$

- When we encounter fractions, it is sometimes expedient to first eliminate the denominators by multiplying the equation with the least common denominator. Here is an example.

$$\begin{aligned}
\frac{2x}{3} - \frac{9}{5} &= \frac{5}{3} && \textit{The least common denominator is } 15;\\
15 \times \left(\frac{2x}{3} - \frac{9}{5}\right) &= 15 \times \left(\frac{5}{3}\right) && \textit{Multiply both sides by } 15;\\
\frac{15}{1} \times \frac{2x}{3} - \frac{15}{1} \times \frac{9}{5} &= \frac{15}{1} \times \frac{5}{3} && \textit{Distribute the multiplication;}\\
\frac{\cancel{15}^{5} \times 2x}{1 \times \cancel{3}^{1}} - \frac{\cancel{15}^{3} \times 9}{1 \times \cancel{5}^{1}} &= \frac{\cancel{15}^{5} \times 5}{1 \times \cancel{3}^{1}} && \textit{Simplify;}\\
10x - 27 &= 25 && \textit{Now, first undo } -27;\\
+27 &= +27 &&
\end{aligned}$$

$$10x = 52 \qquad \textit{and then undo } \times 10;$$

$$x = \frac{10x}{10} = \frac{52}{10} = \frac{\cancel{52}^{26}}{\cancel{10}^{5}} = \frac{26}{5} \qquad \textit{The solution.}$$

- When the variable appears in both sides of the equality sign, then we need to collect the terms with the variable to one side, and the constant terms to the other. Here is an example.

$$\begin{array}{rcll}
4x+7 &=& 2x-8 & \\
-2x \quad &=& -2x & \textit{Subtract } 2x \textit{ from both sides;} \\
\hline
2x+7 &=& -8 & \\
-7 &=& -7 & \textit{Subtract } 7 \textit{ from both sides;} \\
\hline
2x &=& -15 & \\
\frac{2x}{2} &=& \frac{-15}{2} & \textit{Divide both sides by } 2; \\
x &=& -\frac{15}{2} & \textit{The solution.}
\end{array}$$

- Now combine all the rules. First eliminate denominators. Then collect the terms with the variable to one side. Now use the properties of addition and multiplication to solve the equation.

$$\frac{3x}{8} + \frac{9}{4} = -\frac{2x}{3} + \frac{8}{12} \qquad \textit{The least common denominator is } 24.$$

$$24 \times \left(\frac{3x}{8} + \frac{9}{4}\right) = 24 \times \left(-\frac{2x}{3} + \frac{8}{12}\right) \qquad \textit{Multiply both sides by } 24;$$

$$\frac{24}{1} \times \frac{3x}{8} + \frac{24}{1} \times \frac{9}{4} = \frac{24}{1} \times -\frac{2x}{3} + \frac{24}{1} \times \frac{8}{12} \qquad \textit{Distribute the multiplication.}$$

$$\frac{\cancel{24}^{3} \times 3x}{1 \times \cancel{8}^{1}} + \frac{\cancel{24}^{6} \times 9}{1 \times \cancel{4}^{1}} = \frac{\cancel{24}^{8} \times -2x}{1 \times \cancel{3}^{1}} + \frac{\cancel{24}^{2} \times 8}{1 \times \cancel{12}^{1}} \qquad \textit{Simplify;}$$

$$\begin{array}{rcrcll}
9x &+& 54 &=& -16x + 16 & \\
+16x & & &=& +16x & \textit{Add } 16x \textit{ to both sides;} \\
\hline
25x &+& 54 &=& 16 & \\
25x &+& 54 &=& 16 & \\
 & & -54 &=& -54 & \textit{Subtract } 54 \textit{ from both sides;} \\
\hline
 & & 25x &=& -38 &
\end{array}$$

$$x = \frac{25x}{25} = \frac{-38}{25} = -1\frac{13}{25} \qquad \textit{Divide both sides by } 25 \textit{ to get the final answer.}$$

- Here is a tricky one. **There are no solutions to this equation.** No real number can satisfy the following equation.

$$\begin{array}{rcll} 5x+4 &=& 5x-9 & \\ -5x & = & -5x & \textit{Subtract } 5x \textit{ from both sides;} \\ \hline 4 &=& -9. & \textit{This is a } \textbf{false} \textit{ statement.} \end{array}$$

  When the variables cancel, and we have a false statement, then there is no solution to the given equation.

- Here is another tricky case.

$$\begin{array}{rcll} \frac{2x}{3}+8 &=& \frac{2x}{3}+8 & \\ -\frac{2x}{3} &=& -\frac{2x}{3} & \textit{Subtract } \frac{2x}{3} \textit{ from both sides;} \\ \hline 8 &=& 8. & \textit{This is a } \textbf{true} \textit{ statement.} \end{array}$$

  Every real number is a solution of the given equation. When the variables cancel and we have a true statement, then every real number is a solution of the given equation.

Classroom Exercises: Solve the following equations:

(a) $5x-8=7$

(b) $-x+9=6$

(c) $\dfrac{2x}{5}+7=7$

(d) $9x-8=5x+7$

(e) $\dfrac{7x}{4}-\dfrac{2}{3}=-\dfrac{3x}{2}+\dfrac{1}{6}$

(f) $\dfrac{2x}{5}-\dfrac{3}{10}=\dfrac{1}{2}+\dfrac{2x}{5}$

(g) $\dfrac{3x}{4}-\dfrac{3}{10}=-\dfrac{3}{10}+\dfrac{6x}{8}$

(h) $2x-8=2x+5$

3.1.4 Simplify the two sides and then solve

To solve any linear equation, we first simplify the two sides separately to get a simplified equivalent equation and then proceed. Here are some examples.

- $$\begin{aligned}
3(4-(x+3)) &= 8(x+7)+6(3-(2-x)) && \textit{Simplify the two sides;}\\
3(4-x-3) &= 8(x+7)+6(3-2+x) && \textit{Simplify parentheses first;}\\
3(1-x) &= 8(x+7)+6(1+x) && \\
3-3x &= 8x+56+6+6x && \textit{Distribute the multiplication;}\\
3-3x &= 14x+62 && \textit{Combine like terms. Now simplification is over.}\\
\underline{-14x} &\underline{= -14x} && \textit{Subtract } 14x \textit{ from both sides;}\\
3-17x &= 62 && \\
\underline{-3} &\underline{= -3} && \textit{Subtract } 3 \textit{ from both sides;}\\
-17x &= 59 && \\
\frac{-17x}{-17} &= \frac{59}{-17} && \textit{Divide both sides by } -17;\\
x &= -\frac{59}{17} && \textit{The solution.}
\end{aligned}$$

- $$\begin{aligned}
\frac{3x+5}{2}+\frac{8}{3} &= \frac{7x-2}{6}+2 && \textit{Since a fraction bar is a grouping symbol,}\\
\frac{(3x+5)}{2}+\frac{8}{3} &= \frac{(7x-2)}{6}+2 && \textit{place parentheses around the numerators;}\\
6\times\left(\frac{(3x+5)}{2}+\frac{8}{3}\right) &= 6\times\left(\frac{(7x-2)}{6}+2\right) && \textit{The least common denominator is } 6;\\
\frac{6}{1}\times\frac{(3x+5)}{2}+\frac{6}{1}\times\frac{8}{3} &= \frac{6}{1}\times\frac{(7x-2)}{6}+\frac{6}{1}\times\frac{2}{1} && \textit{Distribute the multiplication;}\\
\frac{6\times(3x+5)}{1\times 2}+\frac{6\times 8}{1\times 3} &= \frac{6\times(7x-2)}{1\times 6}+\frac{6\times 2}{1\times 1} && \\
\frac{\overset{3}{\cancel{6}}\times(3x+5)}{1\times \underset{1}{\cancel{2}}}+\frac{\overset{2}{\cancel{6}}\times 8}{1\times \underset{1}{\cancel{3}}} &= \frac{\overset{1}{\cancel{6}}\times(7x-2)}{1\times \underset{1}{\cancel{6}}}+\frac{6\times 2}{1\times 1} && \textit{Simplify;}\\
3(3x+5)+16 &= (7x-2)+12 && \\
9x+15+16 &= 7x-2+12 && \textit{Distribute the multiplication;}\\
9x+31 &= 7x+10 && \textit{Combine like terms;}\\
\underline{-7x} &\underline{= -7x} && \textit{Subtract } 7x \textit{ from both sides;}
\end{aligned}$$

$$\begin{aligned}
2x + 31 &= 10 \\
2x + 31 &= 10 \\
\underline{\quad -31} &\underline{= -31} \\
\frac{2x}{2} &= \frac{-21}{2} \\
x &= -\frac{21}{2} && \textit{The solution.}
\end{aligned}$$

- Here is yet another tricky one:

$$\begin{aligned}
4(x+4) &= 3 - (7 - 4(x+5)) && \textit{Simplify the two sides;} \\
4x + 16 &= 3 - (7 - 4x - 20) && \textit{Simplify parentheses first;} \\
4x + 16 &= 3 - (-4x - 13) \\
4x + 16 &= 3 + 4x + 13 && \textit{Distribute the sign;} \\
4x + 16 &= 4x + 16 \\
\underline{-4x \qquad} &\underline{= -4x \qquad} && \textit{Subtract } 4x \textit{ from both sides;} \\
16 &= 16 && \textit{This is a true statement, independent of the variable.}
\end{aligned}$$

Every real number is a solution of the given equation.

Classroom Exercises : Solve the following equations:

(a) $2(3 - 4(5 - x)) = 7 - (3 + x)$

(b) $\dfrac{2x - 3}{5} + \dfrac{2}{3} = \dfrac{3 - x}{6}$

(c) $\dfrac{3x - 2}{4} + \dfrac{1}{2} = \dfrac{3x}{4}$

(d) $\dfrac{3x - 2}{4} + \dfrac{1}{3} = \dfrac{3x}{4} + \dfrac{1}{2}$

Summary: The steps in solving linear equations are:

Step 1 Clear the denominators by multiplying both sides of the equation by the least common denominator.

Step 2 Simplify both sides separately:

- Remove grouping symbols
- Combine like terms

Step 3 Get all the x-terms on the same side of the equation, and get the constant (no x) terms on the other. At this point, you have three cases:

- You have $ax = b$ for some real numbers a, b with $a \neq 0$. Divide both sides by a to solve for x.
- The x-terms cancel and you have a true statement. Here, every real number is a solution of the original equation.
- The x-terms cancel and you have a false statement. Here, the original equation has no solutions.

3.1.5 Homework Exercises

Check whether the given numbers are solutions of the given equations.

1. $x = -8$. Equation is $2x + 5 = -3 + x$
2. $x = 7$. Equation is $2 + 3x = 5x - (x + 6)$
3. $x = -9$. Equation is $3x + 7 = -10 - 2x$
4. $x = 2, y = 7$. Equation is $8x + y = 4(y - x) + 3$
5. $x = 1, y = 9$. Equation is $2x + y = 7x - y$

Solve the following linear equations.

6. $x + 12 = 15$
7. $x - 9 = -8$
8. $x + 17 = 9$
9. $23 = x - 12$
10. $12 = 2 + x$
11. $x + \dfrac{1}{3} = \dfrac{2}{5}$
12. $-\dfrac{3}{7} = x - \dfrac{2}{9}$
13. $x - \sqrt{10} = 5\sqrt{10}$
14. $x + \dfrac{1}{\sqrt{3}} = \dfrac{7}{\sqrt{3}}$ (Do not forget to rationalize the denominator)

15. $\dfrac{2}{\sqrt{5}} = \dfrac{3}{\sqrt{5}} + x$ (Do not forget to rationalize the denominator)

16. $12x = 19$

17. $-29 = 7x$

18. $-x = 55$

19. $\dfrac{3}{4} = -\dfrac{1}{9}x$

20. $345x = 0$

21. $\sqrt{19}x = 12$ (Do not forget to rationalize the denominator)

22. $\dfrac{4}{5} = -\dfrac{\sqrt{11}x}{3}$ (Do not forget to rationalize the denominator)

23. $\dfrac{2x}{7} = \dfrac{8}{49}$

24. $\dfrac{7}{2}x = \dfrac{9}{5}$

25. $-\dfrac{12x}{7} = \dfrac{9}{4}$

26. $-9x + 3 = 10$

27. $2x - 9 = -4x + 7$

28. $5 + 3x = 8 - 4x$

29. $8 + 9x = 3 + 9x$

30. $\dfrac{8x}{7} + \dfrac{5}{3} = \dfrac{7}{21}$

31. $\dfrac{8x}{3} + \dfrac{5}{3} = \dfrac{5}{3}$

32. $\dfrac{4}{9} + \dfrac{6x}{5} = 8x$

33. $\dfrac{5}{3} + \dfrac{9x}{2} = \dfrac{9}{2} + \dfrac{5x}{3}$

34. $4 + 7(x - 5(2 - x)) = 2(4x - 3)$

35. $(5 + 3x) - 8(2 + x) = -5(x + 7)$

36. $7 - 6(x + 3(x - 5)) = x + 5(2 - x)$

37. $\dfrac{x+5}{4} = \dfrac{6x}{5}$

38. $\dfrac{3+5x}{2} - \dfrac{9}{5} = 2x + 7$

39. $\dfrac{3x+5}{3} - \dfrac{6}{5} = \dfrac{2x+3}{3}$

40. $\dfrac{2x-3}{5} + \dfrac{7}{8} = \dfrac{8+2x}{5} - \dfrac{9}{4}$

3.2 Transition to algebra

3.2.1 Translating phrases

Numerical questions that we encounter in day-to-day life are usually solved using algebra. Colloquially, we call such questions, "word problems." An accurate transition to algebra is the key to solving such problems. The best way to learn to translate a numerical phrase or a statement in English to an algebraic expression or equation is by practising several examples.
Translate the following phrases to algebraic expressions:

1. Thirty plus x.

$$\underbrace{\textit{Thirty}}_{30} \quad \underbrace{\textit{plus}}_{+} \quad \underbrace{x}_{x}$$

   Translation : $30 + x$

2. The sum of a number and 7.

   Note that we are not told which variable to use for the number. So, we choose any variable. Although, since we chose the variable, we must say what it stands for in a **complete sentence.**

   Let the number be x.

$$\underbrace{\textit{The sum of}}_{\text{x +7}} \textit{ a number and } 7$$

Translation : $x + 7$

3. The difference when 5 is subtracted from a number.

   Let the number be x.

$$\underbrace{\textit{The difference}}_{x-5}\text{ when 5 is subtracted from the number}$$

   Translation : $x - 5$

4. The sum of 23 and three times a number

   Let the number be x.

$$\underbrace{\textit{The sum of } 23}_{23 \quad +}\quad\underbrace{\textit{and three times a number}}_{3x}$$

   Translation : $23 + 3x$

5. Three times the sum of 23 and a number (Compare this phrase with the previous one.)

   Let the number be x.

$$\underbrace{\textit{Three times}}_{3 \quad \times}\quad\underbrace{\textit{the sum of } 23 \textit{ and the number}}_{(23+x)}$$

The parentheses indicates that 3 multiplies **the sum** $23 + x$. If we forget to write the parentheses, then 3 multiplies 23 alone: $3 \times 23 + x$. This is an incorrect translation.

   Translation : $3(23 + x)$

6. Four times the difference when 8 is subtracted from twice a number

   Let the number be x.

$$\underbrace{\textit{Four times}}_{4 \quad \times} \quad \underbrace{\textit{the difference when 8 is subtracted from twice a number}}_{(2x-8)}$$

Translation : $4(2x - 8)$

7. Two-fifths the difference when thrice a number is subtracted from 20.

   Let the number be x.

$$\underbrace{\textit{Two-fifths (of) the}}_{\frac{2}{5} \quad \times} \quad \underbrace{\textit{difference when thrice a number is subtracted from } 20.}_{(20-3x)}$$

Translation : $\frac{2}{5}(20 - 3x)$ or $\frac{2(20-3x)}{5}$

8. The sum of three consecutive integers.

   Let the three integers be x, $x + 1$, and $x + 2$.

$$\underbrace{\text{The sum of three consecutive integers}}_{x+(x+1)+(x+2)}$$

Translation : $x + (x + 1) + (x + 2)$

Work :

First integer	Second integer	Third integer
3	4	5
7	8	9
$\vdots$	$\vdots$	$\vdots$
x	$x+1$	$x+2$

9. The sum of three consecutive **even** integers.

   Let the three even integers be x, $x + 2$, and $x + 4$.

$$\underbrace{\text{The sum of three consecutive even integers}}_{x+(x+2)+(x+4)}$$

Translation : $x + (x + 2) + (x + 4)$

Work :

First integer	Second integer	Third integer
4	6	8
10	12	14
$\vdots$	$\vdots$	$\vdots$
x	$x+2$	$x+4$

10. The sum of three consecutive **odd** integers.

Let the three integers be x, $x+2$, and $x+4$.

$$\underbrace{\text{The sum of three consecutive odd integers}}_{x+(x+2)+(x+4)}$$

Translation : $x+(x+2)+(x+4)$

Work :

First integer	Second integer	Third integer
3	5	7
21	23	25
$\vdots$	$\vdots$	$\vdots$
x	$x+2$	$x+4$

11. The quotient of the sum of twice a number and five, and the number

    Let the number be x.

$$\underbrace{\underbrace{\text{The quotient of}}_{\text{The quotient of}}\quad\underbrace{\text{the sum of twice a number and five,}}_{(2x+5)}\quad\underbrace{\text{and the number}}_{\text{and } x}}_{\frac{(2x+5)}{x}}$$

Translation : $\dfrac{(2x+5)}{x}$

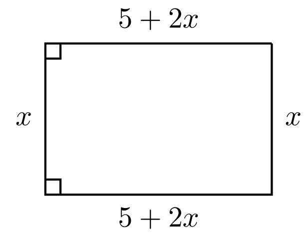
12. The sum of the quotient of twice a number and five, and the number (Do you see the difference between this phrase and the previous one?)

    Let the number be x.

$$\underbrace{\underbrace{\text{The sum of}}_{\text{The sum of}}\quad\underbrace{\text{the quotient of twice a number and five,}}_{\frac{2x}{5}}\quad\underbrace{\text{and the number}}_{\text{and } x}}_{\frac{2x}{5}+x}$$

Translation : $\dfrac{2x}{5}+x$

13. The perimeter of a rectangle whose length is five more than twice its width



Let the width of the rectangle be x

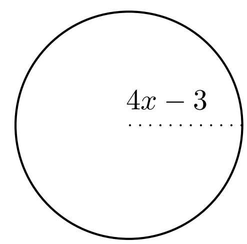
$$\underbrace{\text{The length}}_{\text{The length}} \quad \underbrace{\text{is}}_{=} \quad \underbrace{\text{five more than}}_{5\ +} \quad \underbrace{\text{twice the width.}}_{2x}$$

The perimeter $= x + (5 + 2x) + x + (5 + 2x)$.

Translation : $x + (5 + 2x) + x + (5 + 2x)$.

14. The circumference of a circle whose radius is three less than four times a number.

Let the number be x.



$$\underbrace{\text{The radius}}_{\text{The radius}} \quad \underbrace{\text{is}}_{=} \quad \underbrace{\text{three less than four times a number.}}_{4 \times x - 3}$$

The radius $= 4x - 3$

The circumference $= 2\pi\times$ the radius $= 2\pi(4x - 3)$.

Translation : $2\pi(4x - 3)$.

15. The total value of the coins when the number of nickels is six more than twice the number of dimes.

Let the number of dimes be x.

The number of nickels $\underbrace{\text{is}}_{=}\ \underbrace{\text{six more than}}_{6+}\ \underbrace{\text{twice the number of dimes.}}_{= 2x.}$

Therefore, the number of nickels $= 6 + 2x$.

Note that every nickel is worth 5¢ and every dime is worth 10¢. Therefore,

	Dimes	Nickels
Number of coins	x	$(6 + 2x)$
Value in ¢	$10x$	$5(6 + 2x)$

Hence, the total value of the coins in cents $= 10x + 5(6 + 2x)$.

Translation : $10x + 5(6 + 2x)$.

Classroom Exercises : Write algebraic expressions to translate the following phrases: (Do not forget to write in a **complete sentence** what your variable(s) stand for).

(a) The sum of a number and -9.

(b) The difference when 10 is subtracted from twice a number.

(c) Three times the difference when 7 is subtracted from a number.

(d) Four-fifths the sum of 8 and twice a number.

(e) For three consecutive integers, the sum of the first two integers, minus the third.

(f) For three consecutive even integers, the sum of first and the third.

(g) The sum of two consecutive odd integers.

(h) The quotient of the difference when 7 is subtracted from a number, and 6.

(i) The difference when a number is subtracted from the quotient of 7 and twice the number.

(j) The circumference of a circle whose radius is five times a number.

(k) The perimeter of a rectangle whose width is 7 less than three times its length.

(l) The total value of the coins when the number of quarters is two less than thrice the number of dimes.

3.2.2 Translating statements and solving problems

Firstly, while translating mathematical statements to equations, if we introduce variables, then we must describe in **complete sentences** what they stand for. Upon doing so, immediately focus on finding the correct placement of the symbol =. Here are some examples.

1. Thirty plus x is 28. Find x.

$$\text{Thirty plus } x \underbrace{\text{ is }}_{=} 28.$$

$$\underbrace{\text{Thirty}}_{30}\ \underbrace{\text{plus}}_{+}\ \underbrace{x}_{x}\ \underbrace{\text{is}}_{=}\ \underbrace{28}_{28.}.$$

$$\begin{aligned} \text{That is,} \quad 30 + x &= 28. \\ x &= 28 - 30 \\ x &= -2. \end{aligned}$$

2. The sum of a number and 7 is -25. Find the number

$$\begin{aligned}
&\text{Let the number be } x.\\
&\text{The sum of a number and 7 } \underbrace{\text{is}}_{=} -25.\\
&\underbrace{\text{The sum of a number and 7}}_{x+7}\ \underbrace{\text{is}}_{=}\ \underbrace{-25}_{-25.}.
\end{aligned}$$

$$\begin{aligned}
\text{That is,} \quad x + 7 &= -25.\\
x &= -25 - 7\\
x &= -32.
\end{aligned}$$

3. The difference when 5 is subtracted from a number is 22. Find the number.

$$\begin{aligned}
&\text{Let the number be } x.\\
&\text{The difference when 5 is subtracted from a number } \underbrace{\text{is}}_{=} 22.\\
&\underbrace{\text{The difference when 5 is subtracted from a number}}_{x-5}\ \underbrace{\text{is}}_{=}\ \underbrace{22}_{22.}.
\end{aligned}$$

$$\begin{aligned}
\text{That is,} \quad x - 5 &= 22.\\
x &= 5 + 22\\
x &= 27.
\end{aligned}$$

4. The sum of 23 and three times a number is 54. Find the number.

$$\begin{aligned}
&\text{Let the number be } x.\\
&\text{The sum of 23 and three times a number } \underbrace{\text{is}}_{=} 54.\\
&\underbrace{\text{The sum of 23 and three times a number}}_{23+3x}\ \underbrace{\text{is}}_{=}\ \underbrace{54}_{54.}.
\end{aligned}$$

$$\begin{aligned}
\text{That is,} \quad 23 + 3x &= 54.\\
3x &= 54 - 23.\\
3x &= 31.\\
\frac{3x}{3} &= \frac{31}{3}.\\
x &= \frac{31}{3}.
\end{aligned}$$

5. Three times the sum of 23 and a number is 54. Find the number. (Compare this statement with the previous one.)

Let the number be x.

Three times the sum of 23 and a number $\underbrace{\text{is}}_{=}$ 54.

$$\underbrace{\text{Three times}}_{3\times}\underbrace{\text{the sum of 23 and a number}}_{(23+x)}\underbrace{\text{is}}_{=}\underbrace{54}_{54.}.$$

$$\begin{aligned}\text{That is,}\quad 3(23+x) &= 54.\\ 69+3x &= 54.\\ 3x &= 54-69=-15.\\ x &= \frac{3x}{3}=\frac{-15}{3}.\end{aligned}$$

6. Four times the difference when 8 is subtracted from twice a number is 88. Find the number

Let the number be x.

Four times the difference when 8 is subtracted from twice a number $\underbrace{\text{is}}_{=}$ 88.

$$\underbrace{\text{Four times}}_{4\times}\underbrace{\text{the difference when 8 is subtracted from twice a number}}_{(2x-8)}\underbrace{\text{is}}_{=}\underbrace{88}_{88.}.$$

$$\begin{aligned}\text{That is,}\quad 4(2x-8) &= 88.\\ 8x-32 &= 88.\\ 8x &= 88+32=120.\\ 8x &= 120.\\ x &= \frac{8x}{8}=\frac{120}{8}=15.\end{aligned}$$

7. Two-fifths the difference when thrice a number is subtracted from 20 is 100. Find the number

Let the number be x.

Two-fifths the difference when thrice a number is subtracted from 20 $\underbrace{\text{is}}_{=}$ 100.

$$\underbrace{\text{Two-fifths the}}_{\frac{2}{5}\times}\underbrace{\text{difference when thrice a number is subtracted from 20}}_{(20-3x)}\underbrace{\text{is}}_{=}\underbrace{100}_{100.}.$$

$$\text{That is,}\quad \frac{2}{5} \times (20 - 3x) = 100.$$
$$\frac{2(20-3x)}{5} = 100.$$
$$\overset{1}{\cancel{5}} \times \frac{2(20-3x)}{\underset{1}{\cancel{5}}} = 5 \times 100.$$
$$2(20-3x) = 500.$$
$$40 - 6x = 500.$$
$$-6x = 500 - 40 = 460.$$
$$x = \frac{-6x}{-6} = \frac{\overset{230}{\cancel{460}}}{-\underset{3}{\cancel{6}}} = -\frac{230}{3}.$$

8. The sum of three consecutive integers is 450. Find the three integers.

Let the three consecutive integers be $x, (x+1)$, and $(x+2)$.

The sum of three consecutive integers $\underbrace{\text{is}}_{=}$ 450.

$$\underbrace{\text{The sum of three consecutive integers}}_{x+(x+1)+(x+2)} \underbrace{\text{is}}_{=} \underbrace{450}_{450.}.$$

$$\text{That is,}\quad x + x + 1 + x + 2 = 450.$$
$$3x + 3 = 450.$$
$$3x = 450 - 3.$$
$$3x = 447.$$
$$x = \frac{3x}{3} = \frac{447}{3} = 149.$$

The three integers are 149,150, and 151.

9. The sum of three consecutive **even** integers is 780. Find the three integers.

Let the three consecutive even integers be x, $x+2$, and $x+4$.

The sum of the three consecutive even integers $\underbrace{\text{is}}_{=}$ 780.

$$\underbrace{\text{The sum of the three consecutive even integers}}_{x+(x+2)+(x+4)} \underbrace{\text{is}}_{=} \underbrace{780}_{780.}.$$

That is, $x + x + 2 + x + 4 = 780.$

$$3x + 6 = 780.$$

$$3x = 780 - 6 = 774.$$

$$x = \frac{3x}{3} = \frac{774}{3} = 258. \qquad \textit{(This is the first integer).}$$

The three consecutive even integers are $258, 260,$ and 262.

10. The sum of three consecutive **odd** integers is 927.

Let the three consecutive odd integers be x, $x + 2$, and $x + 4$.

The sum of the three consecutive odd integers $\underbrace{\text{is}}_{=}$ 927.

$$\underbrace{\text{The sum of the three consecutive odd integers}}_{x + (x+2) + (x+4)} \underbrace{\text{is}}_{=} \underbrace{927}_{927.}.$$

That is, $x + x + 2 + x + 4 = 927.$

$$3x + 6 = 927.$$

$$3x = 927 - 6 = 921.$$

$$x = \frac{3x}{3} = \frac{921}{3} = 307. \qquad \textit{(This is the first integer).}$$

The three consecutive even integers are $307, 309,$ and 311.

11. The quotient of the sum of twice a number and five, and the number is 16. Find the number.

Let the number be x.

The quotient of the sum of twice a number and five, and the number $\underbrace{\text{is}}_{=}$ 16.

$$\underbrace{\text{The quotient of } \underbrace{\text{the sum of twice a number and five,}}_{(2x+5)} \text{ and } \underbrace{\text{the number}}_{x}}_{\frac{(2x+5)}{x}} \underbrace{\text{is}}_{=} \underbrace{16}_{16.}.$$

$$
\begin{aligned}
\text{That is,} \quad \frac{(2x+5)}{x} &= 16 && \textit{Multiply both sides by } x. \\
\cancel{x}^{1} \times \frac{(2x+5)}{\cancel{x}^{1}} &= x \times 16 \\
2x + 5 &= 16x. \\
2x - 16x + 5 &= 0. && \textit{Subtract } 16x \textit{ from both sides.} \\
-14x &= -5. && \textit{Subtract } 5 \textit{ from both sides.} \\
x = \frac{-14x}{-14} &= \frac{-5}{-14} = \frac{5}{14}.
\end{aligned}
$$

12. The sum of the quotient of twice a number and five, and the number is 16. Find the number. (Do you see the difference between this phrase and the previous one?)

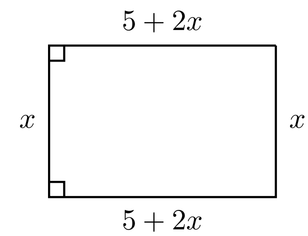
Let the number be x.

The sum of the quotient of twice a number and five, and the number $\underbrace{\text{is}}_{=}$ 16.

$$\underbrace{\text{The sum of } \underbrace{\text{the quotient of twice a number and five,}}_{\frac{2x}{5}} \text{ and } \underbrace{\text{the number}}_{x}}_{\frac{2x}{5}+x} \underbrace{\text{is}}_{=} \underbrace{16}_{16.}.$$

$$
\begin{aligned}
\text{That is,} \quad \left(\frac{2x}{5} + x\right) &= 16. && \textit{Multiply both sides by } 5. \\
5 \times \left(\frac{2x}{5} + x\right) &= 5 \times 16. \\
\frac{5}{1} \times \frac{2x}{5} + 5x &= 90. && \textit{Distribute multiplication} \\
\frac{\cancel{5}^{1} \times 2x}{1 \times \cancel{5}^{1}} + 5x &= 90. \\
2x + 5x &= 90. \\
7x &= 90. \\
x = \frac{7x}{7} &= \frac{90}{7}.
\end{aligned}
$$

13. If the perimeter of a rectangle whose length is five ft. more than twice its width is 32 sq. ft., then find the length.



Let the width of the rectangle be x

The length $\underbrace{\text{is}}_{=}$ $\underbrace{\text{five more than}}_{5+}$ $\underbrace{\text{twice the width.}}_{2x.}$

The length $= 5 + 2x$.

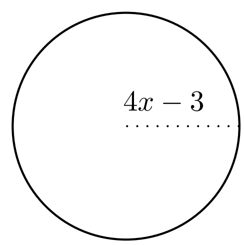
The perimeter is 32 sq. ft. That is,

$$\begin{aligned} x + 5 + 2x + x + 5 + 2x &= 32 \\ 6x + 10 &= 32 \\ 6x &= 32 - 10 = 22 \\ x = \frac{6x}{6} &= \frac{22}{6} = \frac{11}{3} = 3\frac{2}{3} \text{ ft.} \end{aligned}$$

14. The circumference of a circle whose radius is three cm. less than four times a number is 32π cm. Find the number.

Let the number be x.

The radius $\underbrace{\text{is}}_{=}$ $\underbrace{\text{three less than four times the number}}_{4x-3}$.



The radius $= 4x - 3$

The circumference $= 2\pi\times$ the radius $= 2\pi(4x-3)$. That is,

$$2\pi(4x-3) = 32\pi \qquad \textit{Divide both sides by } 2\pi$$

$$\frac{\cancelto{1}{2\pi}(4x-3)}{\cancelto{1}{2\pi}} = \frac{\cancelto{16}{32\pi}}{\cancelto{1}{2\pi}}$$

$$\begin{aligned} 4x - 3 &= 16. \\ 4x &= 16 + 3 = 19. \\ x = \frac{4x}{4} &= \frac{19}{3} \text{ cm.} \end{aligned}$$

15. John has several coins in his pocket. The number of quarters he has is one less than twice the number of nickels, and the number of dimes he has is two more than twice the number of nickels. Suppose that the total value of the coins is \$ 3.70. How many coins of each

kind does John have?

Let the number of nickels be x.

The number of quarters $\underbrace{\text{is}}_{=}$ $\underbrace{\text{one less than twice the number of nickels.}}_{2x-1}$.

The number of dimes $\underbrace{\text{is}}_{=}$ $\underbrace{\text{two more than twice the number of nickels.}}_{2x+2}$.

Therefore, the number of quarters is $2x - 1$, and the number of dimes is $2x + 2$. Note that every nickel is worth 5¢, every dime is worth 10¢, and every quarter is worth 25¢. Therefore,

	Dimes	Nickels	Quarters
Number of coins	$2x + 2$	x	$2x - 1$
Value in ¢	$10(2x + 2)$	$5x$	$25(2x - 1)$

Hence, the total value of the coins in cents $= 10(2x + 2) + 5x + 25(2x - 1)$.

$$\begin{aligned}
\text{That is,}\quad 10(2x+2)+5x+25(2x-1) &= 370 && \textit{The value is in cents.}\\
20x+20+5x+50x-25 &= 370 && \textit{Distribute multiplication.}\\
75x-5 &= 370\\
75x &= 370+5=375\\
x &= \frac{75x}{75}=\frac{375}{75}=5 && \textit{This is the number of nickels.}
\end{aligned}$$

Therefore, the number of nickels is 5, the number of quarters is 9, and the number of dimes is 12.

Classroom Exercises : Answer the following questions (Do not forget to write in a **complete sentence** what your variable(s) stand for).

(a) The sum of a number and -9 is -57. What is the number?

(b) The difference when 10 is subtracted from twice a number is 304. What is the number?

(c) Three times the difference when 7 is subtracted from a number is 423. What is the number?

(d) Four-fifths the sum of 8 and twice a number is 13. What is the number?

(e) For three consecutive integers, the sum of the first two integers, plus twice the third is 285. Find the three integers.

(f) For three consecutive even integers, the sum of the first and the third is 192. Find the second integer.

(g) The sum of two consecutive odd integers is 212. Find the larger integer.

(h) The quotient of the difference when 7 is subtracted from a number, and 6 is 42. Find the number.

(i) The difference when a number is subtracted from the quotient of twice the number and 7 is 23. Find the number.

(j) The circumference of a circle whose radius is five times a number is 50π ft. Find the number.

(k) The perimeter of a rectangle whose width is 7 cm. less than three times its length is 82 cm. Find the width.

(l) Suppose I have some nickels, dimes, and quarters. The number of nickels is twice the number of dimes, and the number of quarters is two less than three times the number of dimes. If the total value of the coins is $ 3.30, how many coins of each kind do I have?

3.2.3 Homework Exercises

Translate the following phrases to algebraic expressions. Describe your variables in **complete sentences.**

1. The sum of a number and 32.
2. The sum of a number and -10.
3. The difference when 12 is subtracted from three times a number.
4. The difference when twice a number is subtracted from 23.
5. Three times the difference when 7 is subtracted from a number.
6. Five times the sum of twice a number and 12.
7. Four-fifths the sum of 8 and twice a number.
8. Three-fourths the difference when 2 is subtracted from five times a number.
9. For three consecutive integers, the sum of the first and the last, minus the second integer.

10. Sum of two consecutive integers.
11. For three consecutive even integers, the sum of first and the second, minus the third.
12. The sum of two consecutive even integers.
13. For three consecutive odd integers, the sum of the first and the third, minus the second integer.
14. The sum of two consecutive odd integers.
15. The quotient of the sum of thrice a number and 7, and 4.
16. The quotient of the difference when 11 is subtracted from twice a number, and 24.
17. The sum of a number and the quotient of 6 and twice the number.
18. The quotient of the sum of a number and 6, and twice the number.
19. The circumference of a circle whose radius is four times a number.
20. The circumference of a circle whose radius is three less than ten times a number.
21. The perimeter of a rectangle whose width is 7 more than three times its length.
22. The perimeter of a rectangle whose length is 3 less than five times its width.
23. The total value of the coins when the number of dimes is three more than the number of nickels, and the number of nickels is four more than the number of quarters.
24. The total value of the coins when the number of quarters is twice the number of dimes, and the number of dimes is three times the number of nickels.

Answer the following. Describe your variables in **complete sentences.**

25. The sum of a number and 32 is -789. Find the number.
26. The sum of a number and -10 is 230. Find the number.
27. The difference when 12 is subtracted from three times a number is 43. Find the number.
28. The difference when twice a number is subtracted from 23 is 98. Find the number.
29. Three times the difference when 7 is subtracted from a number is 212. Find the number.
30. Five times the sum of twice a number and 12 is 67. Find the number.
31. Four-fifths the sum of 8 and twice a number is $\dfrac{3}{7}$. Find the number.

32. Three-fourths the difference when 2 is subtracted from five times a number is $\frac{12}{13}$. Find the number.

33. For three consecutive integers, the sum of the first and the last is 260. Find the second integer.

34. Sum of two consecutive integers is 425. Find the two integers.

35. For three consecutive even integers, the sum of the first and the second integers is 188. Find the third integer.

36. The sum of two consecutive even integers is -110. Find the two integers.

37. For three consecutive odd integers, the sum of the first and the third, minus the second integer is 53. Find the three integers.

38. The sum of two consecutive odd integers is -132. Find the two integers.

39. The quotient of the sum of thrice a number and 7, and 4 is $\frac{1}{12}$. Find the number.

40. The quotient of the difference when 11 is subtracted from twice a number, and 24 is 9. Find the number.

41. The sum of a number and the quotient of twice the number and 5 is 23. Find the number.

42. The quotient of the sum of a number and 6, and twice the number is 18. Find the number.

43. The circumference of a circle whose radius is four times a number is 78 inches. Find the number.

44. The circumference of a circle whose radius is three cm. less than ten times a number is 49 cm. Find the number.

45. The perimeter of a rectangle whose width is 7 ft. more than three times its length is 102 ft. Find the length.

46. The perimeter of a rectangle whose length is 3 miles less than five times its width is 98 miles. Find the length.

47. The total value of the coins when the number of dimes is three more than the number of nickels, and the number of nickels is four more than the number of quarters is \$ 1.70. How many coins of each kind are there?

48. The total value of the coins when the number of quarters is twice the number of dimes, and the number of dimes is three times the number of nickels is \$ 11.10. How many coins of each kind are there?

3.3 Literal equations

An equation that is constructed using one or more variables is called a **literal equation.** To solve a literal equation for one variable, we treat the other variables as constants (numbers). Here are some examples:

1. The slope-intercept form of a line is given by the equation $y = mx + b$ where m stands for the slope, and b stands for the Y-intercept of the line. Solve this equation for x.
Solution : Since we want to solve for x, we treat all the other variables, y, m, b, as constants.

$$\begin{array}{rl} y = mx + b & \textit{We want to isolate } x. \\ \underline{-b \qquad -b} & \textit{Subtract } b \textit{ from both sides.} \\ y - b = mx & \\ \dfrac{y-b}{m} = \dfrac{mx}{m} & \textit{Divide both sides by } m. \\ \dfrac{y-b}{m} = x & \textit{We have solved for } x. \end{array}$$

2. The area of a rectangle is given by the formula $A = lw$ where l stands for the length, and w stands for the width of the rectangle. Solve this equation for l.
Solution : Since we want to solve for l, we treat the other variables, A, w, as constants.

$$\begin{array}{ll} A = lw & \textit{To isolate } l\textit{, we need to undo multiplication by } w. \\ \dfrac{A}{w} = \dfrac{lw}{w} & \textit{Divide both sides by } w. \\ \dfrac{A}{w} = l & \textit{We have solved for } l. \end{array}$$

3. The area of a triangle is given by the formula $A = \dfrac{bh}{2}$ where b stands for the length of the base, and h stands for the height of the triangle. Solve this equation for h.
Solution : Since we want to solve for h, we treat the other variables, A, b, as constants.

$$\begin{array}{rl} A = \dfrac{bh}{2} & \textit{To isolate } h\textit{, we have to undo multiplication by } b \textit{ and division by } 2. \\ \dfrac{A}{1} \times \dfrac{2}{b} = \dfrac{bh}{2} \times \dfrac{2}{b} & \textit{Multiply both sides by } \dfrac{2}{b}. \\ \dfrac{2A}{b} = \dfrac{\cancelto{1}{b}h \times \cancelto{1}{2}}{\cancelto{1}{2} \times \cancelto{1}{b}} & \\ \dfrac{2A}{b} = h. & \textit{We have solved for } h. \end{array}$$

4. The area of a trapezoid is given by the formula $A = \dfrac{(b_1 + b_2)h}{2}$ where b_1 and b_2 are the lengths of the two parallel sides, and h is the height of the trapezoid. Solve this equation for b_1.
   Solution : Since we want to solve for b_1, we treat the other variables, A, b_2, h, as constants.

$$\begin{aligned} A &= \frac{(b_1 + b_2)h}{2} && \textit{To isolate } b_1\textit{, we first isolate } (b_1 + b_2). \\ \frac{A}{1} \times \frac{2}{h} &= \frac{(b_1 + b_2)h}{2} \times \frac{2}{h} && \textit{Multiply both sides by } \frac{2}{h}. \\ \frac{2A}{h} &= \frac{(b_1 + b_2)\cancel{h}^{1} \times \cancel{2}^{1}}{\cancel{2}^{1} \times \cancel{h}^{1}} \\ \frac{2A}{h} &= b_1 + b_2 \\ \underline{-b_2} &= \underline{\quad -b_2} && \textit{Subtract } b_2 \textit{ from both sides.} \\ \frac{2A}{h} - b_2 &= b_1 && \textit{We have solved for } b_1. \end{aligned}$$

5. The perimeter, P, of a rectangle with length l, and width w is given by the formula $P = 2l + 2w$. Solve this equation for l.
   Solution : Since we want to solve for w, we treat the other variables, P, l, as constants.

$$\begin{aligned} P &= 2l + 2w && \textit{We want to isolate } l. \\ \underline{-2w} &= \underline{\quad -2w} && \textit{Subtract } 2w \textit{ from both sides.} \\ P - 2w &= 2l \\ \frac{P - 2w}{2} &= \frac{2l}{2} && \textit{Divide both sides by } 2. \\ \frac{P - 2w}{2} &= l && \textit{We have solved for } l. \end{aligned}$$

6. The circumference of a circle is given by the formula $C = 2\pi r$, where r stands for the radius. Solve this equation for r.
   Solution : Since we want to solve for r, we treat C as a constant.

$$\begin{aligned} C &= 2\pi r && \textit{Divide both sides by } 2\pi. \\ \frac{C}{2\pi} &= r && \textit{We have solved for } r. \end{aligned}$$

7. The volume of a rectangular box with length l, width w, and height h, is given by the formula $V = lwh$. Solve this equation for h.
Solution : Since we want to solve for h, we treat V, l, w as constants.

$$\begin{aligned} V &= lwh && \textit{We want to isolate } h. \\ \frac{V}{lw} &= \frac{lwh}{lw} && \textit{Divide both sides by } lw. \\ \frac{V}{lw} &= h && \textit{We have solved for } h. \end{aligned}$$

8. If the principal P is deposited at a simple annual rate of interest of r for t number of years, the final amount A is given by the formula $A = P(1 + rt)$. Solve this equation for r.
Solution : Since we want to sovle for r, we treat the variables A, P, t as constants.

$$\begin{aligned} A &= P(1 + rt) \\ A &= P + Prt && \textit{To isolate } r, \textit{ we first isolate } Prt. \\ A - P &= Prt \\ \frac{A - P}{Pt} &= r && \textit{Dividing both sides by } Pt. \end{aligned}$$

9. If C denotes the temperature measured in Celsius and F denotes the temperature measured in Fahrenheit, then the two are related by the formula $5(F - 32) = 9C$. Solve this equation for F.
Solution : Since we want to solve for F, we treat C as a constant.

$$\begin{aligned} 5(F - 32) &= 9C \\ 5F - 160 &= 9C \\ 5F &= 9C + 160 && \textit{Add } 160 \textit{ to both sides.} \\ F &= \frac{9C + 160}{5} && \textit{Divide both sides by } 5. \\ F &= \frac{9C}{5} + \frac{160}{5} \\ F &= \frac{9C}{5} + 32. \end{aligned}$$

10. Here is an important kind of a problem which you may see in your credit-bearing courses. Solve the equation $y = \dfrac{3x - 5}{4x - 6}$ for x.

Solution : Since we want to solve for x, we treat y as a constant.

$$\begin{aligned}
y &= \frac{3x-5}{4x-6} && \textit{To eliminate the denominator,}\\
(4x-6)\times y &= (4x-6)\times\frac{3x-5}{4x-6} && \textit{multiply both sides by } (4x-6).\\
(4x-6)\times y &= \cancelto{1}{(4x-6)}\times\frac{3x-5}{\cancelto{1}{4x-6}}
\end{aligned}$$

$$\begin{aligned}
(4x-6)y &= 3x-5.\\
4xy-6y &= 3x-5 && \textit{The goal is to isolate } x\textit{. So, get the } x\textit{-terms on one side,}\\
&&& \textit{and everything else on the other side.}\\
-3x+6y &= -3x+6y && \textit{Subtract } 3x \textit{ from, and add } 6y \textit{ to both sides.}\\
\hline
4xy-3x &= 6y-5\\
x(4y-3) &= 6y-5 && \textit{To isolate } x\textit{, pull it out as a common factor.}\\
\frac{x(4y-3)}{(4y-3)} &= \frac{6y-5}{(4y-3)} && \textit{Divide both sides by } (4y-3).\\
x &= \frac{6y-5}{4y-3}. && \textit{We have solved for } x.
\end{aligned}$$

Here are some problems which require substitutions first.

1. Solve the equation $A = P(1+rt)$ for r, when $P = 2000$, $t = 8$, and $A = 4000$.
   Solution : We first substitute the given values in the equation to get

$$\begin{aligned}
4000 &= 2000(1+r\times 8). && \textit{We now want to isolate } r.\\
4000 &= 2000+16000r && \textit{Distribute the multiplication first.}\\
4000-2000 &= 2000+16000r-2000 && \textit{Subtract } 2000 \textit{ from both sides.}\\
2000 &= 16000r && .\\
\frac{2000}{16000} &= r && \textit{Divide both sides by } 16000.\\
\frac{1}{8} &= r.
\end{aligned}$$

2. The volume of a cone with base radius r, and height h, is given by $V = \dfrac{\pi r^2 h}{3}$. Solve this equation for h when $V = 300\pi\ cm^3$ and $r = 20cm$.

Solution : First substitute the values for the given variables in the formula.

$$\begin{aligned} 300\pi &= \frac{\pi(20)^2 h}{3}. && \textit{Now we solve for } h. \\ \frac{3}{\pi(20)^2} \times 300\pi &= \frac{3}{\pi(20)^2} \times \frac{\pi(20)^2 h}{3}. && \textit{Multiply both sides by } \frac{3}{\pi(20)^2} \\ \frac{3 \times 300\pi}{\pi(20)^2} &= h \\ \frac{900\pi}{\pi 400} &= h \\ \frac{9}{4} &= h. \end{aligned}$$

Classroom Exercises : Solve the following literal equations for the mentioned variable.

(a) Solve the equation $y = mx + b$ for m.

(b) Solve the equation $A = lw$ for the w.

(c) Solve the equation $A = \frac{bh}{2}$ for b.

(d) Solve the equation $A = \frac{(b_1 + b_2)h}{2}$ for b_2.

(e) Solve the equation $y = \frac{6x - 3}{x + 5}$ for x.

Here are some problems which will require you to first substitute values.

1. Suppose $y = \frac{x - 5}{2x + 3}$. Then solve for x when $y = 1$.
2. If $\frac{1}{y} = 2 + \frac{x}{3}$, then solve for x when $y = 4$.

3.3.1 Homework Exercises

Solve the literal equations for the mentioned variables.

1. $N = \frac{a + b}{c}$ for b.
2. $K = ch + dh - prh$ for h.
3. $F = \frac{9C}{5} + 32$ for C.

4. $P = 2l + 2w$ for w.

5. $V = lwh$ for l

6. $A = P(1 + rt)$ for t.

7. $S = \dfrac{k+t}{r+m}$ for k.

8. $G = \dfrac{3x+y}{a+b}$ for x.

9. $y = \dfrac{x+7}{8x-9}$ for x.

10. $x = \dfrac{2y-1}{3y+1}$ for y.

11. $N = \dfrac{a+b}{c}$ for a when $N = 3$, $b = 1$, and $c = 2$.

12. $K = ch + dh - prh$ for h when $K = 2$, $c = 1$, $d = 5$, $p = -1$, and $r = 4$.

13. $F = \dfrac{9C}{5} + 32$ for C when $F = 70$.

14. $y = \dfrac{x+7}{8x-9}$ for x when $y = 2$.

15. $x = \dfrac{2y-1}{3y+1}$ for y when $x = 5$.

3.4 Linear inequalities in one variable

3.4.1 Introduction

An **inequality** is a mathematical statement involving inequality symbols ($<, \leq, >, \geq$). A **linear inequality** is an inequality where the the two sides consist of linear polynomials. Let us understand what this means using examples. First, the meaning of the inequality symbols are given.

Symbol	Translation
$<$	*Less than*
$\leq$	*Less than or equal to*
$>$	*Greater than*
$\geq$	*Greater than or equal to*

Here are some examples of inequalities.

$xy + 5 < 3x^2 - y$ *This is not a linear inequality.*
$x^2 + y^3 \leq 4x + y$ *This is not a linear inequality.*
$x + 4y > y - x$ *This is a linear inequality in two variables, x, y.*
$4x - 6 \geq 3x + 12$ *This is a linear inequality in one variable, x.*

In our lesson we will be working with linear inequalities in one variable. A **solution set** of a linear inequality in one variable is a set of numbers each of which satisfies the given inequality. For instance, consider the inequality

$$2x \leq 6.$$

We see that 3, 2, 1, 0, -1, -2, $\cdots$ are all solutions of the inequality $2x \leq 6$. Moreover, 2.5, $2\frac{1}{3}$, $-\pi$, $\sqrt{8}$, 2.9 ,2.99, $\cdots$ are also solutions of $2x \leq 6$. In fact, we have infinitely many solutions to the inequality $2x \leq 6$. By now you have probably recognized that $2x \leq 6$ is equivalent to $x \leq 3$. All these solutions are put together in a collection called, the **set of solutions** for this particular inequality. We can describe this set in a **set-builder** notation as follows:

$\{x$ *a real number* $\mid x \leq 3\}$.

such that

The set of

So, we read $\{x$ *a real number* $\mid x \leq 3\}$ as

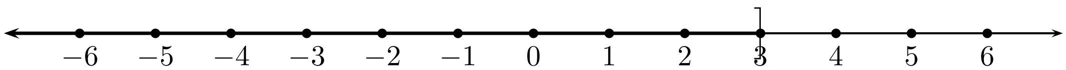
"The set of those real numbers x such that

x is less than or equal to three."

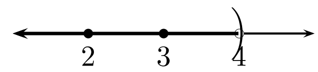
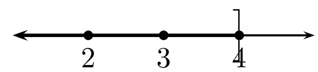
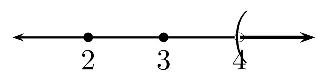
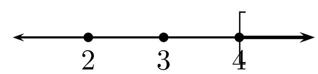
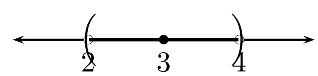
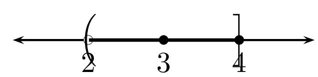
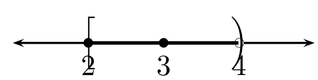
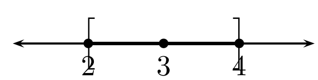
The same set can be described in an interval notation as follows:

$(-\infty, 3]$, *which is read, "Open at negative infinity to closed at three."*

The graph of this interval on the real number line is



Note that $(,)$ stand for "open" and $[,]$ stand for "closed" in interval notations. Likewise, an $\circ$ represents an "open" (or an excluded)point while $\bullet$ denotes "closed" (or an included) point on the number line. The infinities, $+\infty, -\infty$, are always excluded, or open, since they are not real numbers. Here are various situations.

The set	set-builder form	Interval form	Graph on the number line
The set of real numbers less than 4.	$\{x \text{ a real number } \mid x < 4\}$	$(-\infty, 4)$	
The set of real numbers less than or equal to 4.	$\{x \text{ a real number } \mid x \leq 4\}$	$(-\infty, 4]$	
The set of real numbers greater than 4.	$\{x \text{ a real number } \mid x > 4\}$	$(4, +\infty)$	
The set of real numbers greater than or equal to 4.	$\{x \text{ a real number } \mid x \geq 4\}$	$[4, +\infty)$	
The set of real numbers greater than 2 and less than 4.	$\{x \text{ a real number } \mid 2 < x < 4\}$	$(2, 4)$	
The set of real numbers greater than 2 and less than or equal to 4.	$\{x \text{ a real number } \mid 2 < x \leq 4\}$	$(2, 4]$	
The set of real numbers greater than or equal to 2 and less than 4.	$\{x \text{ a real number } \mid 2 \leq x < 4\}$	$[2, 4)$	
The set of real numbers greater than or equal to 2 and less than or equal to 4.	$\{x \text{ a real number } \mid 2 \leq x \leq 4\}$	$[2, 4]$	

Classroom Exercises : Describe the following in set-builder form, in interval form, and graph the set on the number line.

(a) The set of all real numbers less than 5 and greater than or equal to -3.

(b) The set of all real numbers less than or equal to -3.

(c) The set of all real numbers greater than 0.

Note that *the set of all real numbers greater than* 3 *and less than* 0 has nothing in it. Such sets are called **empty** sets. That is, an empty set is a set with no elements. It is denoted by the symbol ϕ.
Let us now understand these inequalities better.

Addition property for inequalities
When the same number is added to or subtracted from the two sides of an inequality, the inequality stays the same.

For example, we know that $3 < 5$. Adding 6 to both sides gives us $9 < 11$. The inequality stayed the same. Likewise, when we multiply or divide the two sides of an inequality by a positive real number, then the inequality stays the same. But the situation is different when we multiply or divide the two sides of an inequality by a negative real number. Observe the following:

$3 < 5$	*and when we multiply both sides by* -1,	$-3 > -5$.
$-3 < 5$	*and when we multiply both sides by* -2,	$6 > -10$.
$-15 < -3$	*and when we divide both sides by* -3,	$5 > 1$.
$0 < 5$	*and when we multiply both sides by* -6,	$0 > -30$.
$-3 < 0$	*and when we divide both sides by* -1,	$3 > 0$.

Multiplication property for inequalities
When the two sides of an inequality are multiplied or divided by the same positive real number, then the inequality stays the same.
When the two sides of an inequality are multiplied or divided by the same **negative** real number, then the inequality changes to its opposite.

3.4.2 Solving Linear Inequalities

We use these rules to solve linear inequalities in one variable. We further describe the solution set in set-builder form, interval form, and graph the solution set on the number line.

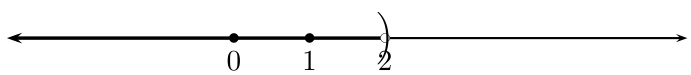
1.

$$\begin{aligned} x+6 &< 4 && \textit{To solve for } x \textit{ we undo } +6. \\ x+6-6 &< 4-6 && \textit{Subtract } 6 \textit{ from both sides.} \\ x &< -2 \end{aligned}$$

The solution set in set-builder form : $\{x$ *a real number* $\mid x < -2\}$

in interval form : $(-\infty, -2)$

The graph of the solution set on the number line :



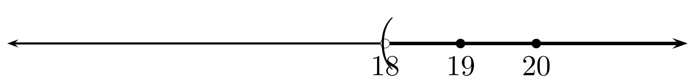
2.

$$\begin{aligned} x-11 &> 7 && \textit{To solve for } x \textit{ we undo } -11. \\ x-11+11 &< 7+11 && \textit{Add } 11 \textit{ to both sides.} \\ x &> 18. \end{aligned}$$

The solution set in set-builder form : $\{x$ *a real number* $\mid x > 18\}$

in interval form : $(18, \infty)$

The graph of the solution set on the number line :



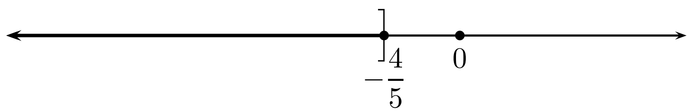
3.

$$\begin{aligned} 5x-6 &\le -10 && \textit{First undo } -6. \\ 5x-6+6 &\le -10+6 && \textit{Add } 6 \textit{ to both sides.} \\ 5x &\le -4 \\ \frac{5x}{5} &\le \frac{-4}{5} && \textit{Dividing both sides by } +5 \textit{ keeps the inequality same.} \\ x &\le -\frac{4}{5} \end{aligned}$$

The solution set in set-builder form : $\left\{x \text{ a real number } \mid x \leq -\frac{4}{5}\right\}$

in interval form : $\left(-\infty, -\frac{4}{5}\right]$

The graph of the solution set on the number line :



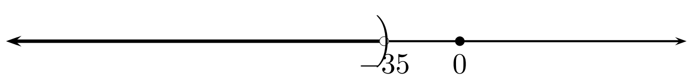
4.

$$\begin{aligned} -3x - 12 &> 93 \\ -3x - 12 + 12 &> 93 + 12 && \textit{Add 12 to both sides.} \\ -3x &> 105 \\ \frac{-3x}{-3} &< \frac{105}{-3} && \textit{Divide both sides by } -3 \\ x &< -35 && \textit{and change the inequality.} \end{aligned}$$

The solution set in set-builder form : $\{x \text{ a real number } \mid x < -35\}$

in interval form : $(-\infty, -35)$

The graph of the solution set on the number line :



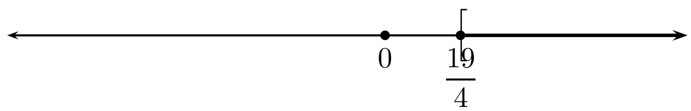
5. When the variable appears in both sides of the inequality, try to collect the terms with the variable on one side, and the constants on the other.

$$\begin{aligned} 5x + 7 &\leq 9x - 12 \\ 5x - 9x + 7 &\leq 9x - 9x - 12 && \textit{Subtract } 9x \textit{ from both sides.} \\ -4x + 7 &\leq -12 \\ -4x + 7 - 7 &\leq -12 - 7 && \textit{Subtract 7 from both sides.} \\ -4x &\leq -19 \\ \frac{-4x}{-4} &\geq \frac{-19}{-4} && \textit{Divide both sides by } -4 \\ x &\geq \frac{19}{4} && \textit{and change the inequality.} \end{aligned}$$

$$\textit{The solution set in set-builder form : } \left\{x \text{ a real number } \mid x \geq \frac{19}{4}\right\}$$

$$\textit{in interval form : } \left[\frac{19}{4}, +\infty\right)$$

The graph of the solution set on the number line :



6. Some inequalities require simplifications first.

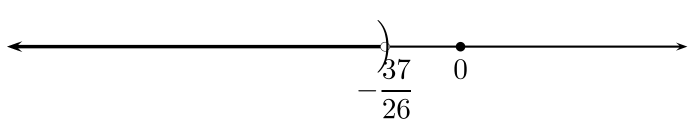
$$\begin{aligned}
3x + 5 &< \frac{x+8}{9} \\
9 \times (3x+5) &< \cancel{9}^{1} \times \frac{x+8}{\cancel{9}_{1}} && \textit{Multiply both sides by } 9. \\
27x + 45 &< x + 8 \\
27x + 45 - x &< x + 8 - x && \textit{Subtract } x \textit{ from both sides.}
\end{aligned}$$

$$\begin{aligned}
26x + 45 &< 8 \\
26x + 45 - 45 &< 8 - 45 && \textit{Subtract } 45 \textit{ from both sides.} \\
26x &< -37 \\
\frac{26x}{26} &< \frac{-37}{26} \\
x &< -\frac{37}{26} && .
\end{aligned}$$

$$\textit{The solution set in set-builder form : } \left\{x \text{ a real number } \mid x < -\frac{37}{26}\right\}$$

$$\textit{in interval form : } \left(-\infty, -\frac{37}{26}\right)$$

The graph of the solution set on the number line :



7.

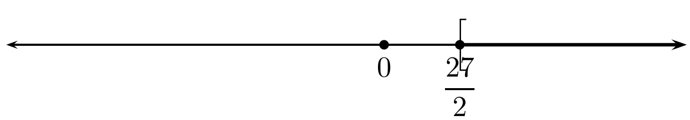
$$
\begin{aligned}
3-(2(5-2x)+x) &\geq \frac{4x+6}{3}+x \\
3-(10-4x+x) &\geq \frac{4x+6}{3}+x && \textit{Simplify parentheses first.} \\
3-(10-3x) &\geq \frac{4x+6}{3}+x \\
3-10+3x &\geq \frac{4x+6}{3}+x \\
-7+3x &\geq \frac{4x+6}{3}+x \\
3\times(-7+3x) &\geq 3\times\left(\frac{4x+6}{3}+x\right) && \textit{Multiply both sides by } 3. \\
-21+9x &\geq \cancel{3}^{1}\times\frac{4x+6}{\cancel{3}^{1}}+3x \\
-21+9x &\geq 4x+6+3x \\
-21+9x &\geq 7x+6 \\
-21+9x-7x &\geq 7x+6-7x && \textit{Subtract } 7x \textit{ from both sides.} \\
-21+2x &\geq 6 \\
-21+2x+21 &\geq 6+21 && \textit{Add } 21 \textit{ to both sides.}
\end{aligned}
$$

$$
\begin{aligned}
2x &\geq 27 \\
\frac{2x}{2} &\geq \frac{27}{2} && \textit{Divide both sides by } 2. \\
x &\geq \frac{27}{2}
\end{aligned}
$$

The solution set in set-builder form : $\left\{x \text{ a real number } \mid x \geq \frac{27}{2}\right\}$

in interval form : $\left[\frac{27}{2}, +\infty\right)$

The graph of the solution set on the number line :



8. If an inequality simplifies to a **false statement**, then there are no solutions. In this case,

the solution set is just empty. For example:

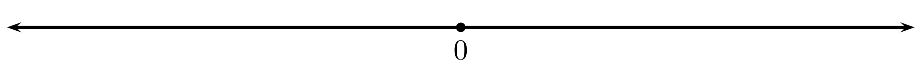
$$\begin{aligned} 5x - 3 &\leq 2(3x-5) - x && \textit{First simplify the right hand side.} \\ 5x - 3 &\leq 6x - 10 - x \\ 5x - 3 &\leq 5x - 10 \\ 5x - 3 - 5x &\leq 5x - 10 - 5x && \textit{Subtract } 5x \textit{ from both sides.} \\ -3 &\leq -10 && \textit{This is a false statement.} \end{aligned}$$

The solution set is empty, denoted by ϕ. There is no graph for this set.

9. If an inequality simplifies to a **true statement**, then every real number is a solution. In this set, the solution set consists of all real numbers. This set is denoted by $\mathbb{R}$.

$$\begin{aligned} 3 - (4 + x) &> \frac{-3x-5}{3} \\ 3 - 4 - x &> \frac{-3x-5}{3} \\ -1 - 3 &> \frac{-3x-5}{3} && \textit{Now multiply both sides by } 3. \\ 3 \times (-1 - x) &> \cancel{3}^{1} \times \frac{(-3x-5)}{\cancel{3}^{1}} \\ -3 - 3x &> -3x - 5 \\ -3 - 3x + 3x &> -3x - 5 + 3x && \textit{Add } 3x \textit{ to both sides.} \\ -3 &> -5 && \textit{This is a true statement} \\ & && \textit{independent of any variable.} \end{aligned}$$

Every real number is a solution. The solution set is $\mathbb{R}$. The solution set in descriptive form is $\{x \mid x \textit{ is a real number }\}$ and in interval form is $(-\infty, +\infty)$. The graph of the solution set is



10. Sometimes, we encounter two inequalities simultaneously. The rules of solving such in-

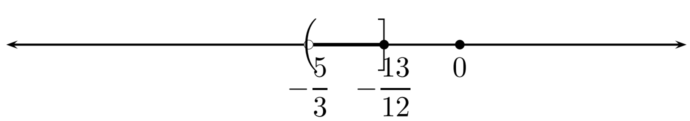
equalities are the same as before.

$$\begin{array}{ccccc}
5 & \leq & -4(3x+2) & < & 12 \\
5 & \leq & -12x-8 & < & 12 \\
5+8 & \leq & -12x-8+8 & < & 12+8 & \textit{Add 8 to the three parts.} \\
13 & \leq & -12x & < & 20 \\
\dfrac{13}{-12} & \geq & \dfrac{-12x}{-12} & > & \dfrac{\cancelto{5}{20}}{-\cancelto{3}{12}} & \textit{Divide the three parts by } -12 \\
-\dfrac{13}{12} & \geq & x & > & -\dfrac{5}{3} & \textit{and change the inequalities} \\
-\dfrac{5}{3} & < & x & \leq & -\dfrac{13}{12}
\end{array}$$

The solution set in set-builder form : $\left\{ x \text{ a real number } \mid -\dfrac{5}{3} < x \leq -\dfrac{13}{12} \right\}$

in interval form : $\left(-\dfrac{5}{3}, -\dfrac{13}{12} \right]$

The graph of the solution set on the number line :



Remark : In a course on pre-calculus or calculus, you will see more interesting inequalities.
Classroom Exercises : Solve the following inequalities. Describe the solution set in set-builder form, in interval form, and graph the set on the number line.

(a) $3x+5 \leq 8x-12$

(b) $6(x+3(2-3x)) > 3(x+5)$

(c) $4(x+2(5-x)) \leq 2(3-2x)$

(d) $3(x-(3-2x)) \geq 9(x-1)$

(e) $2 \leq \dfrac{6x+2}{-5} \leq 7$.

Here are a few questions which require transition to algebra.

1. Four more than three times a number is at least seven. What is the number?

   Let the number be x.

   $$\underbrace{\textit{Four more than}}_{4+}\ \underbrace{\textit{three times a number}}_{3x}\ \underbrace{\textit{is at least}}_{\geq}\ \underbrace{\textit{seven.}}_{7.}$$

   $$\begin{aligned}
   \textit{That is,}\quad 4+3x &\geq 7 \\
   4+3x-4 &\geq 7-4 && \textit{Subtract } 4 \textit{ from both sides.} \\
   3x &\geq 3 \\
   x &\geq 1 && \textit{Divide both sides by } 3.
   \end{aligned}$$

   The number has to be at least 1.

2. To qualify for a certain scholarship, Rob's monthly income should be no more than \$ 900.00. He receives \$ 350.00 per month from his weekend job at a bookstore. In addition, he gets paid \$ 12.00 per hour at his work in the local library. How many whole hours can he work in a month to be eligible for the scholarship?

   Suppose Rob works x number of hours every month in the library.
   Then he earns $12x$ dollars from his work at the library, and \$ 350 from his work at the bookstore. His total monthly income is $12x + 350$.

   $$\underbrace{\textit{Rob's monthly income}}_{12x+350}\ \underbrace{\textit{should be no more than}}_{\leq}\ \underbrace{\$900.}_{900.}$$

   $$\begin{aligned}
   \textit{That is,}\quad 12x+350 &\leq 900 \\
   12x+350-350 &\leq 900-350 && \textit{Subtract } 350 \textit{ from both sides.} \\
   12x &\leq 550 \\
   \frac{12x}{12} &\leq \frac{550}{12} && \textit{Divide both sides by } 12. \\
   x &\leq 45\frac{5}{6}
   \end{aligned}$$

   Rob can work at most 45 whole hours every month at the library to be eligible for the scholarship.

Classroom Exercises:

(a) Five less than two-thirds of a number is at most 5. What is the number?

(b) Matt needs to save no less than \$ 1,200 to travel to Brazil next month. His present weekend job pays him \$ 150 per month. He has another \$ 230 saved. How many whole hours should he put this month as a tutor if the tutoring job pays him \$ 15 per hour?

3.4.3 Homework Exercises

Describe the following sets in set-builder form, in interval form, and graph the set on the number line.

1. The set of all real numbers less than 7 and greater than 1.
2. The set of all real numbers less than or equal to 7 and greater than 1.
3. The set of all real numbers less than 7 and greater than or equal to 1.
4. The set of all real numbers less than or equal to 7 and greater than or equal to 1.
5. The set of all real numbers less than 3 and greater than or equal to 1.
6. The set of all real numbers less than or equal to -2.
7. The set of all real numbers less than 2.
8. The set of all real numbers greater than 7.
9. The set of all real numbers greater than or equal to 7.
10. The set of all real numbers greater than 7 and less than 2.

Solve the following inequalities. Describe the solution set in set-builder form, in interval form, and graph the solution set on the number line.

11. $x - 5 > 7$
12. $x + 8 \leq 9$
13. $2x + 7 < 6$
14. $-3x + 5 \geq 9$
15. $4x + 3 < 6x - 8$

16. $4(3-2(3x+4)) > 4(x-2(x-6))$
17. $\dfrac{4x+3}{2} - \dfrac{8}{3} \leq \dfrac{5x-1}{6}$
18. $\dfrac{5-2x}{5} + \dfrac{8}{15} \geq \dfrac{3x+1}{3}$
19. $3(2-3(5-x)) \geq 9(x+8)$
20. $3(2-3(5-x)) \leq 9(x+8)$
21. $0 \leq 4(5-2x) < 7$
22. $2 < \dfrac{3-2x}{3} \leq 9$

Answer the following:

23. Six less than three-fifths of a number is at most two. What is the number?
24. To afford a certain house, Sashsa needs an income of no less than \$ 1,800 per month. Her current job pays her \$ 1,200 per month after taxes. How many whole hours does she need to tutor every month if her tutoring charges are \$ 20 per hour?

3.5 Absolute value equations

Recall that the absolute value of a real number x, denoted by $|x|$, is its distance from 0 on the numberline. Therefore, $|4| = 4$ and so is $|-4| = 4$, while $|0| = 0$. In other words,

$$|x| = \begin{cases} x & \textit{if } x > 0; \\ 0 & \textit{if } x = 0; \\ -x & \textit{if } x < 0. \end{cases}$$

Thus,

- $|x| = 2$ implies that $x = 2$ or $x = -2$.
- $|x| = 0$ implies that $x = 0$.
- $|x| = -3$ does not have a solution, since absolute value of a number cannot be negative.

With these ideas in mind we can solve certain absolute value equations.

Solve for x and describe the solution set:

1. $|x-10|=12$.

Case 1:	**Case 2:**
$x-10=12$	$x-10=-12$
$x=22$	$x=-2$

Solution set $=\{22,-2\}$.

2. $|2x+6|=7$

Case 1:	**Case 2:**
$2x+6=7$	$2x+6=-7$
Solve this...	*Solve this...*
$x=\frac{1}{2}$	$x=-\frac{13}{2}=-6\frac{1}{2}$

Solution set $=\left\{\frac{1}{2},-6\frac{1}{2}\right\}$.

3. $\left|\frac{2x}{5}-7\right|=-23$.

This equation has no solution because absolute value cannot be negative.

Solution set $=\phi$.

4. $\left|\frac{3x}{7}-23\right|=0$. This has only one case:

$$\frac{3x}{7}-23=0 \qquad \textit{Solve the linear equation;}$$
$$x=\frac{161}{3}=53\frac{2}{3}$$

Solution set $=\left\{53\frac{2}{3}\right\}$.

5. $\left|\dfrac{2x}{3} - 5\right| = |3x+5|$

Case 1:	**Case 2:**
$\dfrac{2x}{3} - 5 = 3x + 5$	$\dfrac{2x}{3} - 5 = -(3x+5)$
Solve this...	*Solve this...*
$x = -4\dfrac{2}{7}$	$x = 0$

Solution set $= \left\{-4\dfrac{2}{7}, 0\right\}$.

6. $|2(3x-5)| = |3(2x-5)|$

Case 1:	**Case 2:**
$2(3x-5) = 3(2x-5)$	$2(3x-5) = -3(2x-5)$
Solve this...	*Solve this...*
No solutions here.	$x = 2\dfrac{1}{12}$

Solution set $= \left\{2\dfrac{1}{12}\right\}$.

7. $|3(x-1)| = |-3x+3|$

Case 1:	**Case 2:**
$3(x-1) = -3x+3$	$3(x-1) = -(-3x+3)$
Solve this...	*Solve this...*
$x = 0$	Every real number is a solution here.

Solution set $= \mathbb{R}$. Since $x = 0$ is included in $\mathbb{R}$, we do not need to mention $x = 0$ separately.

Classroom Exercises: Solve the absolute value equations and describe the solution set.

(a) $|x - 7| = 32$

(b) $|3x + 11| = 45$

(c) $|3(2x + 7) - 8| = 0$

(d) $\left|\frac{4x + 5}{23} + 7\right| = -234$

(e) $|2(x + 7)| = |x + 5|$

(f) $\left|\frac{2x + 5}{8}\right| = \left|\frac{3x}{5} + 1\right|$

(g) $|4(x - 9)| = |2(2x + 7)|$

(h) $|2(3x + 1)| = |-6x - 2|$

3.5.1 Homework Exercises

Solve the absolute value equations and describe the solution set.

1. $|x - 23| = 42$
2. $|5x + 1| = 75$
3. $|8(3x + 7) - 9| = 0$
4. $\left|\frac{x + 7}{2} - 2\right| = -234$
5. $|3(x - 9)| = |5x - 8|$
6. $\left|\frac{3x + 5}{3}\right| = \left|\frac{2x}{3} - 7\right|$
7. $|3(4x - 5)| = |12x - 15|$
8. $|3(4x - 5)| = |12x + 15|$

Chapter 4

Graphing linear equations

4.1 Linear equations in two variables

4.1.1 Introduction

Linear equations in two variables x, y can be written in the form

$$ax + by = c \quad \textit{for some real numbers } a, b, c.$$

Here are examples of linear equations in two variables x and y:

$$3x + 4y = 7; \quad \sqrt{3}x - \frac{1}{4}y = -20; \quad 4y = 10 - 7x.$$

Here are examples of equations which are **not linear**:

$$3xy = 7; \quad \sqrt{x} + y = 8; \quad \frac{3x}{4y} = 7.$$

While we have used variables x and y, a linear equation in two variables can be written in any two variables. For example, $3p + 4q = 7$ represents the same linear equation as $3x + 4y = 7$.
A pair of real numbers, written within parentheses and separated by a comma is called an **ordered pair of real numbers**. For example, $(3, 4)$ and $\left(\sqrt{2}, -\frac{1}{4}\right)$ are examples of ordered pairs of real numbers. The order in an ordered pair is important. For instance, $(3, 4) \neq (4, 3)$ as ordered pairs.

4.1.2 Solutions of linear equations in two variables

Given a linear equation in variables x and y, we say that an ordered pair (a, b) is a **solution of the linear equation** if when we substitute $x = a$ and $y = b$, the linear equation results in a true statement. For example, $(3,2)$ is a solution of the linear equation $4x+5y=22$ because

$$\begin{aligned}&4(3)+5(2) \quad (\textit{substitute } x=3, y=2)\\&=12+10 \quad (\textit{simplify})\\&=22.\end{aligned}$$

More examples :

- Show that $(2,3)$ is a solution of $5x-y=7$.

$$\begin{aligned}&5(2)-(3) \quad (\textit{substitute } x=2, y=3)\\&=10-3 \quad (\textit{simplify})\\&=7 \quad (\textit{the right hand side}).\end{aligned}$$

- Is $(3,2)$ a solution of $5x-y=7$?

$$\begin{aligned}&5(3)-(2) \quad (\textit{substitute } x=2, y=3)\\&=15-2 \quad (\textit{simplify})\\&=13\neq 7 \quad (\textit{the right hand side}).\end{aligned}$$

  Hence, $(3,2)$ is not a solution of $5x-y=7$.

- Is $\left(\sqrt{2},-\dfrac{1}{4}\right)$ a solution of $\sqrt{8}x+4y=3$?

$$\begin{aligned}&\sqrt{8}\left(\sqrt{2}\right)+4\left(-\frac{1}{4}\right) \quad (\textit{substitute } x=2, y=3)\\&=\sqrt{16}-\frac{4\times 1}{4} \quad (\textit{simplify})\\&=4-1=3 \quad (\textit{the right hand side}).\end{aligned}$$

  Yes, $\left(\sqrt{2},-\dfrac{1}{4}\right)$ a solution of $\sqrt{8}x+4y=3$.

Classroom Exercises :

(a) Is $(4,3)$ a solution of $-5x+2y=-14$?

(b) Is $(3,4)$ a solution of $-5x+2y=-14$?

(c) Is $(1,\sqrt{3})$ a solution of $-5x+\sqrt{12}y=1$?

There are infinitely many solutions to a linear equation in two variables.
Example : Consider the linear equation $2x + 4y = 10$.
Set $x = 0$. We then get

$$\begin{aligned}
&2(0) + 4y = 10 && \text{(substitute } x = 0\text{)}\\
&0 + 4y = 10 && \text{(simplify)}\\
&\frac{4y}{4} = \frac{10}{4} && \text{(divide both sides by 4)}\\
&y = \frac{5}{2} = 2\frac{1}{2} && \text{(we have solved for } y\text{).}
\end{aligned}$$

Thus, $\left(0, 2\frac{1}{2}\right)$ is a solution of the given equation.
Now set $x = 1$. We then get

$$\begin{aligned}
&2(1) + 4y = 10 && \text{(substitute } x = 1\text{)}\\
&2 + 4y = 10 && \text{(simplify)}\\
&4y = 10 - 2 && \text{(simplify)}\\
&\frac{4y}{4} = \frac{8}{4} && \text{(divide both sides by 4)}\\
&y = 2 && \text{(we have solved for } y\text{).}
\end{aligned}$$

Thus, $(1, 2)$ is also a solution of the given equation.
The reader can see that one can set x to any real number, and one can solve for y thereby finding other solutions. This is described as follows:
Let x be any real number. Then

$$\begin{aligned}
&4y = 10 - 2x && \text{(subtract } 2x \text{ from both sides)}\\
&\frac{4y}{4} = \frac{10 - 2x}{4} = \frac{2(5 - x)}{4} && \text{(divide both sides by 4)}\\
&y = \frac{5 - x}{2} && \text{(we have solved for } y\text{).}
\end{aligned}$$

Thus $\left(x, \frac{5 - x}{2}\right)$ is a solution of the given equation for any real number x.
Classroom Exercises :

1. Complete the ordered pairs so that each is a solution to the given equation.
   (a) $5x - 4y = 3$ $\qquad (0, \quad), (1, \quad), (-1, \quad), (2, \quad).$
   (b) $x - 2y = 6$ $\qquad (\quad ,0), \left(\quad , \frac{1}{3} \right), (\quad , -3), (\quad , 1).$
2. Find five solutions to each of the equations :
   (a) $2x - 5y = 10$
   (b) $-3x + 2y = 7$
   (c) $x = 4$
   (d) $y = 8$

Linear equations are routinely observed in real life.
Examples :

- Temperature is measured in Celcius (C) or in Fahrenheit (F). The two are related by a linear equation, $F = \frac{9}{5}C + 32$. When $C = 12$, we can calculate

$$\begin{aligned} F &= \frac{9}{5} \cdot 12 + 32 \\ &= \frac{9 \times 12}{5} + 32 \\ &= \frac{108}{5} + 32 \\ &= 21\frac{3}{5} + 32 = 53\frac{3}{5}. \end{aligned}$$

  When $F = 43$, we can calculate

$$\begin{aligned} 43 &= \frac{9}{5}C + 32 \\ 43 - 32 &= \frac{9}{5}C \\ 11 &= \frac{9}{5}C \\ 11 \times \frac{5}{9} &= \frac{9}{5}C \times \frac{5}{9} \\ \frac{55}{9} &= C \\ 6\frac{1}{9} &= C. \end{aligned}$$

- Suppose a certain factory manufactures widgets. The setting up cost is \$ 125.00, and the cost of producing one widget is \$ 5.25. Then the cost (C in dollars) of producing x number of widgets is given by the linear equation is $C = 5.25x + 125$.

  If 100 widgets are to be manufactured, then we can calculate the total cost to be

$$\begin{aligned} C &= 5.25(100) + 125 \\ &= 525 + 125 = \$650. \end{aligned}$$

  On the other hand, if we are allowed to spend \$ 1,000 then we can count the number of widgets which can be produced.

$$\begin{aligned} 1000 &= 5.25x + 125 \\ 1000 - 125 &= 5.25x \\ 875 &= 5.25x = 5\frac{1}{4}x = \frac{21}{4}x \\ 875 \times \frac{4}{21} &= \frac{21}{4}x \times \frac{4}{21} \\ 166\frac{2}{3} &= x. \end{aligned}$$

  That is, 166 entire widgets can be manufactured for \$ 1,000.00

Classroom Exercises:

1. A taxi charges \$ 12.00 for pick-up and \$ 4.00 per mile thereafter. Set up a linear equation to describe the taxi-charges with respect to the distance travelled. How much will it cost to travel $5\frac{1}{4}$ miles? How many whole miles can be travelled in \$ 100.00?

2. An ant had stored 100 gms. of sugar last week. This week, she has been adding 1.2 gms. of sugar every hour. Set up a linear equation relating the total amount of sugar accumulated and the amount of time. How long will the ant have to work to have a total of 300 gms. of sugar?

4.1.3 Homework Exercises

1. Check whether the given points are solutions to the respective equations:

   (a) $-2x + y = 5$ $\qquad$ $(0, 0), (2, 1), (-1, 1), (0, 5)$.

   (b) $3x - y = 12$ $\qquad$ $(0, 0), (2, -6), (-1, 1), (0, -12)$.

   (c) $5x - y = 8$ $\qquad$ $(0, 0), \left(\frac{2}{5}, -6\right), (-1, -3), \left(1\frac{3}{5}, 0\right)$.

2. Describe all the infinitely many solutions of the following equations:

   (a) $-2x + y = 5$

   (b) $3x - y = 12$

   (c) $5x - y = 8$

3. Complete the ordered pairs so that each is a solution to the given equation.

   (a) $2x - 7y = 3$ $\quad (0, \quad), (1, \quad), (-1, \quad), (2, \quad).$

   (b) $x - 2y = 9$ $\quad (\quad , 0), \left(\quad , \frac{1}{3} \right), (\quad , -3), (\quad , 1).$

4. Find five solutions to each of the equations :

   (a) $2x - 9y = 10$

   (b) $5x + 2y = 7$

   (c) $x = -2$

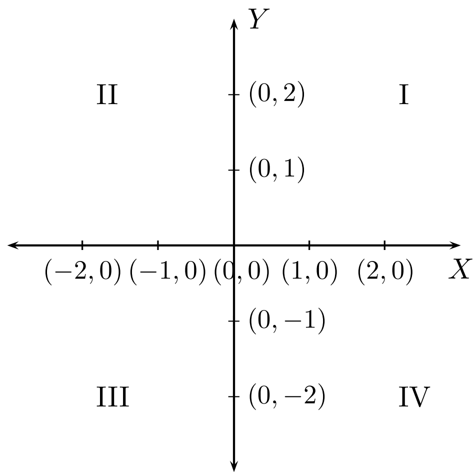
   (d) $y = -6$

5. One foot is twelve inches. Write a linear equation describing this conversion. How many inches constitute $5\frac{2}{3}$ feet?

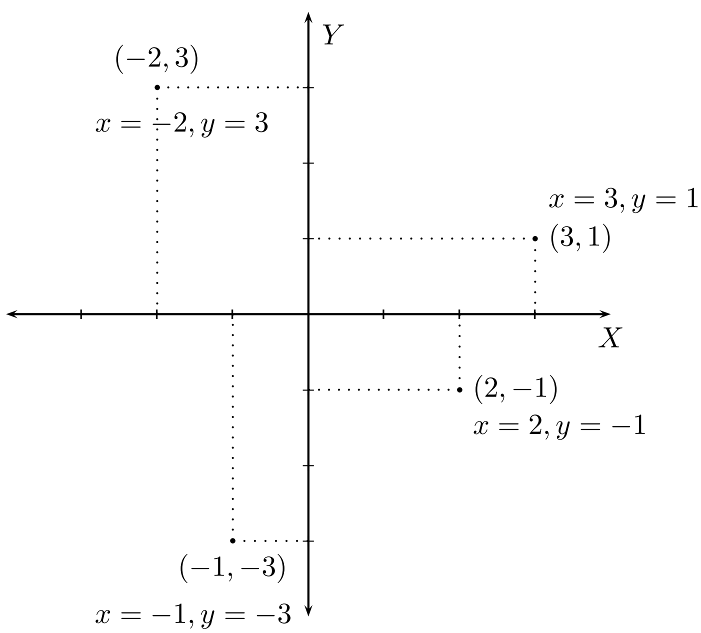
6. An icecream maker costs \$ 52.00. Further, it costs \$ 2.50 in materials to make one gallon of icecream. Set up a linear equation relating the cost (C) in dollars to make x gallons of icecream. How many gallons of icecream can be made with an investment of \$ 200.00?

4.2 The cartesian coordinate system

Real numbers are arranged on a number-line. Every point on the line corresponds to precisely one real number. In the same way, we would like to understand points on a flat plane. The French mahtematician Rene Descartes (1596-1650) introduced the concept of coordinate plane. In his honour, the coordinate plane is called the Cartesian plane.



Two number lines are drawn on the plane; one is drawn horizontally, and the other vertically. They intersect at the number 0 on each line. The two lines are called axes; the horizontal line is called the X-axis, while the vertical line is called, the Y-axis. Points on the X-axis are named with an additional **second coordinate** of 0; points on the Y-axis are named with an additional **first coodinate** of 0. The two axes divide the plane into four parts, the **quadrants**. The quadrants are numbered anti-clockwise, starting with the upper right, as shown. The quadrants are traditionally written in Roman numerals. The following figure shows how points on the coordinate plane are named. The first coordinate is the x**-coordinate**, and the second one is the y**-coordinate**.

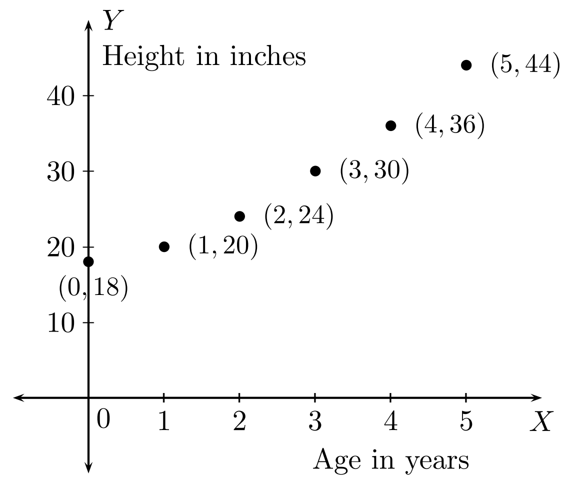


The coordinate plane is a good way of visualizing information. We usually use the y-coordinate to represent the dependent variable, while the x-coordinate is used to denote the independent variable, where the dependent and independent variables are to be understood from the context as shown in the following example.

Example : The following table gives the height of a child in inches from birth until the age of five years:

Age (in years)	0	1	2	3	4	5
Height (in inches)	18	20	24	30	36	44

These data can be visualized as a collection of points on the coordinate plane.
Note that since height is dependent on the age, we let y coordinate represent the height and let x coordinate represent the age.



4.2.1 Homework Exercises

1. What can you say about the coordinates of points on the X-axis?
2. What can you say about the coordinates of points on the Y-axis?
3. What can you say about the coordinates of points in the first quadrant?
4. What can you say about the coordinates of points in the second quadrant?
5. What can you say about the coordinates of points in the third quadrant?
6. What can you say about the coordinates of points in the fourth quadrant?
7. Plot the following points on a coordinate plane:
$$(0,0),(1,2),(0,3),(0,-5),(2,0),(-3,0),(2,-1),(-4,2),(-2,-5)$$
8. The following table is the data recorded on a particular morning on the number of speeding cars:

Time (A.M.)	6:00	6:30	7:00	7:30	8:00	8:30	9:00	9:30	10:00
Number of cars	5	6	9	15	16	14	12	10	3

Plot this information on a coordinate plane. Explain your choice of coordinates (dependent or independent) for the two items.

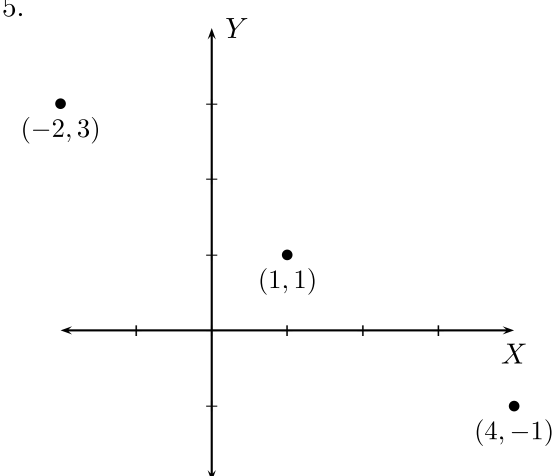
4.3 The graph of a linear equation

Recall that a linear equation in two variables, x, y, can be written in the form of $ax + by = c$ for some real numbers a, b, c. Linear equations can be understood algebraically as well as geometrically. To understand a linear equation geometrically, we draw its graph in the coordinate plane.

The graph of a linear equation in two variables x, y is its set of solutions (x, y) in the coordinate plane.

Example : Consider the linear equation $2x + 3y = 5$.

We can guess some solutions right away:
$(1,1), (-2,3), (4,-1), (-5,5), \cdots$
Notice how these points are arranged.

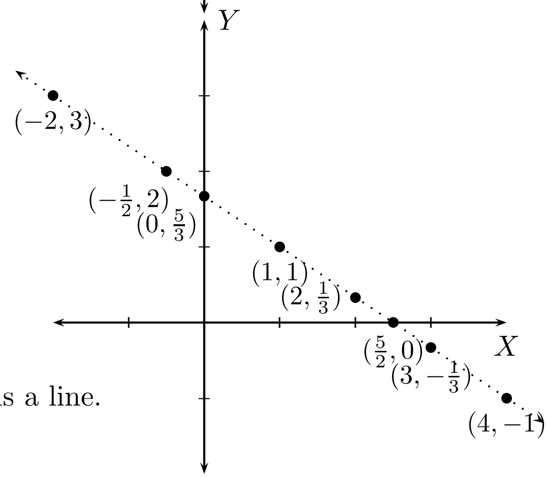


If we persist some more, and use fractions, then we see many more points:

$$\left(0, \frac{5}{3}\right), \left(\frac{5}{2}, 0\right), \left(2, \frac{1}{3}\right), \left(-\frac{1}{2}, 2\right), \left(3, -\frac{1}{3}\right) \cdots$$

All the solutions are arranged in a line.

The graph of a linear equation in two variables is a line.



A line is determined by two distinct points. Therefore, to draw the graph of a linear equation, we need to find two distinct solutions. A third point is usually taken to make sure that no mistakes were made while finding two points. We now introduce the concepts of **intercepts**.

The x-coordinate of the point where a line intersects the X-axis is called its **x-intercept**. Similarly, the y-coordinate of the point where a line intersects the Y-axis is called its **y-**

intercept. In the previous example, the graph of $2x + 3y = 5$, has x-intercept of $\frac{5}{2}$, and its y-intercept is $\frac{5}{3}$. This can be seen algebraically as follows:

To find the x-intercept, set $y = 0$ and solve for x:

$$
\begin{aligned}
&2x + 3y = 5 && \text{(the given equation)} \\
&2x + 3(0) = 5 && \text{(set } y = 0\text{)} \\
&\frac{2x}{2} = \frac{5}{2} && \text{(solve for } x\text{)} \\
&x = \frac{5}{2} && \text{(this is the } x\text{-intercept).}
\end{aligned}
$$

To find the y-intercept, set $x = 0$ and solve for y:

$$
\begin{aligned}
&2x + 3y = 5 && \text{(the given equation)} \\
&2(0) + 3y = 5 && \text{(set } x = 0\text{)} \\
&\frac{3y}{3} = \frac{5}{3} && \text{(solve for } y\text{)} \\
&y = \frac{5}{3} && \text{(this is the } y\text{-intercept).}
\end{aligned}
$$

Classroom Exercises :

1. Find the x and y intercepts of

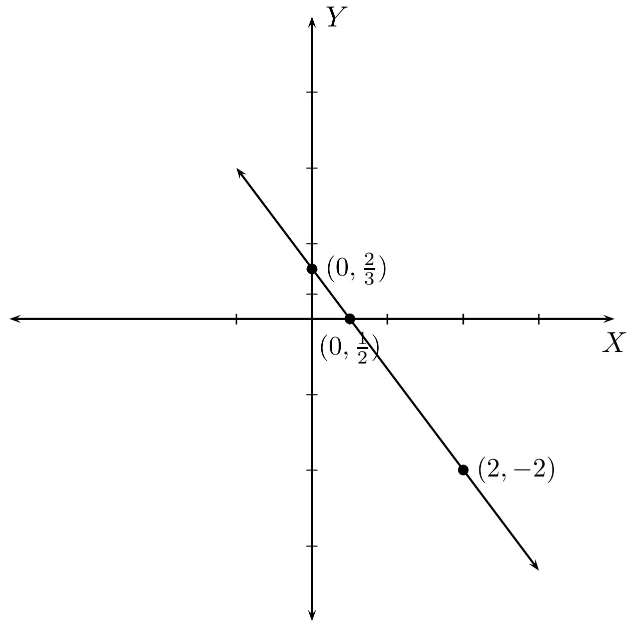
$$3x + 4y = 7; \quad 4x - 2y = 3; \quad x - 3y = 7.$$

We are now ready to use the intercepts to draw graphs of linear equations.

Example : Graph the linear equation $4x + 3y = 2$. We find the two intercepts, and one other solution.

x	y
0	$\frac{2}{3}$
$\frac{1}{2}$	0
2	-2

← y-intercept
← x-intercept
← a third solution



Classroom Exercises: For each of the linear equations given below, find the x, y-intercepts if they exist (if one of them does not exist, then explain why), find three points and graph on a coordinate plane.

2. $x - y = 2$

3. $2x - 3y = 6$

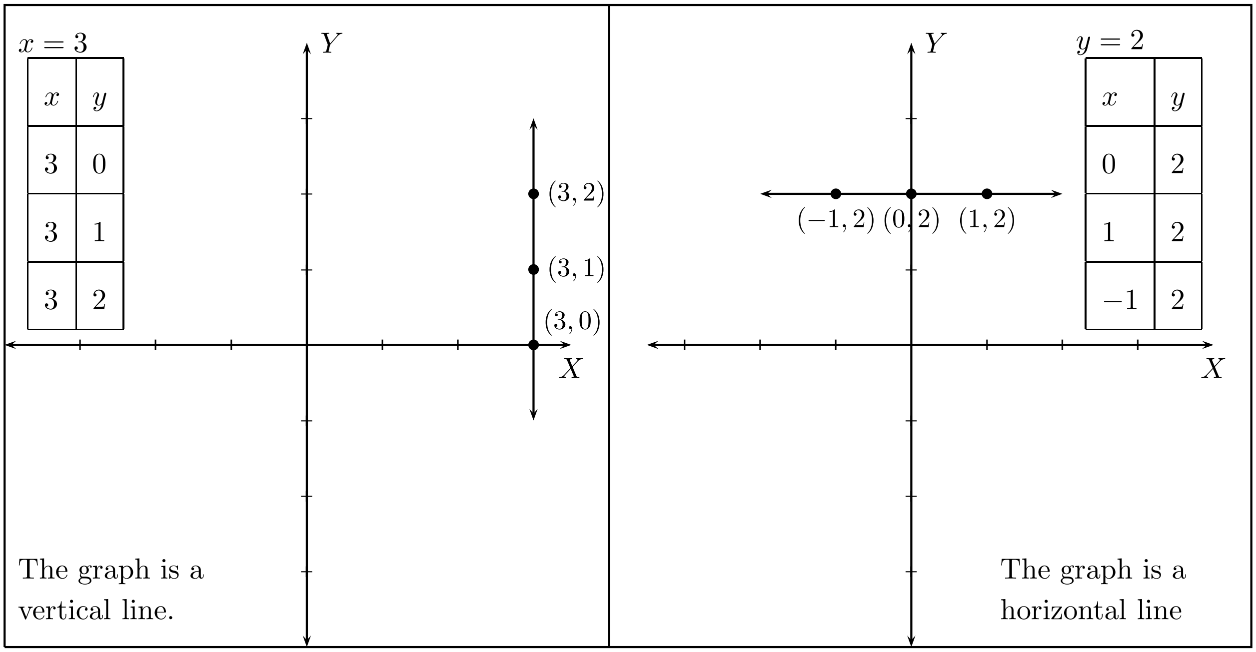
4. $3x - 2y = 5$

Certain lines do not have x and y intercepts. A line which is parallel to the X-axis will not have an x-intercept, while a line which is parallel to the Y-axis will not have a y-intercept. Algebraically this can be noticed when we cannot solve for x or for y.

For example, in the linear equation $x = 3$, setting $x = 0$ gives us $0 = 3$ which is false. In other words, we cannot solve for y (since the variable y is absent) and the graph of $x = 3$ will not have a y-intercept. However, the graph of the equation $x = 3$ will have an x-intercept of 3.

Likewise, consider the equation $y = 2$. Here, setting $y = 0$ gives us $0 = 2$ which is false. In other words, we cannot solve for x (since the variable x is absent) and the graph of $y = 2$ will not have an x-intercept. However, the graph of the equation $y = 2$ will have an y-intercept of 2.

Their graphs are given below:



Classroom Exercises: For each of the linear equations given below, find the x, y-intercepts if they exist (if one of them does not exist, then explain why), find three points and graph on a coordinate plane.

(a) $x = 4$ (b) $y = 4$

(c) $x = 0$ (d) $y = 0$

4.3.1 Homework Exercises

For each of the linear equation given below,

- find the x-intercept if it exists, and if it does not exist, then explain why it does not exist;
- find the y-intercept if it exists, and if it does not exist, then explain why it does not exist;
- find three solutions including the intercepts if they exist;
- graph on a coordinate plane.

1. $x + 2y = 6$
2. $4x - y = 8$
3. $2x - 3y = 9$

4. $3x + 2y = 1$

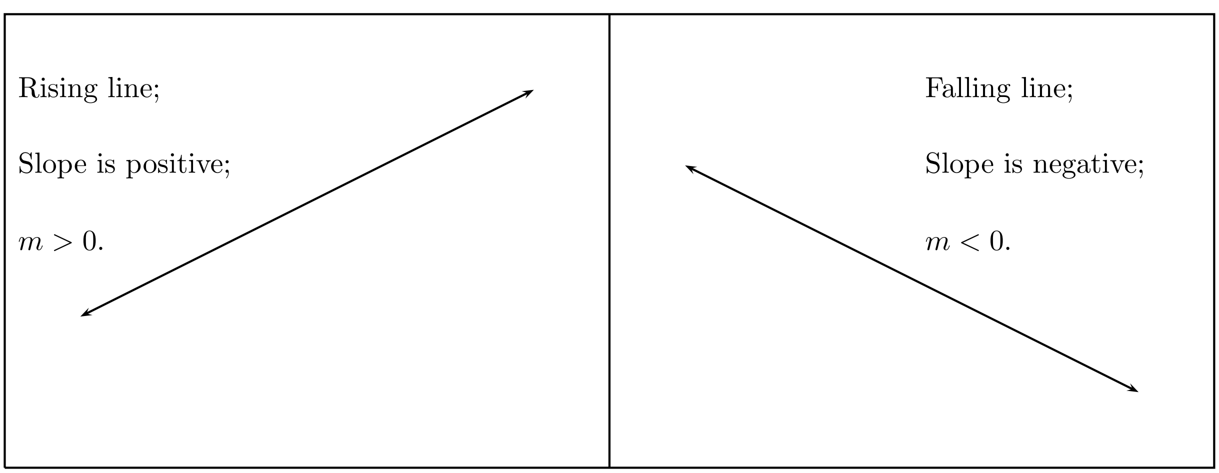
5. $x = -2$

6. $y = -3$

7. $2x + 3y = 4$

4.4 Slope

In the previous section we learned about graphing linear equations. In this section we will learn about the slope of a line. Intuitively, we understand that a rising line should have positive slope, while a falling line should have negative slope. It is also clear that a horizontal line should have zero slope. One can imagine trying to bike up a street, even a steep street. But one cannot bike up a wall. The slope of a vertical line is so large that it is undefined. We denote the slope of a line by m.

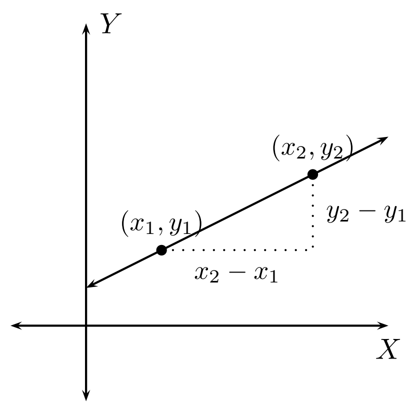


Horizontal line; Slope is zero; $m = 0$. 	 Vertical line; Slope is undefined; m is undefined.

The **slope** of a line joining two points is the ratio of the difference in their y-coordinates to their corresponding difference in their x-coordinates. We will understand this statement in a short while. Intuitively speaking, slope is a measure of the rise of the line with respect to its run. Informally speaking,

$$\textit{Slope} = \frac{\textit{rise}}{\textit{run}}.$$

The adjoining figure shows a line joining two points (x_1, y_1) and (x_2, y_2).



Notice that the rise of the line is $y_2 - y_1$ while the run of the line is $x_2 - x_1$. We thus have the formula for slope,

$$\textit{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

Examples : Find the slope of the line joining the points

- $(-2, -1)$ and $(3, 2)$.
  Here $(x_1, y_1) = (-2, -1)$ and $(x_2, y_2) = (3, 2)$. So,
  $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - (-1)}{3 - (-2)} = \frac{2 + 1}{3 + 2} = \frac{3}{5}$$
  Check for yourself that in this case the line is rising.
- $(-2, 5)$ and $(3, 2)$. Here $(x_1, y_1) = (-2, 5)$ and $(x_2, y_2) = (3, 2)$. So,
  $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 5}{3 - (-2)} = \frac{-3}{3 + 2} = -\frac{3}{5}$$

Check for yourself that in this case the line is falling.

- $(-2,-1)$ and $(3,-1)$. Here $(x_1,y_1)=(-2,-1)$ and $(x_2,y_2)=(3,-1)$. So,

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{-1-(-1)}{3-(-2)}=\frac{-1+1}{3+2}=\frac{0}{5}=0$$

Check for yourself that in this case the line is horizontal.

- $(-2,-1)$ and $(-2,2)$. Here $(x_1,y_1)=(-2,-1)$ and $(x_2,y_2)=(-2,2)$. So,

$$m=\frac{y_2-y_1}{x_2-x_1}=\frac{2-(-1)}{-2-(-2)}=\frac{2+1}{-2+2}=\frac{3}{0}=\textit{undefined}$$

Check for yourself that in this case the line is vertical.

Classroom Exercises : Find the slope of the line joining the points

(a) $(-3,2)$ and $(4,5)$

(b) $(-2,6)$ and $(3,-1)$

(c) $(-3,4)$ and $(4,4)$

(d) $(-3,4)$ and $(-3,-3)$

The slope of a line can also be detected from its defining linear equation. A linear equation is said to be in **standard form** if it is written

$$ax+by=c \quad \textit{for some real numbers } a,b,c.$$

A linear equation is said to be in **slope-intercept** form if it is written

$$y=mx+b \quad \textit{for some real numbers } m,b.$$

Note that the slope-intercept equation for a line is unique. But the standard form equation for a line need not be unique.

Note that a line given by the equation $y=mx+b$ has the following two solutions:

$$x=0 \Rightarrow y=m(0)+b \Rightarrow y=b \quad (\textit{this is the } y \textit{ intercept since } x=0,) \textit{ and}$$

$$x=1 \Rightarrow y=m(1)+b \Rightarrow y=m+b.$$

The two solutions are $(0,b)$ and $(1,m+b)$. Therefore,

$$\textit{Slope}=\frac{y_2-y_1}{x_2-x_1}=\frac{(m+b)-b}{1-0}=\frac{m}{1}=m.$$

That is, the line given by the equation $y=mx+b$ has y-intercept b and slope m.

Converting the linear equation from its standard form to its slope-intercept form can thus be used to find the slope and y-intercept algebraically.

Examples : Find the slope and y-intercept of the line given by the equation

- $2x + 3y = 4$. This is the standard form of the equation. We first convert the equation to its slope-intercept form. To do so, we need to isolate y:

  $$\begin{aligned} &2x + 3y = 4 && \text{(given equation);} \\ &3y = -2x + 4 && \text{(subtract } 2x \text{ from both sides);} \\ &\frac{3y}{3} = \frac{-2x+4}{3} && \text{(divide both sides by 3);} \\ &y = -\frac{2}{3}x + \frac{4}{3} && \text{(division has to be distributed).} \end{aligned}$$

  This gives us the slope-intercept form of the equation. We then get

  $$\textit{Slope} = m = -\frac{2}{3}; \quad y\textit{-intercept } = \frac{4}{3}.$$

- $3y = 4$. This is in standard form.

  $$\begin{aligned} &3y = 4 && \text{(given equation);} \\ &\frac{3y}{3} = \frac{4}{3} && \text{(divide both sides by 3);} \\ &y = 0x + \frac{4}{3} && \text{(since there is no } x \text{ we write } 0x\text{).} \end{aligned}$$

  This gives us the slope-intercept form of the equation. We then get

  $$\textit{Slope} = m = 0; \quad y\textit{-intercept } = \frac{4}{3}.$$

- $2x = 4$. This is in standard form. Since there is no y in this expression, we can at best simplify the given equation to $x = 2$ by dividing both sides by 2. This equation does not have a slope-intercept form. The reader is asked to check for himself/herself that the resulting line is a vertical line passing through $(2, 0)$. Therefore, the slope of the line is undefined, and the line has no y-intercept.

Classroom Exercises : Find the slope and y-intercept, if they exist, by converting the given equation to slope-intercept form:

(a) $3x + 4y = 7$

(b) $3x - 4y = 9$

(c) $3x = 9$

(d) $-4y = 9$

A linear equation in its slope-intercept form can be graphed in an intuitive fashion. Recall that a line is determined by two distinct points. Likewise, a line is determined by one point and its slope.

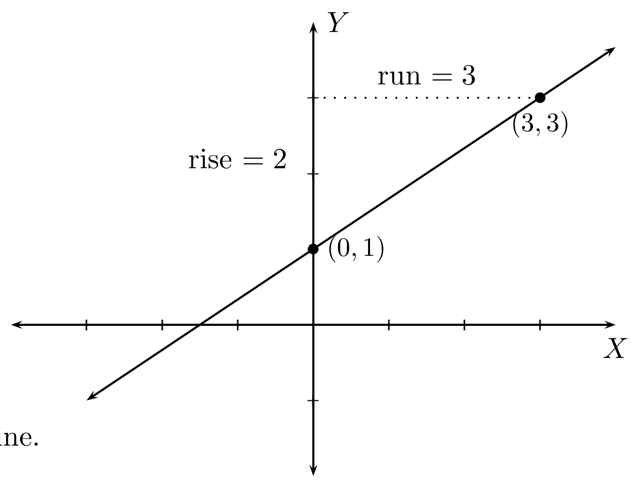
Example 1: $y = \frac{2}{3}x + 1$

Here, $m = \frac{2}{3}$ and $b = 1$.

That is, the y-intercept is 1.
We thus have a point $(0, 1)$ on the graph.

Since $slope = \frac{rise}{run}$ we have

$rise = 2$ and $run = 3$.
We thus get another point $(3, 3)$.
Joining the two points gives us the required line.



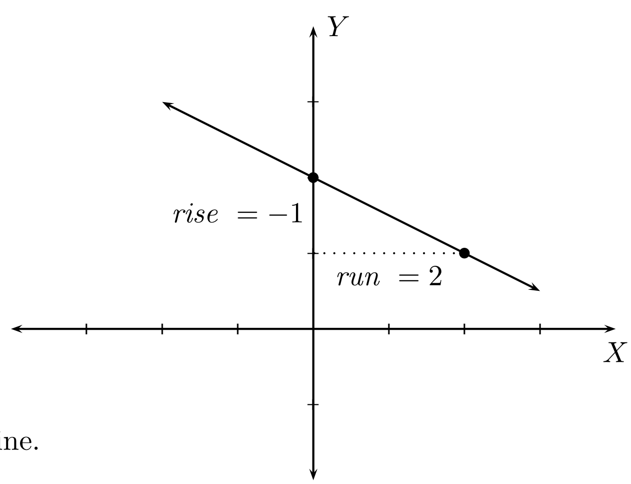
Example 2: $y = -\frac{1}{2}x + 2$

Here, $m = -\frac{1}{2} = \frac{-1}{2}$ and $b = 2$.

That is, the y-intercept is 2.
We thus have a point $(0, 2)$ on the graph.

Since $slope = \frac{rise}{run}$ we have

$rise = -1$ and $run = 2$.
We thus get another point $(2, 1)$.
Joining the two points gives us the required line.



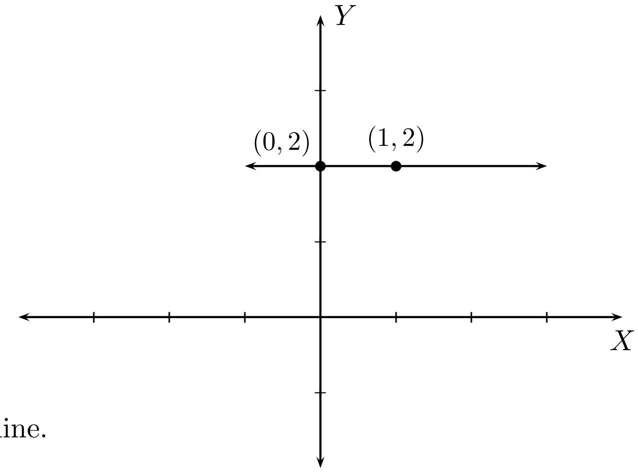
Example 3: $y = 2$

Here, $m = 0 = \frac{0}{1}$ and $b = 2$.

That is, the y-intercept is 2.
We thus have a point $(0, 2)$ on the graph.

Since $slope = \frac{rise}{run}$ we have

$rise = 0$ and $run = 1$.
We thus get another point $(1, 2)$.
Joining the two points gives us the required line.



Classroom Exercises : Graph the lines given by the linear equations in slope-intercept form:

(a) $y = \frac{2}{3}x - 1$

(b) $y = -\frac{2}{3}x + 2$

(c) $y = 3$

(d) $y = \frac{1}{3}x + \frac{2}{3}$

(e) $y = -\frac{1}{3}x + \frac{2}{3}$

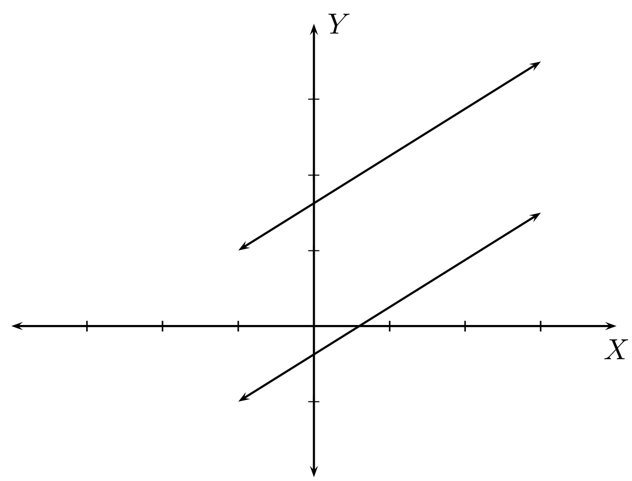
(f) $y = \frac{2}{3}$

Parallel Lines:

Two non-vertical lines are said to

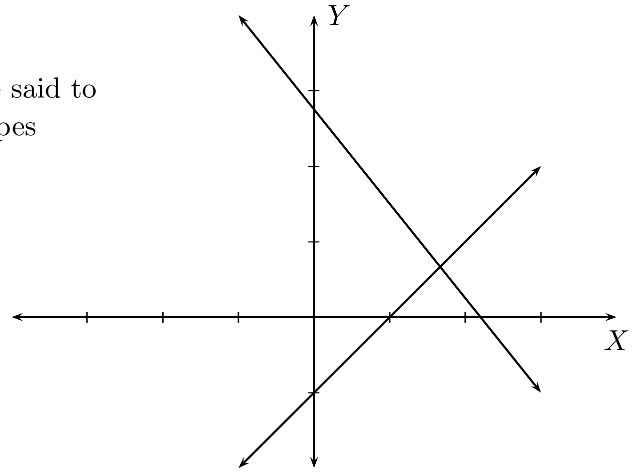
be **parallel** if their slopes are equal.

Two **vertical lines** are parallel.



Perpendicular Lines:

Two non-vertical lines with slopes m_1, m_2 are said to be **perpendicular** if the product of their slopes is equal to -1. That is, $m_1 m_2 = -1$. Geometrically, perpendicular lines form right angles where they cross.



A **vertical line** and a **horizontal line** are perpendicular to each other.

Examples : Check whether the lines given by the equations are parallel, perpendicular, or neither.

- $2x + 3y = 12$ and $4x + 6y = -5$: We find the slopes of the two lines.

$2x + 3y = 12$	(the first equation)	$4x + 6y = -5$	(the second equation)
$3y = -2x + 12$	(subtract $2x$ from both sides)	$6y = -4x - 5$	(subtract $4x$ from both sides)
$\frac{3y}{3} = \frac{-2x+12}{3}$	(divide both sides by 3)	$\frac{6y}{6} = \frac{-4x-5}{6}$	(divide both sides by 6)
$y = -\frac{2}{3}x + 4$	(slope-intercept form)	$y = -\frac{2}{3}x - \frac{5}{6}$	(slope-intercept form).
$m_1 = -\frac{2}{3}$	(slope)	$m_2 = -\frac{2}{3}$	(slope)

The two lines are parallel.

- $3x - 2y = 12$ and $4x + 6y = -5$: We find the slopes of the two lines.

$3x - 2y = 12$	(the first equation)	$4x + 6y = -5$	(the second equation)
$-2y = -3x + 12$	(subtract $3x$ from both sides)	$6y = -4x - 5$	(subtract $4x$ from both sides)
$\frac{-2y}{-2} = \frac{-3x+12}{-2}$	(divide both sides by -2)	$\frac{6y}{6} = \frac{-4x-5}{6}$	(divide both sides by 6)
$y = \frac{3}{2}x + 4$	(slope-intercept form)	$y = -\frac{2}{3}x - \frac{5}{6}$	(slope-intercept form).
$m_1 = \frac{3}{2}$	(slope)	$m_2 = -\frac{2}{3}$	(slope)

Since $m_1 \times m_2 = \frac{3}{2} \times \left(-\frac{2}{3}\right) = -1$. The two lines are perpendicular to each other.

- $y = 3$ and $x = 2$: The line given by $y = 3$ is a horizontal line, while the line given by $x = 2$ is a vertical line. Hence, the two lines are perpendicular to each other.
- $x = 2$ and $x = 5$: Here, both the lines are vertical, and therefore they are parallel.

Classroom Exercises : Check whether the lines given by the equations are parallel, perpendicular, or neither.

(a) $4x + 5y = 8$ and $10x - 8y = 3$

(b) $3x + 2y = 7$ and $9x + 6y = 17$

(c) $3x + 2y = 5$ and $4x + 5y = 8$

(d) $3x = 5$ and $4x = 7$

(e) $3x = 5$ and $4y = 7$

4.4.1 Homework Exercises

1. Find the slope of the line joining the two points:
   (a) $(4, 7)$ and $(-2, -2)$
   (b) $(-5, -1)$ and $(2, 8)$
   (c) $(-5, 8)$ and $(2, -1)$
   (d) $(4, -2)$ and $(-2, 7)$
   (e) $(4, -2)$ and $(4, 7)$

(f) $(-5, 8)$ and $(-5, 1)$

(g) $(4, -2)$ and $(10, -2)$

(h) $(-5, 8)$ and $(0, -8)$

2. Find the slope and the y-intercept, if they exist, of the line given by the equation:

   (a) $4x + 5y = 10$

   (b) $7x - 6y = 11$

   (c) $7x = 12$

   (d) $7y = 12$

3. Graph the lines given by the linear equations in slope-intercept form:

   (a) $y = \frac{1}{3}x - 2$ (b) $y = -\frac{3}{5}x + 4$

   (c) $y = \frac{2}{3}$ (d) $y = \frac{1}{4}x - \frac{2}{3}$

   (e) $y = -\frac{2}{3}x + \frac{3}{5}$ (f) $y = \frac{3}{7}$

4. Check whether the lines given by the equations are parallel, perpendicular, or neither.

   (a) $5x - 6y = 11$ and $6x + 5y = 10$

   (b) $5x - 3y = 2$ and $10x - 6y = 7$

   (c) $5x + 2y = 7$ and $5x - 3y = 11$

   (d) $5x = 7$ and $-3y = 11$

   (e) $2x = 7$ and $4x = 9$

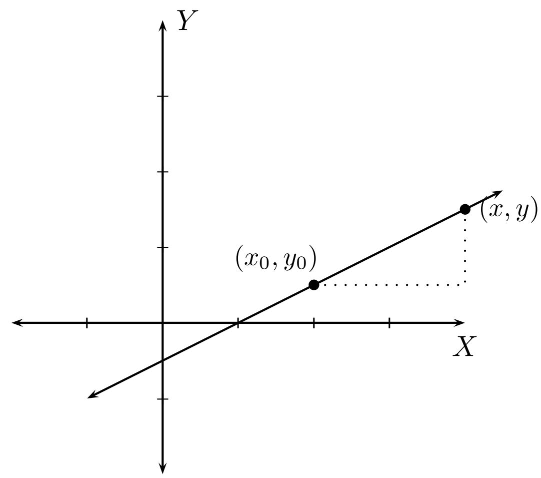
4.5 The point-slope form of the equation of a line

While we have used the notion of a line, we had not defined a line. **A line is a set of points on a plane with a fixed slope from a fixed point.** That is, a line is determined by a point and a slope. That is, there is one and only one line passing through a given point and a given slope. Suppose (x_0, y_0) is a point on the coordinate plane, and we draw a non-vertical line with slope m passing through (x_0, y_0).

Let (x, y) be a general point on the line.

The slope of the line is $m = \dfrac{y - y_0}{x - x_0}$.

$m(x - x_0) = y - y_0$
$y - y_0 = m(x - x_0)$
is the equation of a line through point (x_0, y_0) with slope m.



The equation $y - y_0 = m(x - x_0)$ is called the **point-slope** form of the equation for the line. This proves that the points on a line satisfy a linear equation. Note that the point-slope form is not unique, since it depends on the choice of a point. Given the slope and a point on a line, we can say many things about the line.

Examples : For each line which passes through the given point with the given slope,

(a) find an equation of the line;

(b) write the equation of the line in slope-intercept form;

(c) write the equation of the line in standard form;

(d) find the x and y intercepts, if any, of the line;

(e) graph the line.

Example 1: $P = (1, 3)$, and $m = \frac{2}{3}$.

- Use **point-slope** form to get

$$y - 3 = \frac{2}{3}(x - 1).$$

  This is an equation of the line.

- To convert this equation into the slope-intercept form, we use the following steps:

$$\begin{aligned}
y - 3 &= \frac{2}{3}(x - 1) && \text{(point-slope form)}\\
y - 3 &= \frac{2}{3}x - \frac{2}{3} && \text{(distributive law of multiplication over addition/subtraction)}\\
y &= \frac{2}{3}x - \frac{2}{3} + 3 && \text{(add 3 to both sides)}\\
y &= \frac{2}{3}x - \frac{2}{3} + \frac{9}{3} && \text{(common denominator is 3)}\\
y &= \frac{2}{3}x + \frac{7}{3} && \text{(this is the \textbf{slope-intercept} form).}
\end{aligned}$$

- To write the equation in **standard form**, we merely subtract $\frac{2}{3}x$ from both sides to get

$$-\frac{2}{3}x + y = \frac{7}{3} \quad (\textit{standard form}).$$

- To find the x-intercept, set $y = 0$ in any of the equations and solve for x. For instance, let us use the standard form.

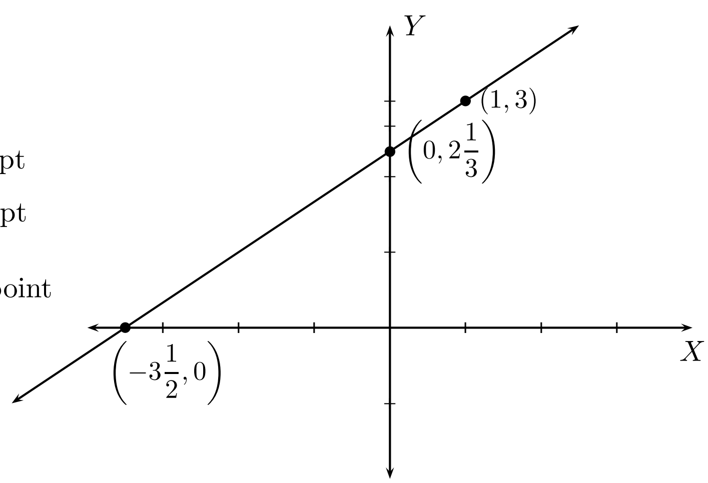
$$
\begin{aligned}
&-\frac{2}{3}x + (0) = \frac{7}{3} && \text{(substitute } y = 0\text{)}\\
&-\frac{2}{3}x = \frac{7}{3} && \\
&-\frac{2}{3}x \times \left(-\frac{3}{2}\right) = \frac{7}{3} \times \left(-\frac{3}{2}\right) && \text{(multiply both sides by } \left(-\frac{3}{2}\right)\text{)}\\
&x = -\frac{7}{2} = -3\frac{1}{2} && \text{(the } x\text{-intercept).}
\end{aligned}
$$

To find the y-intercept, we set $x = 0$ and solve for y from any of the equations; or, we could just use the slope-intercept form of the equation, $y = \frac{2}{3}x + \frac{7}{3}$.

The y-intercept is $y = \frac{7}{3} = 2\frac{1}{3}$.

- For graphing the line, it helps to find a third point. Use $x = 1$ in the slope-intercept form to get $y = \frac{2}{3} + \frac{7}{3} = \frac{9}{3} = 3$.

x	y	
0	$2\frac{1}{3}$	⟵ y-intercept
$-3\frac{1}{2}$	0	⟵ x-intercept
1	3	⟵ a third point



Example 2: $P = (-1, 3)$ and $m = -\frac{1}{2}$.

- Use **point-slope** form to get an equation:

$$
\begin{aligned}
y - 3 &= -\frac{1}{2}(x - (-1)),\\
y - 3 &= -\frac{1}{2}(x + 1).
\end{aligned}
$$

- To convert this equation to the slope-intercept form,

$$
\begin{aligned}
y - 3 &= -\frac{1}{2}(x+1) && \text{(point-slope form)}\\
y - 3 &= -\frac{1}{2}x - \frac{1}{2} && \text{(multiplication distributes)}\\
y &= -\frac{1}{2}x - \frac{1}{2} + 3 && \text{(add 3 to both sides)}\\
y &= -\frac{1}{2}x - \frac{1}{2} + \frac{6}{2} && \text{(common denominator)}\\
y &= -\frac{1}{2}x + \frac{5}{2} && (\textbf{slope-intercept}\ \text{form}).
\end{aligned}
$$

- To write the equation in **standard form**, add $\frac{1}{2}x$ to both sides to get

$$\frac{1}{2}x + y = \frac{5}{2}.$$

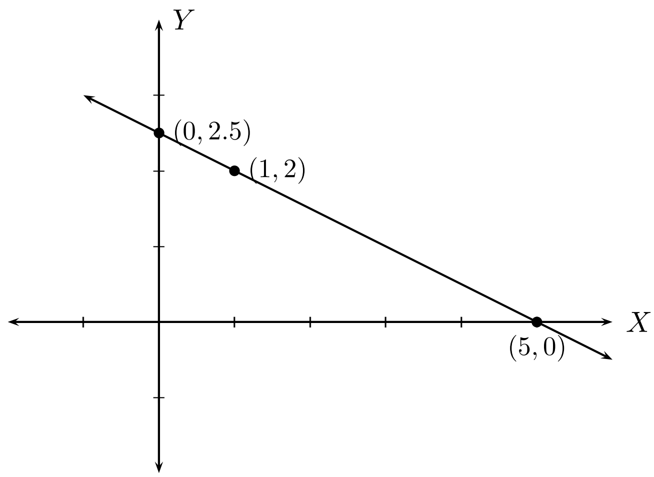
- To find the x-intercept, set $y = 0$ in any of the equations of the line. We use standard form to get

$$\frac{1}{2}x + 0 = \frac{5}{2} \Rightarrow \frac{1}{2}x \times \frac{2}{1} = \frac{5}{2} \times \frac{2}{1} \Rightarrow x = 5 \quad (\textit{the } x\textit{-intercept}).$$

  Using the slope-intercept form, the y-intercept is $\frac{5}{2} = 2\frac{1}{2}$.

- For graphing the line, it helps to find a third point. Set $x = 1$ in the slope-intercept form to get $y = -\frac{1}{2} + \frac{5}{2} = \frac{4}{2} = 2$. The third point is $(1, 2)$.

x	y	
0	2.5	⟵ y-intercept
5	0	⟵ x-intercept
1	2	⟵ a third point



Here are some interesting questions.

Example 1: Write an equation in standard form of a line which passes through point $(2, -1)$ and is parallel to the line given by the equation $2x - 3y = 7$.

The required line is parallel to the line given by the equation $2x - 3y = 7$. We find its slope first by converting this equation to its slope-intercept form:

$$\begin{aligned}
2x - 3y &= 7 \quad (\textit{given equation}) \\
-3y &= -2x + 7 \quad (\textit{subtract } 2x \textit{ from both sides}) \\
\frac{-3y}{-3} &= \frac{-2x+7}{-3} \quad (\textit{divide both sides by } -3) \\
y &= \frac{2}{3}x - \frac{7}{3} \quad (\textit{slope-intercept form}).
\end{aligned}$$

Therefore, we need to find an equation of a line passing through $(2, -1)$ with slope $\frac{2}{3}$.

$$\begin{aligned}
y - (-1) &= \frac{2}{3}(x - 2) \quad (\textbf{point-slope form}) \\
y + 1 &= \frac{2}{3}x - \frac{2}{3} \cdot 2 \quad (\textit{simplify}) \\
y + 1 &= \frac{2}{3}x - \frac{4}{3} \quad (\textit{simplify}) \\
y &= \frac{2}{3}x - \frac{4}{3} - 1 \\
y &= \frac{2}{3}x - \frac{7}{3} \quad (\textbf{slope-intercept form}) \\
-\frac{2}{3}x + y &= -\frac{7}{3} \quad (\textbf{standard form})
\end{aligned}$$

Example 2: Write an equation in standard form of a line which passes through point $(2, -1)$ and is perpendicular to the line given by the equation $2x - 3y = 7$.

The required line is perpendicular to the line given by the equation $2x - 3y = 7$ which, as we have seen in the previous example, has slope $\frac{2}{3}$. Since the required line is perpendicular to this one, the slope of the required line is $-\frac{3}{2}$ and it passes through point $(2, -1)$. Hence, we write equations of the required line:

$$\begin{aligned}
y-(-1) &= -\frac{3}{2}(x-2) \quad (\textbf{point-slope form})\\
y+1 &= -\frac{3}{2}x+\frac{3}{2}\cdot 2 \quad (\textit{simplify})\\
y+1 &= -\frac{3}{2}x+3 \quad (\textit{simplify})\\
y &= -\frac{3}{2}x+3-1\\
y &= -\frac{3}{2}x+2 \quad (\textbf{slope-intercept form})\\
\frac{3}{2}x+y &= 2 \quad (\textbf{standard form})
\end{aligned}$$

Example 3: Write an equation in standard form of a horizontal line passing through point $(2,-1)$.

A horizontal line has equation of the form $y = a$ for some real number a. Since the line passes through point $(2,-1)$, the required equation is $y = -1$.

Example 4: Write an equation in standard form of a vertical line passing through point $(2,-1)$.

A vertical line has equation of the form $x = a$ for some real number a. Since the line passes through point $(2,-1)$, the required equation is $x = 2$.

Example 5: Write an equation in standard form of a line passing through points $(3,2)$ and $(-2,5)$.

In this case, two points are given. So we can find the slope of the line.

$$\textit{Slope } = m = \frac{5-2}{-2-3} = \frac{3}{-5} = -\frac{3}{5}.$$

Now we use any of the points to proceed with point-slope equation of the line. Let us use the point $(3,2)$. The point-slope form will then be

$$y-2 = -\frac{3}{5}(x-3).$$

Simplifying this equation (check this for yourself), we get the slope-intercept form

$$y = -\frac{3}{5}x+\frac{1}{5}.$$

Now, an equation in standard form for the line is

$$\frac{3}{5}x+y = \frac{1}{5}.$$

Classroom Exercises: For each line with the given properties,

(a) find an equation of the line;

(b) write the equation of the line in slope-intercept form, if it exists;

(c) write an equation of the line in standard form;

(d) find the x and y intercepts, if any, of the line;

(e) graph the line.

1. The line passes through point $(3, -4)$ with slope $\frac{3}{5}$.
2. The line passes through point $(-2, 4)$ with slope $\frac{2}{3}$.
3. The line passes through point $(1, 0)$ and is parallel to the line given by the equation $4x - y = 7$.
4. The line passes through point $(0, -3)$ and is perpendicular to the line given by the equation $4x - y = 7$.
5. The line is horizontal and passes through the point $(1, -3)$.
6. The line is vertical and passes through the point $(-1, 3)$.
7. The line passes through points $(-2, 4)$ and $(0, 5)$.

4.5.1 Homework Exercises

For each line with the given properties,

(a) find an equation of the line;

(b) write the equation of the line in slope-intercept form, if it exists;

(c) write the equation of the line in standard form;

(d) find the x and y intercepts, if any, of the line;

(e) graph the line.

1. The line passes through point $(3, 4)$ with slope $\frac{2}{5}$.

2. The line passes through point $(3, -4)$ with slope $-\frac{3}{5}$.

3. The line passes through point $(-1, -3)$ and is parallel to the line given by the equation $3x - 4y = 8$.

4. The line passes through point $(-1, -3)$ and is perpendicular to the line given by the equation $3x - 4y = 8$.

5. The line is horizontal and passes through the point $(-1, -3)$.

6. The line is vertical and passes through the point $(-1, -3)$.

7. The line passes through points $(-2, 5)$ and $(0, 0)$.

4.6 Graphing linear inequalities in two variables

In the previous sections we learned about linear equations. A **linear inequality** in two variables x and y can be written in the form of

$$ax + by < c, \quad ax + by \leq c, \quad ax + by > c, \quad or \quad ax + by \geq c$$

for some real numbers a, b, c (a or b nonzero).

An ordered pair (x_0, y_0) is said to be a solution of a linear inequality in two variables x, y if the substitution $x = x_0$ and $y = y_0$ gives a true statement.

Examples : Consider the inequality $2x - 3y < 5$. Some of the solutions are

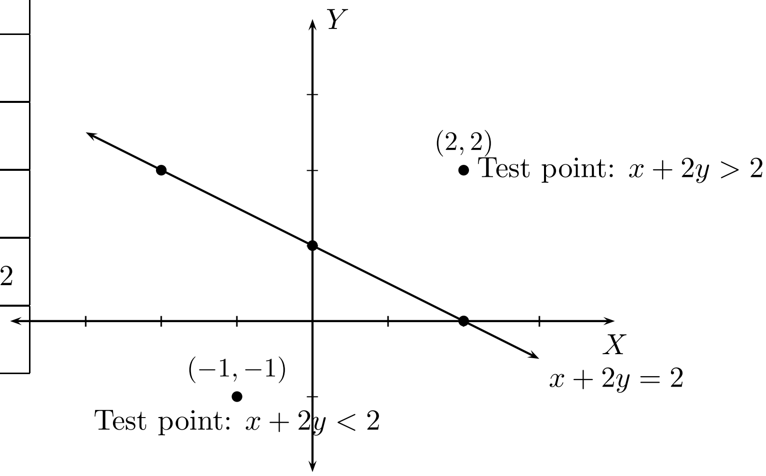
$$(0,1), (1,0), (1,1), (2,0), \left(\frac{1}{2}, 0\right), \cdots .$$

Some of the solutions of the inequality $3x + 2y \geq 5$ are

$$(1,1), (2,0), (3,-2), (0,3), \cdots .$$

The **graph of a linear inequality in two variables** is the set of points in the coordinate plane whose coordinates, as ordered pairs, are solutions of the linear inequality. Note that the graph of a linear **equality** in the coordinate plane is a line. A line divides the coordinate plane into two halves. Each half represents a strict inequality, where a strict inequality could be $<$ (less than) or $>$ (greater than) but equality is not permitted. The graph is determined using test points. For example, let us consider the line $x + 2y = 2$

(x, y)	$x + 2y$
$(-2, 2)$	$-2 + 2(2) = 2$
$(0, 1)$	$0 + 2(1) = 2$
$(2, 0)$	$2 + 2(0) = 2$
$(-1, -1)$	$-1 + 2(-1) = -3 < 2$
$(2, 2)$	$2 + 2(2) = 6 > 2$



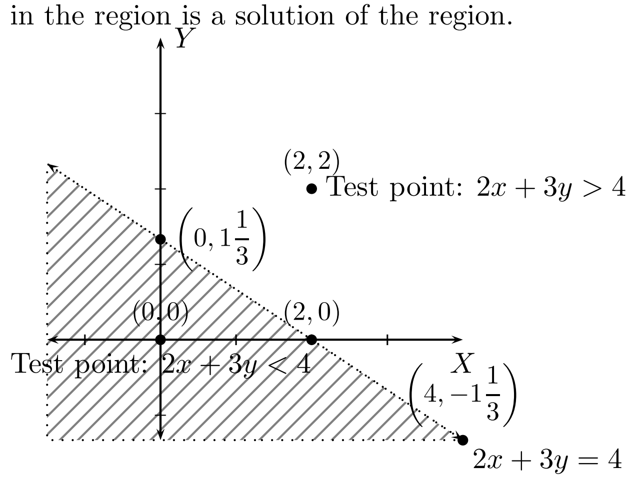
In the above figure, the graph of $x + 2y = 2$ is first drawn using its x-intercept $(2, 0)$, its y-intercept $(0, 1)$, and a third point $(-2, 2)$. This line divides the coordinate plane into two halves. Let us choose a **test point** from each of the halves. We have chosen the points $(-1, -1)$ and $(2, 2)$. Substituting the coordinates of $(-1, -1)$ into the expression $x+2y$ gives us the inequality $x + 2y < 2$. Likewise, the test point $(2, 2)$ satisfies the inequality $x + 2y > 2$.

Now we are ready to graph linear inequalities.

Example 1: Graph the inequality $2x + 3y < 4$. We first draw the line $2x + 3y = 4$ using techniques learnt in the previous sections. The x-intercept for the line is $(2, 0)$, while the y-intercept is $\left(0, 1\frac{1}{3}\right)$, and a third point $\left(4, -1\frac{1}{3}\right)$. We draw a **dotted line** because the inequality is $<$, which does not allow for equality. The shaded region is the required graph. Note that the shading indicates that **every point** in the region is a solution of the region.

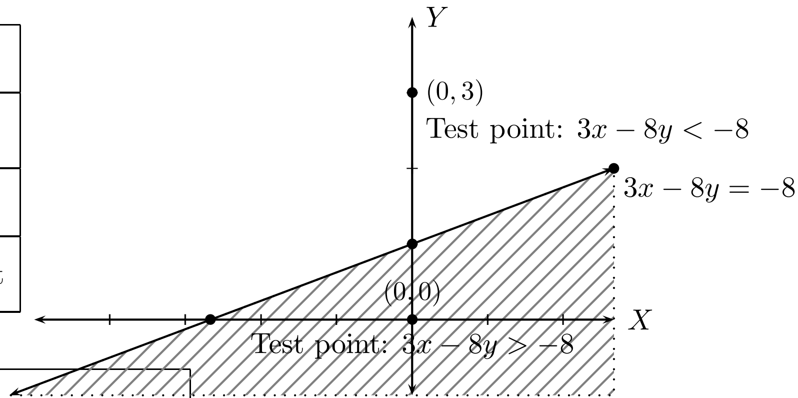
(x, y)	
$(2, 0)$	x-intercept
$\left(0, 1\frac{1}{3}\right)$	y-intercept
$\left(4, -1\frac{1}{3}\right)$	Third point

Test points	$2x + 3y$
$(0, 0)$	$2(0) + 3(0) = 0 < 4$
$(2, 2)$	$2(2) + 3(2) = 10 > 4$



Example 2: Graph the inequality $3x - 8y \geq -8$. We first draw the line $3x - 8y = -8$ using the x-intercept $\left(-2\frac{2}{3}, 0\right)$, the y-intercept $(0, 1)$, and a third point $\left(2\frac{2}{3}, 2\right)$. We draw a **solid line** because the inequality is $\geq$, which allows for equality. The shaded region along with the line is the required graph.

(x, y)	
$\left(-2\frac{2}{3}, 0\right)$	x-intercept
$(0, 1)$	y-intercept
$\left(2\frac{2}{3}, 2\right)$	Third point



Test points	$3x - 8y$
$(0, 0)$	$3(0) - 8(0) = 0 > -8$
$(0, 3)$	$3(0) - 8(3) = -24 < -8$

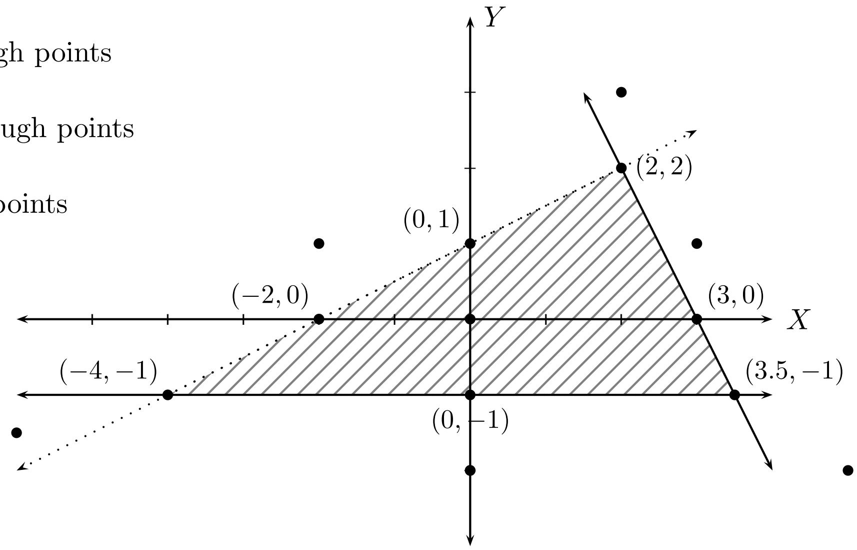
Sometimes several inequalities are presented. **A system of inequalities** is a set of more than one inequality. A solution in this case is an ordered pair (x_0, y_0) which satisfies **every** inequality. A graph of a system of linear inequalities is the set of all points in the coordinate plane whose coordinates satisfy every inequality.

In this case, we draw a line for each inequality first keeping in mind that the inequalities $<, >$ produce dotted lines, while the inequalities $\leq, \geq$ produce solid lines. Note that more lines may divide the coordinate plane into several pieces. We need to use test points from each piece and proceed as before.

Example 3: Graph the system of linear inequalities

$$2x + y \leq 6, \quad x - 2y > -2, \quad \textit{and} \quad y \geq -1.$$

The line $2x + y = 6$ passes through points $(2, 2), (3, 0), (3.5, -1)$.
The line $x - 2y = -2$ passes through points $(-4, -1), (-2, 0), (0, 1), (2, 2)$.
The line $y = -1$ passes through points $(-4, -1), (0, -1), (3.5, -1)$.



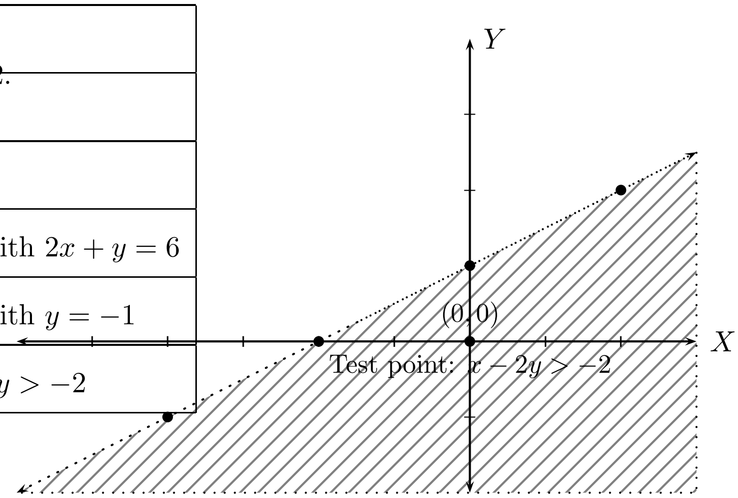
These three lines divide the coordinate plane into seven pieces. We choose a test point from each piece. Our test points are $(0, 0), (3, 1), (2, 3), (-2, 1), (-6, -1.5), (0, -2)$, and $(5, -1)$.

Only the point $(0, 0)$ satisfies all the three inequalities. Hence the solution set is the region containing $(0, 0)$.

Another method is to individually shade each inequality, and take the common region shaded. The three relevant graphs are given below.

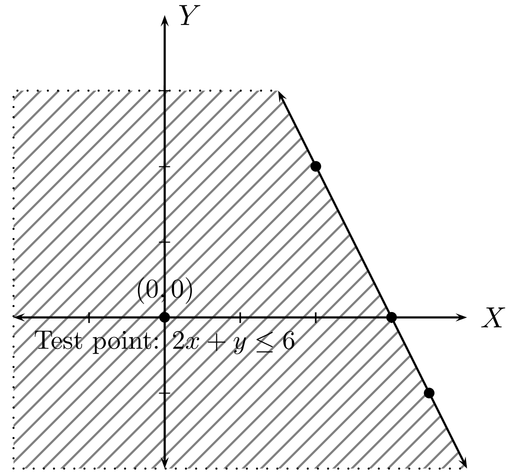
The dotted line is of $x - 2y = -2$.

(x, y)	
$(-2, 0)$	x-intercept
$(0, 1)$	y-intercept
$(2, 2)$	Intersection point with $2x + y = 6$
$(-4, -1)$	Intersection point with $y = -1$
$(0, 0)$	Test point for $x - 2y > -2$

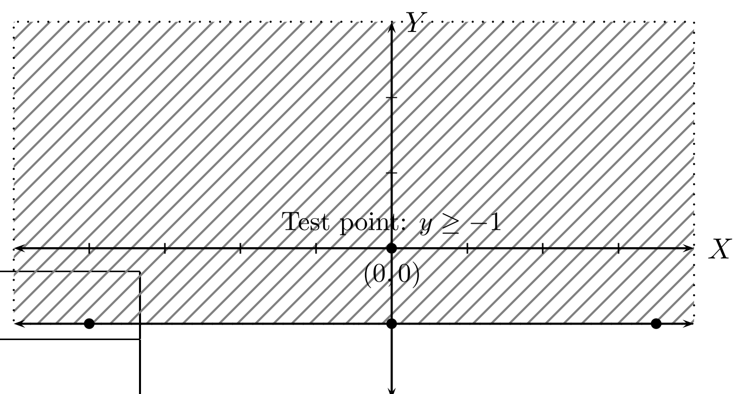


The line $2x + y = 6$ is first drawn.

(x, y)	
$(3, 0)$	x-intercept
$(0, 6)$	y-intercept (not shown)
$(2, 2)$	Intersection point with $x - 2y = -2$
$(3.5, -1)$	Intersection point with $y = -1$
$(0, 0)$	Test point for $2x + y < 6$



First, the line $y = -1$ is drawn.



(x, y)	
	No x-intercept
$(0, -1)$	y-intercept
$(-4, -1)$	Intersection point with $x - 2y = -2$
$(3.5, -1)$	Intersection point with $2x + y = 6$
$(0, 0)$	Test point for $y > -1$

The common region among the three regions is the required graph.

Classroom Exercises : Graph the following inequalities:

(a) $(a)\quad 3x + 4y \geq 2.\quad (b)\quad 4x + 3x > 2.\quad (c)\quad 3x - 4y \leq 5.$

(b) $(a)\quad x - 2y < 1.\quad (b)\quad 2x - 3y \geq 5.\quad (c)\quad -2x + 3y > 2.$

(c) $-2 < y \leq 3$ (Hint: $-2 < y$ and $y \leq 3$).

(d) $-3 \leq x \leq 2$ (Hint: $-3 \leq x$ and $x \leq 2$).

(e) $2x - y > 3$, $x \geq 1$, and $y < 3$.

4.6.1 Homework Exercises

Graph the following inequalities:

1. $2x + 3y < 5$
2. $2x + 3y \leq 5$
3. $4x - 3y > 5$
4. $4x - 3y \geq 5$
5. $x \geq 3$
6. $x > 3$
7. $x < 3$
8. $x \leq 3$
9. $y \geq -2$
10. $y \leq -2$
11. $y > -2$
12. $y < -2$
13. $-3 < x \leq 5$
14. $-2 \leq y < 4$
15. $2x - y > -2, \quad y \geq -1$, and $x \leq 3$.
16. $x + y \leq 4, \quad x \geq -2$, and $y \geq -3$

4.7 Solving systems of linear equations in two variables

A **system** is a collective term for equations just as a "bunch" is a collective term for roses, or a "flock" is a collective term for birds. A **system of linear equations in two variables** is a collection of two or more linear equations in two variables.

Here are three systems of equations:

$$\begin{cases} 2x - 3y = 1 \\ x + y = 4 \end{cases} \qquad \begin{cases} x - 4y = 3 \\ x = 5 \end{cases} \qquad \begin{cases} 2x + 3y = 1 \\ y = 4 \end{cases}$$

A **solution to a system of linear equations in two variables** is an ordered pair (x_0, y_0) which solves every equation in the system.

Example 1 : The ordered pair $(2, 1)$ is a solution of the system $\begin{cases} x + 2y = 4 \\ 2x - y = 3 \end{cases}$

We check this by substituting $x = 2$ and $y = 1$ in both the equations.

$(2) + 2(1) = 4 \quad$ and $\quad 2(2) - (1) = 3.$

The ordered pair $(0, 2)$ is **not a solution** of the system because

$(0) + 2(2) = 4 \quad$ but $\quad 2(0) - (2) \neq 3.$

Example 2 : The ordered pair $(-3, 4)$ is a solution of the system $\begin{cases} 2x + y = -2 \\ 3x - 2y = -17 \end{cases}$

We check this by substituting $x = -3$ and $y = 4$ in both the equations.

$2(-3) + (4) = -6 + 4 = -2 \quad$ and $\quad 3(-3) - 2(4) = -9 - 8 = -17.$

The ordered pair $(-1, 0)$ is **not a solution** of the system because

$2(-1) + (0) = -2 \quad$ but $\quad 3(-1) - 2(0) = -3 \neq -17.$

Classroom Exercise : Which of the following ordered pairs solve the system of equations $\begin{cases} 2x + 3y = -1 \\ 3x - 4y = 7 \end{cases}$?

$$(-2, 1), \left(\frac{1}{2}, -\frac{2}{3}\right), (5, 2), (1, -1).$$

Example 1: Consider the system of equations

$$x - 2y = 1$$

$$x + y = 4$$

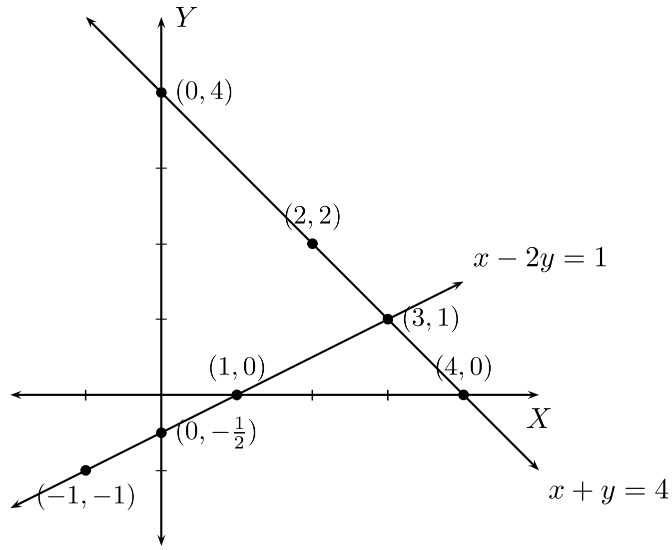
We learnt to draw graphs of linear equations in the previous sections using the x and y-intercepts and a third point for each equation.

$x - 2y = 1$

x	y	
0	$-\frac{1}{2}$	y-intercept
1	0	x-intercept
-1	-1	third point

$x + y = 4$

x	y	
0	4	y-intercept
4	0	x-intercept
2	2	third point



The two lines intersect at the point $(3, 1)$. The ordered pair is a solution to the given system: $(3) - 2(1) = 1$ and $(3) + (1) = 4$.

Example 2:

$$2x - y = -4$$

$$2x + y = 0$$

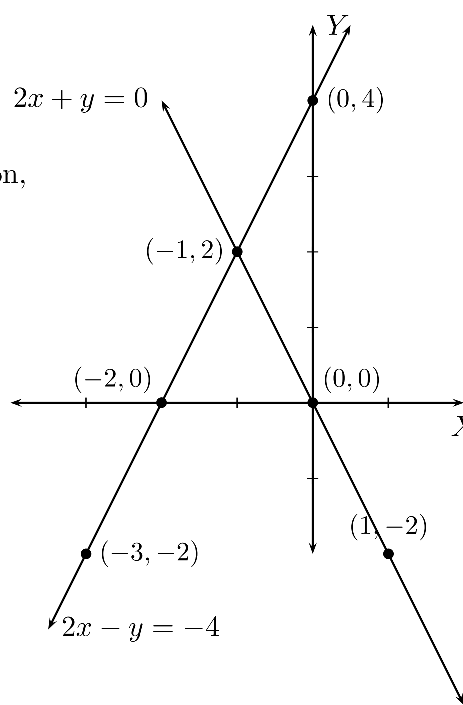
Using the x and y-intercepts and a third point for each equation, we draw the two lines.

$2x - y = -4$

x	y	
0	4	y-intercept
-2	0	x-intercept
-3	-2	third point

$2x + y = 0$

x	y	
0	0	y-intercept
0	0	x-intercept
1	-2	second point



The two lines intersect at the point $(-1, 2)$. That ordered pair is a solution to the given system:

$$2(-1) - (2) = -4 \quad \text{and } 2(-1) + (2) = 0.$$

These two examples show us how to solve a system of linear equations in two variables. This procedure is called, **the graphing method** of solving.

Classroom Exercises: Solve the following systems using graphing method.

(a) $\begin{array}{l} 3x - 2y = 5 \\ 2x - y = 3 \end{array}$ (b) $\begin{array}{l} x + y = -2 \\ 2x - 3y = 1 \end{array}$

The graphing method of solving a system of linear equations has its limitations as the next example will show.

Example 3: Consider the system of equations:

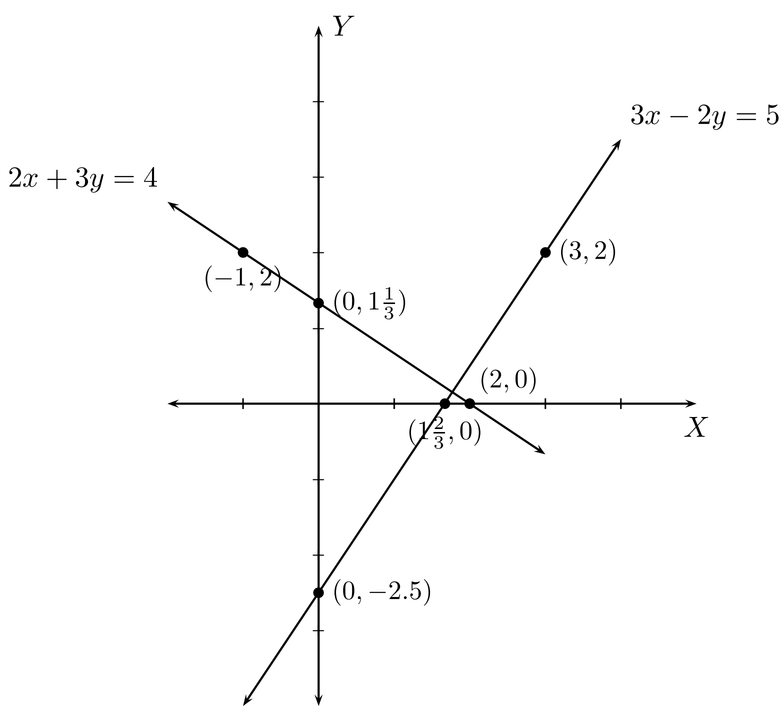
$$2x + 3y = 4$$
$$3x - 2y = 5.$$

$2x + 3y = 4$

x	y	
0	$1\frac{1}{3}$	y-intercept
2	0	x-intercept
-1	2	third point

$3x - 2y = 5$

x	y	
0	$-2\frac{1}{2}$	y-intercept
$1\frac{2}{3}$	0	x-intercept
3	2	third point



Notice that the intersection point does not have integer coordinates. So the graphing method does not give us an accurate solution. We thus turn to an algebraic method, called the **addition**

method or the **elimination method**. We eliminate one of the variables. For example, in the same example as above, we eliminate the variable x. To do so, we multiply the equations by appropriate numbers so that the coeffiencts of x are of **equal absolute values, but opposite signs**.

$$\begin{aligned} 2x+3y=4 &\quad \Longrightarrow^{\times 3} \quad && 6x+9y=12 \\ 3x-2y=5 &\quad \Longrightarrow^{\times(-2)} \quad && -6x+4y=-10 \end{aligned}$$

We add the resulting two equations to get a new equation. This means, we add the left hand sides of the two equations to each other to get the left hand side of the new equation, and we add the right hand sides of the two equations to each other to get the right hand side of the new equation. That is, $13y=2 \Longrightarrow y=\frac{2}{13}$.

Now substitute $y=\frac{2}{13}$ in any of the two original equations and solve for x. We use the first equation:

$$\begin{aligned} 2x+3\left(\frac{2}{13}\right) &= 4, \\ 2x+\frac{6}{13} &= 4, \\ 13\times\left(2x+\frac{6}{13}\right) &= 13\times 4, \quad \textit{Multiply both sides by } 13 \\ 26x+6 &= 52, \\ 26x &= 52-6, \\ 26x &= 46, \\ x &= \frac{46}{26}=\frac{23}{13}=1\frac{10}{13}. \end{aligned}$$

Now we **check** whether the ordered pair $\left(\frac{23}{13},\frac{2}{13}\right)$ is indeed the solution of the given system. First check that $2x+3y=4$.

$$L.H.S=2\left(\frac{23}{13}\right)+3\left(\frac{2}{13}\right)=\frac{46}{13}+\frac{6}{13}=\frac{52}{13}=4=R.H.S.$$

Next check that $3x-2y=5$,

$$L.H.S.=3\left(\frac{23}{13}\right)-2\left(\frac{2}{13}\right)=\frac{69}{13}-\frac{4}{13}=\frac{65}{13}=5=R.H.S.$$

Example 4 : We solve the system of equations

$$\begin{aligned} 3x - y &= 5 \\ 5x + 2y &= 1. \end{aligned}$$

We eliminate the variable x as follows:

$$\begin{aligned} 3x - y = 5 &\Longrightarrow^{\times 5} 15x - 5y = 25 \\ 5x + 2y = 1 &\Longrightarrow^{\times(-3)} -15x - 6x = -3. \end{aligned}$$

Adding the resulting equations gives us

$$-11y = 22 \Longrightarrow y = \frac{22}{-11} \Longrightarrow y = -2.$$

Now substitute $y = -2$ in any of the two original equations and solve for x. We use the second equation:

$$\begin{aligned} 5x + 2(-2) &= 1; \\ 5x - 4 &= 1; \\ 5x &= 1 + 4; \\ 5x &= 5; \\ x &= \frac{5}{5} = 1. \end{aligned}$$

Now **check** that $x = 1$ and $y = -2$ solve the given system of equations:

$$(3x - y = 5) \quad L.H.S = 3(1) - (-2) = 3 + 2 = 5 = R.H.S., \qquad \text{and}$$

$$(5x + 2y = 1) \quad L.H.S. = 5(1) + 2(-2) = 5 - 4 = 1 = R.H.S.$$

We can solve the same system by eliminating variable y as follows:

$$\begin{aligned} 3x - y = 5 &\Longrightarrow^{\times 2} 6x - 2y = 10 \\ 5x + 2y = 1 &\Longrightarrow^{\times 1} 5x + 2y = 1. \end{aligned}$$

Adding the resulting equations gives us

$$11x = 11 \Longrightarrow x = \frac{11}{11} = 1.$$

Substitute $x = 1$ in any of the original equations to solve for y. We use the first equation:

$$\begin{aligned} 5(1) + 2y &= 1; \\ 5 + 2y &= 1; \\ 2y &= 1 - 5; \\ 2y &= -4; \\ y &= \frac{-4}{2} = -2; \end{aligned}$$

we have arrived at the same solution. Since we have already checked that $(1, -2)$ is indeed the solution, we do not check again.

Classroom Exercises : Solve the following systems of equations and **check** that your solutions are indeed correct.

(a) $$\begin{aligned} 3x + y &= 5 \\ 2x - 3y &= 1 \end{aligned}$$

(b) $$\begin{aligned} 2x - 3y &= 4 \\ x &= 5 \end{aligned}$$

(c) $$\begin{aligned} 3x + 5y &= 1 \\ y &= 4 \end{aligned}$$

(d) $$\begin{aligned} 4x + 5y &= 3 \\ 3x + 2y &= 7 \end{aligned}$$

(e) $$\begin{aligned} -2x + 3y &= 4 \\ 4x + 2y &= 3 \end{aligned}$$

In all the above cases, we found a **unique** solution (x_0, y_0) for any given system of two equations in two variables. This is not always the case. Some systems have **infinitely many solutions**. This happens to be the case if, when **using the addition procedure, both the variables are eliminated, and we get a true statement**.

Example 5: Consider the system of equations

$$\begin{aligned} 2x - 2y &= 8 \\ 3x - 3y &= 12. \end{aligned}$$

We try to eliminate variable x as follows:

$$\begin{aligned} 2x - 2y = 8 &\Longrightarrow^{\times 3} 6x - 6y = 24 \\ 3x - 3y = 12 &\Longrightarrow^{\times(-2)} -6x + 6y = -24. \end{aligned}$$

Adding the resulting equations eliminates both the variables, and we get $0 = 0$ which is a true statement. The solution then can be described as

$$2x - 2y = 8 \Longrightarrow 2x = 2y + 8 \Longrightarrow x = \frac{2y+8}{2} \Longrightarrow x = \frac{2y}{2} + \frac{8}{2} \Longrightarrow x = y + 4$$

where y is any real number.

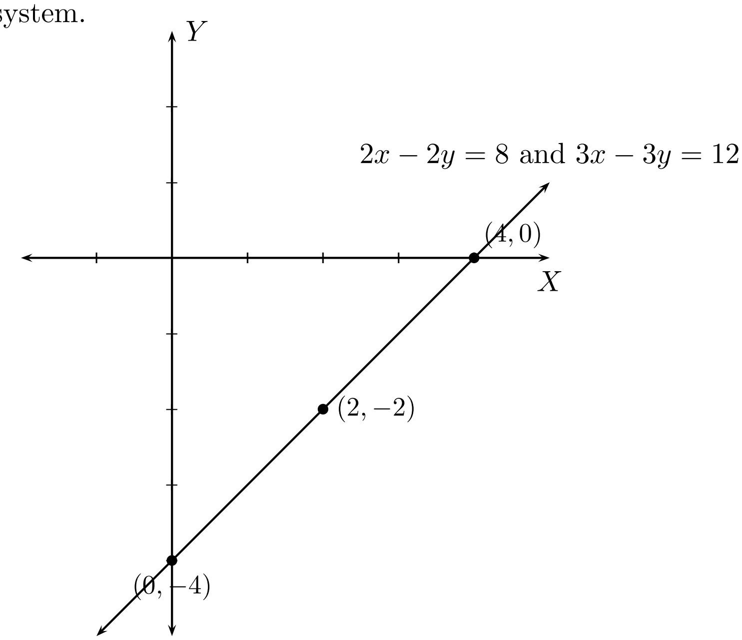
In this case, the graphs of the two linear equations give us one and the same line. That is, every point on the line is a solution of the system.

$2x - 2y = 8$

x	y	
0	-4	y-intercept
-4	0	x-intercept
2	-2	third point

$3x - 3y = 12$

x	y	
0	-4	y-intercept
4	0	x-intercept
2	-2	third point



A system of linear equations in two variables which has a unique solution is called, **an independent system**. A system of linear equations in two variables which has an infinitely many solutions is called, **a dependent system**. In the latter case, both equations determine one and the same line. A system of linear equations in two variables with at least one solution is called, **a consistent system**.

Classroom Exercises : Solve the system of equations. If the system is dependent, then describe the solution set. Draw the graph of the two lines.

(a) $$\begin{aligned} 6x - 9y &= 18 \\ 4x - 6y &= 12 \end{aligned}$$

(b) $$\begin{aligned} 5x + 10y &= 20 \\ -3x - 6y &= -12 \end{aligned}$$

Some systems **do not have any solutions**. Such a system is called **an inconsistent system**. The addition method in such cases **elminates both the variables and results in a false statement**.

Example 6: Consider the system

$$\begin{aligned} 2x - y &= 4 \\ 6x - 3y &= -12. \end{aligned}$$

We eliminate variable x as follows:

$$\begin{aligned} 2x - y = 4 &\Longrightarrow^{\times(-3)} -6x + 3y = -12 \\ 6x - 3y = -12 &\Longrightarrow^{\times 1} 6x - 3y = -12. \end{aligned}$$

Adding the two systems gives us $0 = -24$ which is a **false statement**. Therefore, there is no solution to the given system. Here is the geometric explanation:

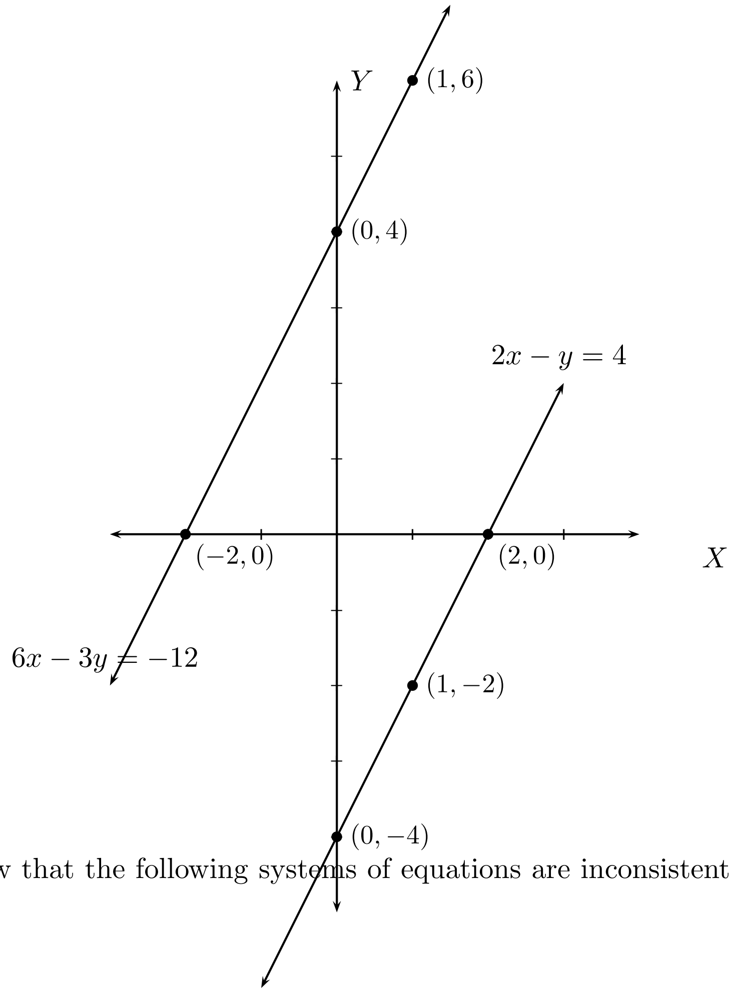
Graphing the two linear equations gives us two parallel lines. Hence, the two lines do not intersect.

$2x - y = 4$

x	y	
0	-4	y-intercept
2	0	x-intercept
1	-2	third point

$6x - 3y = -12$

x	y	
0	4	y-intercept
-2	0	x-intercept
1	6	third point



Classroom Exercises : Show that the following systems of equations are inconsistent. Draw the two graphs.

(a)
$$3x - 9y = 12$$
$$x - 3y = 3$$

(b)
$$2x + y = 4$$
$$6x + 3y = 6$$

The following table summarizes what we have observed.

System	Graphs of lines	Number of solutions
Consistent and Independent	The two lines intersect at a unique point	A unique solution
Consistent and Dependent	The two lines are one and the same line	Infinitely many solutions
Inconsistent	The two lines are parallel	No solution

Systems of equations can be used to solve interesting problems.

Example 1: There are some quarters and nickels in a jar. The total value of the coins is \$ 4.00. If there are two more nickels than quarters, then find the number of quarters.

Let the number of quarters $= q.$

Let the number of nicles $= n.$

The value of q number of quarters $= 25q$ ¢.

The value of n number of quarters $= 5n$ ¢.

Total value is \$ 4.00 (or 400 ¢).

That is, $25q + 5n = 400.$

There are two more nickels than quarters.

That is, $n = q + 2.$

Subtract q from both sides to get $-q + n = 2.$

This gives us a system of linear equations $\begin{cases} 25q + 5n = 400 \\ -q + n = 2. \end{cases}$

The reader can now solve this system to find q and n.

The answer is $q = 13$ and $n = 15$.

To answer the question, the number of quarters is 13.

Example 2: Rob wants to buy flour and sugar at a grocery store. If two pounds of flour and six pounds of sugar cost \$ 15.16, while three pounds of flour and two pounds of sugar cost \$ 7.83, then what is the price per pound of sugar?

We convert the dollar amounts to cents so as to maintain consistency.

Let the price per pound of flour $= f$ ¢.

Let the price per pound of sugar $= s$ ¢.

The cost of two pounds of flour $= 2f$ ¢.

The cost of six pounds of sugar $= 6f$ ¢.

Two pounds of flour and six pounds of sugar cost \$ 15.16 (or, 1516 ¢).

That is, $2f + 6s = 1516.$

The cost of three pounds of flour $= 3f$ ¢.

The cost of two pounds of sugar $= 2f$ ¢.

Three pounds of flour and two pounds of sugar cost \$ 7.83 (or, 783 ¢).

That is, $3f + 2s = 783.$

Thus, we get a system of equations $\begin{cases} 2f + 6s = 1516 \\ 3f + 2s = 783. \end{cases}$

The reader can now solve this system of equations to find f and s.
The answer is, $f = 119$ and $s = 213$.
To answer the question, the price per pound of sugar is \$ 2.13.

Classroom Exercises:

(a) I have some dimes and quarters in my pocket. The number of dimes is 5 more than the number of quarters. The total amount of money is \$ 4.70. Find the number of dimes and quarters.

(b) Seven pounds of wheat flour and five pounds of rice flour together cost \$ 10.30, whereas, five pounds of wheat flour and seven pounds of rice flour togehter cost \$ 10.10. Find the price per pound of wheat flour.

4.7.1 Homework Exercises

1. Solve by the graphing method and check that your solutions are indeed correct:

(a)
$$3x - 4y = -5$$
$$2x + y = 4$$

(b) $$\begin{aligned} x - 2y &= 1 \\ 2x - y &= -1 \end{aligned}$$

(c) $$\begin{aligned} 4x + y &= -7 \\ -3x - 2y &= 4 \end{aligned}$$

2. Solve by the addition method and check that your solutions are indeed correct:

(a) $$\begin{aligned} 3x - 4y &= 2 \\ 2x + y &= -4 \end{aligned}$$

(b) $$\begin{aligned} x - 2y &= 11 \\ 2x - y &= -5 \end{aligned}$$

(c) $$\begin{aligned} 4x + y &= 7 \\ -3x - 2y &= 6 \end{aligned}$$

3. Show that the following systems are dependent by the addition method and describe their solutions. Further, graph their lines.

(a) $$\begin{aligned} 3x - 4y &= 2 \\ 6x - 8y &= 4 \end{aligned}$$

(b) $$\begin{aligned} 3x - 3y &= -15 \\ 2x - 2y &= -10 \end{aligned}$$

(c) $$\begin{aligned} 6x + 12y &= -12 \\ -3x - 6y &= 6 \end{aligned}$$

4. Show that the following systems are inconsistent by the addition method. Further, graph their lines.

(a) $$\begin{aligned} 3x - 4y &= 2 \\ 6x - 8y &= -4 \end{aligned}$$

(b) $$\begin{aligned} 3x - 3y &= -15 \\ 2x - 2y &= 5 \end{aligned}$$

(c) $$\begin{aligned} 6x + 12y &= -12 \\ -3x - 6y &= -6 \end{aligned}$$

5. There are some nickels and dimes in a jar. The number of nickels is one less than twice the number of dimes. If the total value of the coins is \$ 8.55, then find the number of dimes.

6. A gardener wants to buy two kinds of fertilizers, say, fertilizer A and fertilizer B. If two pounds of fertilizer A and three pounds of fertilizer B cost \$ 33.52, while one pound of fertilizer A and two pounds of fertilizer B cost \$ 20.27, then find the price per pound of fertilizer A.

Chapter 5

Polynomials

5.1 Integer exponents

By now you are familiar with the concepts of a *variable*, *an algebraic expression*, and *evaluating of algebraic expressions*. In this chapter we will be studying polynomials over real numbers. Before working towards polynomials, let us understand integer exponents of variables.

For integer n positive,

$$\begin{aligned} x^n &= x \cdot x \cdot x \cdots x && \textit{Here } x \textit{ appears } n \textit{ times} \\ x^0 &= 1 \\ x^{-n} &= \frac{1}{x^n} = \frac{1}{x \cdot x \cdot x \cdots x} && \textit{Here } x \textit{ appears } n \textit{ times} \end{aligned}$$

We thus have the following rules for n, m integers, which can be carefully checked, but we skip the details:

$$\begin{aligned} (xy)^n &= x^n y^n; \\ \left(\frac{x}{y}\right)^n &= \frac{x^n}{y^n}; \\ x^n \cdot x^m &= x^{n+m}; \\ (x^n)^m &= x^{nm}; \\ \frac{x^n}{x^m} &= x^{n-m}; \\ x^{-n} = \frac{1}{x^n} \quad &\textit{and} \quad \frac{1}{x^{-n}} = x^n. \end{aligned}$$

Using these rules we can simplify some algebraic expressions.

Examples :

- $x^3 \cdot x^4 = x^7$
- $(y^2)^{12} = y^{2 \cdot 12} = y^{24}$
- $(y^{78})^0 = 1$
- $\dfrac{x^2 \cdot x^{13} \cdot x^{-7}}{x^{12} \cdot x^{-4}} = \dfrac{x^2 \cdot x^{13} \cdot x^4}{x^{12} \cdot x^7} = \dfrac{x^{2+13+4}}{x^{12+7}} = \dfrac{x^{19}}{x^{19}} = 1.$
- $\dfrac{x^{20} \cdot y^{14} \cdot x^{-7} \cdot y^{12} \cdot (z^2)^{-3}}{y^{-6} \cdot y^7 \cdot x^{10}}$

$$\begin{aligned}
\frac{x^{20} \cdot y^{14} \cdot x^{-7} \cdot y^{12} \cdot (z^2)^{-3}}{y^{-6} \cdot y^7 \cdot x^{10}} &= \frac{x^{20} \cdot y^{14} \cdot x^{-7} \cdot y^{12} \cdot z^{2\times -3}}{y^{-6} \cdot y^7 \cdot x^{10}} \\
&= \frac{x^{20} \cdot y^{14} \cdot y^6 \cdot y^{12} \cdot z^{-6}}{x^7 \cdot y^7 \cdot x^{10}} \\
&= \frac{x^{20} \cdot y^{14+6+12}}{x^{7+10} \cdot y^7 \cdot z^6} = \frac{x^{20} \cdot y^{32}}{x^{17} \cdot y^7 \cdot z^6} \\
&= \frac{x^{20-17} y^{32-7}}{z^6} = \frac{x^3 y^{25}}{z^6}.
\end{aligned}$$

Classroom Exercises : Simplify and rewrite with only positive exponents:

$$y^{12} \cdot y \cdot y^0 \cdot y^{23}; \quad \frac{y^{32}x^{19}}{x^3 y^{42}}; \quad (a^{412}a^{22})^0; \quad \frac{(r^2)^3(s^{-3})^4 t^{12}}{(r^3)^2 s}; \quad \frac{a^{12} \cdot b^{-7} \cdot a^{-9} \cdot c^{17}}{a^{-12} \cdot b^3 \cdot c^{12} \cdot b^{18}}.$$

5.1.1 Homework Exercises

1. Simplify and rewrite with only positive exponents:

   (a) $x^2 \cdot x^{13} \cdot x^0 \cdot x \cdot x^{21}$

   (b) $y^3 \cdot y^{-2} \cdot y^{-5} \cdot x$

   (c) $(x^4)^{-2} \cdot (y^2)^5$

   (d) $\left(\dfrac{x^3 + y^3}{x^2 - y^{22}}\right)^0$

   (e) $\dfrac{x^{14} \cdot x^{-9} \cdot y^{19} \cdot z^{-3}}{(x^{-2})^4 \cdot z^5 \cdot y^{22}}$

5.2 Introduction to Polynomials

Polynomials are algebraic expressions obtained by applying addition, subtraction, or multiplication to variables and real numbers. Notice that division is missing from the above list of operations.

Examples :

$3x^2y^4 - 12xy^4 - \frac{1}{12}y + \sqrt{5}x$	is a polynomial in variables x, y. Notice that $4, -12, -\frac{1}{12}, \sqrt{5}$ are all real numbers.
$\frac{x^{12}}{2+13y}$	is not a polynomial since an expression in y is in the denominator.
$3x^2 + 4x - 7$	is a polynomial in one variable x.
3	is a polynomial which can be thought of as $3x^0$. A polynomial with no variable is called a **constant polynomial**.
t	is a polynomial in one variable t.
$2e + 4$	is a polynomial in one variable e.
$2\|e+4\|$	is not a polynomial because the variable is inside $\|\quad\|$.
$x^{100} - 25x^{29} + \sqrt{3}x^4 - \sqrt{x}$	is not a polynomial because of the term $\sqrt{x}$.
$x^{100} - 25x^{29} + \sqrt{3}x^4 - \sqrt{2}x$	is a polynomial.

That is, a polynomial is a finite sum of terms where each term can be written as $ax^ny^m\cdots t^k$ where a is a real number (poitive or negative or zero), and $n, m, \cdots, k$ are whole numbers $(0, 1, 2, \cdots)$ and $x, y, \cdots, t$ are a finite set of variables. The real number a is called the **coefficient** of the term $ax^ny^m\cdots t^k$.

In particular, a polynomial in one variable x can be written in the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad \textit{where the coefficients, } a_0, a_1, \cdots, a_n, \textit{ are real numbers.}$$

Classroom Exercises : Which of the following are polynomials. If a certain expression is not a polynomial, then explain why.

(1) $2xt - 4t^2 + 12xy - \sqrt{2}$ (2) $\frac{3x-y}{x}$ (3) $-\frac{1}{24}$

(4) $45|z+5|$ (5) 0 (6) $2x^2 - 7x + \sqrt{y^2+9}$

A polynomial with one nonzero term is called a **monomial**. A polynomial with two nonzero terms is called a **binomial**. A polynomial with three nonzero terms is called a **trinomial**.

Examples

Monomials	$2x, \quad 3t^2, \quad 4y^{100}, \quad -\sqrt{3}e^{12}, \quad 10, \quad -\frac{2}{7}$
Binomials	$3x+2y, \quad 4x+5, \quad -\sqrt{5}e^4+y, \quad \frac{1}{3}x^7-x$
Trinomials	$22x^3-4x+5, \quad \sqrt{7}x+y+z, \quad 2+x+x^2$

Classroom Exercises : Write down three examples each of monomials, binomials, and trinomials.
The **degree** of a term, $ax^n y^m \cdots t^k$ is $n+m+\cdots+k$. For example,

Term	Degree
$-3x^4y^5z^7$	$4+5+7=16$
$7x^5$	5
32	0, since $32=32x^0$
$3z$	1, since $3z=3z^1$

Classroom Exercises : What are the degrees of the following terms?

$$2x, \quad 3t^2, \quad 4y^{100}, \quad -\sqrt{3}e^{12}, \quad 10, \quad -\frac{2}{7}.$$

The **degree of a polynomial** is the maximum of the degrees of its nonzero terms.

Examples :

Polynomial	Degree
$-2x^2y^3z+7xyz+10t^5$	6, since the degree of $-2x^2y^3z$ is 6.
$-2x^2y^3z+7xyz+10t^9$	9, since the degree of $10t^9$ is 9.

Classroom Exercises : Write down a polynomial each of degree

- 20
- 3
- 0
- 1

A polynomial of degree 1 is called a **linear** polynomial. A polynomial of degree 2 is called a **quadratic** polynomial. A polynomial of degree 3 is called a **cubic** polynomial. Notice that, we could have a linear polynomial in many variables. For example, $x+y+z$ is a linear polynomial in three variables, while xyz is a cubic.

Two terms are said to be **like terms** when they have the same variables, and the corresponding variables have the same exponent. For example, $3x^2y^2z$ and $\frac{1}{2}x^2y^2z$ are like terms. The order in which the variables are written is not important. For example, $4x^2y$ and $-10yx^2$ are like terms. Notice that $3x^2y$ and $3xy^2$ are **not like** terms since in the first term, x has exponent 2, while in the second term, x has exponent 1; similarly, in the first term, y has exponent 1, while in the second term, y has exponent 2.

We can **combine like terms**:

- $$\begin{aligned} 3x^2y^2z + 7x^2y^2z &= (3+7)x^2y^2z \quad \text{(Distributive law)} \\ &= 10x^2y^2z \end{aligned}$$

- $$\begin{aligned} -4x^3y^2 + \frac{2}{7}x^3y^2 &= \left(-4 + \frac{2}{7}\right)x^3y^2 \quad \text{(Distributive law)} \\ &= \left(-\frac{28}{7} + \frac{2}{7}\right)x^3y^2 \quad \text{(Common denominator)} \\ &= -\frac{26}{7}x^3y^2 \\ &= -\left(3\frac{5}{7}\right)x^3y^2 \quad \text{(Final answer)} \end{aligned}$$

- $$\sqrt{3}xyz + 7xyz = (\sqrt{3}+7)xyz \quad \text{(Distributive law and the final answer.)}$$

Notice that $(\sqrt{3}+7)$ cannot be simplified any further.

Classroom Exercises : Combine like terms in the following:

(a) $-12t^2r + 14t^2r + 6t^2r$

(b) $\frac{2}{5}xyz - 4xyz$

(c) $\sqrt{7}e + 4e - 7e$

Given a polynomial in which all the like terms are combined, we can write it in ascending order of the degrees of the terms, or in descending order of the degrees of the terms. When a polynomial consists of more than one variable, then a variable needs to be specified.

Examples :

$$3x^2 + 5x^3 - 7x + 11 - 20x^4 = 11 - 7x + 3x^2 + 5x^3 - 20x^4 \quad \text{(Ascending order)}$$

$$= -20x^4 + 5x^3 + 3x^2 - 7x + 11 \quad \text{(Descending order)}$$

In the following example, we arrange the given polynomial in the ascending order of the degrees of x.

$$11x^2y^3 + 4xy - 5x^2y + 7x^3 + 6xy^2 - y^3 = -y^3 + 4xy + 6xy^2 - 5x^2y + 11x^2y^3 + 7x^3$$

$$= -y^3 + (4y + 6y^2)x + (-5y + 11y^3)x^2 + 7x^3$$

Notice that within the parantheses, the polynomials themselves are in ascending order of the degrees of y.

Similarly, the same polynomial can be written in the descending order of the degrees of x (with the polynomials within the parantheses written in descending order also) as follows:

$$11x^2y^3 + 4xy - 5x^2y + 7x^3 + 6xy^2 - y^3 = 7x^3 + (11y^2 - 5y)x^2 + (6y^2 + 4y)x - y^3$$

Classroom Exercises : Write the following polynomials as indicated:

(a) $-9x^{10} + x + 4 - 6x^2 + 3x^4$ (in ascending order).

(b) $-9x^{10} + x + 4 - 6x^2 + 3x^4$ (in descending order).

(c) $4m^2n - 7mn^3 + 11m^2n^3 - 14 - 2mn + 3n^2 - m$ (in ascending order of n).

(d) $4m^2n - 7mn^3 + 11m^2n^3 - 14 - 2mn + 3n^2 - m$ (in descending order of m).

Given two polynomials, we can add them, subtract one from the other, multiply them, and in the case of polynomials in one variable, divide one by the (nonzero) other with a possible remainder just as in the case of integers. In the following sections we will learn about these operations.

5.2.1 Homework Exercises

1. Which of the following are polynomials. If a certain expression is not a polynomial, then explain why.

   (a) $23x^5 - 7x^4 - \frac{2}{3}x^3 + \sqrt{2}x^2 + x - 7$

   (b) $32|t|$

   (c) $27r^2s^7 - \frac{1}{rs}$

   (d) z

(e) $\sqrt{x^2 + y^2}$

(f) $2t^2s^3 + 5ts - 89$

2. Write down three examples of monomials.

3. Write down three examples of binomials.

4. Write down three examples of trinomials.

5. Divide the following into three groups: monomials, binomials, and trinomials.

- $3x^2 - 5x + 2$
- 43
- $-23t^4 + x - 11$
- $33x + y$
- $-7x + \sqrt{4}$
- $\sqrt{5} - \frac{1}{3} + 8$
- $(44 + \sqrt{2})x + y$
- $2x + 7y + 2$
- x

6. What are the degrees of the following terms?

$$x, \quad -5t^3, \quad 4y^{1,200}, \quad -\sqrt{3}z^7, \quad 0, \quad -\frac{5}{12}.$$

7. Write down a polynomial each of degree

- 12
- 4
- 0
- 0

8. Combine like terms in the following:

(a) $-s^2t + 4s^2t + 5s^2t$

(b) $-\frac{3}{7}x - 8x$

(c) $-\sqrt{2}f - 4f + 11f$

9. Write down the given polynomials as indicated:

(a) $-2x + x^4 - 4x^3 + 7x^2 + 11$ (in ascending order).

(b) $-2x + x^4 - 4x^3 + 7x^2 + 11$ (in descending order).

(c) $3rn - 7rn^2 + 15r^3n^3 - 14r - 2r^2n + 6n^2 - n$ (in ascending order of n).

(d) $3rn - 7rn^2 + 15r^3n^3 - 14r - 2r^2n + 6n^2 - n$ (in descending order of r).

5.3 Addition and Subtraction of polynomials

Here we will first learn to add two polynomials. This process is the same as combining like terms.
Examples :

- $$\begin{aligned}&(4+2x-3y)+(-3+4x+8y)\\=&(4+(-3))+(2x+4x)+(-3y+8y)\\=&(4+(-3))+(2+4)x+(-3+8)y\\=&1+6x+5y.\end{aligned}$$

- $$\begin{aligned}&\left(2x^2y^3-\frac{2}{3}x^2y+4xy+11\right)+\left(-2x^2y^3+3x^2y+xy-y+\sqrt{3}\right)\\=&(2+(-2))x^2y^3+\left(-\frac{2}{3}+3\right)x^2y+(4+1)xy-y+(11+\sqrt{3})\\=&0x^2y^3+\left(-\frac{2}{3}+\frac{9}{3}\right)x^2y+5xy-y+(11+\sqrt{3})\\=&\frac{7}{3}x^2y+5xy-y+(11+\sqrt{3})\\=&\left(2\frac{1}{3}\right)x^2y+5xy-y+(11+\sqrt{3})\end{aligned}$$

Classroom Exercises : Find the following sums of polynomials:

(a) $(12x^2 - 3x + 5) + (-3x^2 + 4)$

(b) $(3x^2y - 5xy + y^2) + (7x^2y + 5xy + 7y)$

(c) $\left(\frac{1}{5}x^2z^2 + \sqrt{3}z^2\right) + \left(-2x^2z^2 + 12xz + z - \sqrt{3}\right)$

(d) $(4s^2t - 7st^2 + 11st + 7) + \left(\frac{2}{3}s^2t - 3st^2 + 8st - 9\right)$

The process of subtraction is similar, except, one needs to **distribute** the subtraction.
Examples :

- $$\begin{aligned} &(4 + 2x - 3y) - (-3 + 4x + 8y) \\ =&4 + 2x - 3y - (-3) - (+4x) - (+8y) \qquad \textit{(Distribute the subtraction)} \\ =&4 + 2x - 3y + 3 - 4x - 8y \\ =&(4 + 3) + (2x - 4x) + (-3y - 8y) \\ =&7 - 2x - 11y. \end{aligned}$$

- $$\begin{aligned} &\left(2x^2y^3 - \frac{2}{3}x^2y + 4xy + 11\right) - \left(-2x^2y^3 + 3x^2y + xy - y + \sqrt{3}\right) \\ =&2x^2y^3 - \frac{2}{3}x^2y + 4xy + 11 - (-2x^2y^3) - (+3x^2y) - (+xy) - (-y) - (+\sqrt{3}) \\ &\qquad\qquad \textit{(Distribute the subtraction)} \\ =&2x^2y^3 - \frac{2}{3}x^2y + 4xy + 11 + 2x^2y^3 - 3x^2y - xy + y - \sqrt{3} \\ =&(2 + 2)x^2y^3 + \left(-\frac{2}{3} - 3\right)x^2y + (4 - 1)xy + y + (11 - \sqrt{3}) \\ =&4x^2y^3 + \left(-\frac{2}{3} - \frac{9}{3}\right)x^2y + 3xy + y + (11 - \sqrt{3}) \\ =&4x^2y^3 + \left(-\frac{11}{3}\right)x^2y + 3xy + y + (11 - \sqrt{3}) \\ =&4x^2y^3 - \left(3\frac{2}{3}\right)x^2y + 3xy + y + (11 - \sqrt{3}) \end{aligned}$$

Classroom Exercises : Find the following differences of polynomials:

(a) $(12x^2 - 3x + 5) - (-3x^2 + 4)$

(b) $(3x^2y - 5xy + y^2) - (7x^2y + 5xy + 7y)$

(c) $\left(\frac{1}{5}x^2z^2 + \sqrt{3}z^2\right) - \left(-2x^2z^2 + 12xz + z - \sqrt{3}\right)$

(d) $(4s^2t - 7st^2 + 11st + 7) - \left(\frac{2}{3}s^2t - 3st^2 + 8st - 9\right)$

5.3.1 Homework Exercises

1. Find the following sums of polynomials:

   (a) $(z^2 - 5z + 7) + (-3z^2 - 4z + 7)$
   (b) $(-5w^3 + 7w^2 - 3w + 4) + (11w^3 + w^2 - w + 1)$
   (c) $(4r^2t^2 - 9rt^2 - 2t^2) + (7r^2t^2 + 5rt + 7t^2)$
   (d) $(7x^3y^2 - 5x^2y^3 + 4x^2y - x^2 + y + 2) + (-x^3y^2 + 3x^2y^3 + x + y - 5)$
   (e) $\left(\frac{3}{5}st^2 + \sqrt{3}st + t^2 - 6\right) + \left(-2st^2 + 2st - \sqrt{3}t^2 + 3\right)$

2. Find the following differences of polynomials:

   (a) $(z^2 - 5z + 7) - (-3z^2 - 4z + 7)$
   (b) $(-5w^3 + 7w^2 - 3w + 4) - (11w^3 + w^2 - w + 1)$
   (c) $(4r^2t^2 - 9rt^2 - 2t^2) - (7r^2t^2 + 5rt + 7t^2)$
   (d) $(7x^3y^2 - 5x^2y^3 + 4x^2y - x^2 + y + 2) - (-x^3y^2 + 3x^2y^3 + x + y - 5)$
   (e) $\left(\frac{3}{5}st^2 + \sqrt{3}st + t^2 - 6\right) - \left(-2st^2 + 2st - \sqrt{3}t^2 + 3\right)$

5.4 Multiplication of polynomials

Here we will first learn to multiply two polynomials. In this process we will use the distributive laws, and then combine like terms, if any.

Examples :

- $3xyz \cdot 5x^2y^3z = 15x^3y^4z^2$ *(Multiply the coefficients and the variables separately)*

- $\frac{2}{3}yz \cdot 5x^2y^7 = \frac{10}{3}x^2y^8z = 3\frac{1}{3}x^2y^8z$ *(Multiply the coefficients and the variables separately)*

- $4(3x^2 + 5x - 7)$ *(Use distributive law)*

  $= 12x^2 + 20x - 28$ *(This is the final answer.)*

- $\frac{1}{4}x^3(3x^2 + 5y - 7)$ *(Use distributive law)*
  $= \left(\frac{1}{4}x^3 \cdot 3x^2\right) + \left(\frac{1}{4}x^3 \cdot 5y\right) - \left(\frac{1}{4}x^3 \cdot 7\right)$
  $= \frac{3}{4}x^5 + \frac{5}{4}x^3y - \frac{7}{4}x^3$
  $= \frac{3}{4}x^5 + \left(1\frac{1}{4}\right)x^3y - \left(1\frac{3}{4}\right)x^3$ *(This is the final answer.)*

- $(4x^2 - 3y)(3x^2 + 5y)$ *(Use distributive law)*

  $= (4x^2 - 3y) \cdot 3x^2 + (4x^2 - 3y) \cdot 5y$ *(Use distributive law again)*

  $= 12x^4 - 9x^2y + 20x^2y - 15y^2$ *(Combine the like terms $-9x^2y + 20x^2y$)*

  $= 12x^4 + 11x^2y - 15y^2$ *(This is the final answer.)*

- $(4x^2 + 3xy - 5y^2)(5x^2 - y^2 + 3)$ *(Use distributive law)*
  $= (4x^2 + 3xy - 5y^2) \cdot 5x^2 - (4x^2 + 3xy - 5y^2) \cdot y^2 + (4x^2 + 3xy - 5y^2) \cdot 3$
  (Use distributive law again)
  $= 20x^4 + 15x^3y - 25x^2y^2 - (4x^2y^2 + 3xy^3 - 5y^4) + 12x^2 + 9xy - 15y^2$
  (We pay careful attention to the negative)
  $= 20x^4 + 15x^3y - 25x^2y^2 - 4x^2y^2 - 3xy^3 + 5y^4 + 12x^2 + 9xy - 15y^2$
  (Now combine like terms)
  $= 20x^4 + 15x^3y - 29x^2y^2 - 3xy^3 + 5y^4 + 12x^2 + 9xy - 15y^2$
  (This is the final answer.)

- $\left(\frac{2}{5}x^2 - \frac{1}{3}x + y\right)(-2y - 3x + 4)$ *(Use distributive law)*

$$= -\left(\frac{2}{5}x^2 - \frac{1}{3}x + y\right) \cdot 2y - \left(\frac{2}{5}x^2 - \frac{1}{3}x + y\right) \cdot 3x + \left(\frac{2}{5}x^2 - \frac{1}{3}x + y\right) \cdot 4$$

(Use distributive law again)

$$= -\left(\frac{4}{5}x^2y - \frac{2}{3}xy + 2y^2\right) - \left(\frac{6}{5}x^3 - \frac{3}{3}x^2 + 3xy\right) + \frac{8}{5}x^2 - \frac{4}{3}x + 4y$$

(We pay careful attention to the negatives)

$$= -\frac{4}{5}x^2y + \frac{2}{3}xy - 2y^2 - \frac{6}{5}x^3 + \frac{3}{3}x^2 - 3xy + \frac{8}{5}x^2 - \frac{4}{3}x + y \quad \textit{(Combine like terms)}$$

$$= -\frac{4}{5}x^2y + \left(\frac{2}{3} - 3\right)xy - 2y^2 - \frac{6}{5}x^3 + \left(\frac{3}{3} + \frac{8}{5}\right)x^2 - \frac{4}{3}x + 4y$$

$$= -\frac{4}{5}x^2y - \left(2\frac{1}{3}\right)xy - 2y^2 - \left(1\frac{1}{5}\right)x^3 + \left(2\frac{3}{5}\right)x^2 - \left(1\frac{1}{3}\right)x + 4y \quad \textit{(The answer.)}$$

Note, $\left(\frac{2}{3} - 3\right) = -2\frac{1}{3}$, $\left(\frac{3}{3} + \frac{8}{5}\right) = \left(1 + 1\frac{3}{5}\right) = 2\frac{3}{5}$, *and* $\frac{4}{3} = 1\frac{1}{3}$.

Some important formulae :

- Square of a sum

$$\begin{aligned}
(x+y)^2 &= (x+y)(x+y) \\
&= (x+y)\cdot x + (x+y)\cdot y && \textit{(Use distributive law)} \\
&= x^2 + xy + xy + y^2 && \textit{(Use distributive law again)} \\
&= x^2 + 2xy + y^2 && \textit{(Combining like terms gets us the final answer.)} \\
(x+y)^2 &= x^2 + 2xy + y^2
\end{aligned}$$

- Square of a difference

$$\begin{aligned}
(x-y)^2 &= (x-y)(x-y) \\
&= (x-y)\cdot x - (x-y)\cdot y && \textit{(Use distributive law)} \\
&= x^2 - xy - (xy - y^2) && \textit{(Use distributive law again)} \\
&= x^2 - xy - xy + y^2 && \textit{(Pay careful attention to the negative)} \\
&= x^2 - 2xy + y^2 && \textit{(Combining like terms gets us the final answer.)} \\
(x-y)^2 &= x^2 - 2xy + y^2
\end{aligned}$$

- Difference of squares

$$
\begin{aligned}
(x+y)(x-y) &= (x+y)\cdot x - (x+y)\cdot y && \textit{(Use distributive law)}\\
&= x^2 + xy - (xy + y^2) && \textit{(Use distributive law again)}\\
&= x^2 + xy - xy - y^2 && \textit{(Pay careful attention to the negative)}\\
&= x^2 - y^2 && \textit{(Combining like terms gets us the final answer.)}\\
(x+y)(x-y) &= x^2 - y^2
\end{aligned}
$$

Classroom Exercises : Perform the following multiplications:

(a) $2x^3y^2 \cdot 5xyz$

(b) $\dfrac{3}{5}abc \cdot \dfrac{2}{7}a^2bcd$

(c) $2(5x^2 - 6x + 2)$

(d) $-4(3x^2 + 7x - 5)$

(e) $-2x(3x^2 + 7x - 5)$

(f) $\dfrac{1}{3}(2y^2 + 5y - 3)$

(g) $\dfrac{2}{3}y^3(2y^2 + 5y - 3)$

(h) $(3x - 2y)(x^2 - 2xy + 3y)$

(i) $\left(\dfrac{2}{3}y + 5\right)(4xy + x + y)$

(j) $(x - y)(x^2 + xy + y^2)$ (Difference of cubes formula)

(k) $(x + y)(x^2 - xy + y^2)$ (Sum of cubes formula)

5.4.1 Homework Exercises

Find the following products:

1. $3x^2y^9z^3 \cdot 5x^2y^2z^8$
2. $-7a^2b^7 \cdot 4a^7b^9$
3. $-3(5y^2 + 3y - 7)$
4. $\dfrac{2}{5}\left(2x^2 - 3x + 5\right)$

5. $-\dfrac{1}{2}n(4m^2 + 7mn)$

6. $\dfrac{2}{3}y(5y^3 + 4y^2 - 6y + 3)$

7. $\left(\dfrac{2}{3}x + y\right)(3x^2 + 4xy - y^2)$

8. $\left(\dfrac{1}{5}x + \dfrac{2}{3}y\right)(2x + 7y)$

9. (Square of a sum) $(a + b)^2$

10. (Square of a difference) $(a - b)^2$

11. (Difference of squares) $(a + b)(a - b)$

12. (Difference of cubes) $(a - b)(a^2 + ab + b^2)$

13. (Sum of cubes) $(a + b)(a^2 - ab + b^2)$

14. (Cube of a sum) $(a + b)^3$

15. (Cube of a difference) $(a - b)^3$

5.5 Division by a monomial

Here we learn to divide a polynomial by a monomial. Recall that division by a number is multiplication by its reciprocal. In the same way, division by a monomial is multiplication by its reciprocal.

Examples :

- $$\begin{aligned} 12a^4b^3c^5 \div (3a^2b^3c) &= 12a^4b^3c^5 \times \frac{1}{3a^2b^3c} && \textit{(Division is multiplication by the reciprocal)} \\ &= \frac{12a^4b^3c^5}{3a^2b^3c} \\ &= 4a^2c^4 && \textit{(This is the final answer.)} \end{aligned}$$

- $(a+b+c) \div (3abc) = (a+b+c) \times \dfrac{1}{3abc}$ *(Division is multiplication by the reciprocal)*

$$= a \cdot \frac{1}{3abc} + b \cdot \frac{1}{3abc} + c \cdot \frac{1}{3abc} \quad \textit{(Use distributive law)}$$
$$= \frac{a}{3abc} + \frac{b}{3abc} + \frac{c}{3abc}$$
$$= \frac{1}{3bc} + \frac{1}{3ac} + \frac{1}{3ab} \quad \textit{(This is the final answer.)}$$

- $(2x^2y + 3xy^2 - 4xy) \div (6xy) = (2x^2y + 3xy^2 - 4xy) \times \dfrac{1}{6xy}$

(Division is multiplication by the reciprocal)

$$= 2x^2y \cdot \frac{1}{6xy} + 3xy^2 \cdot \frac{1}{6xy} - 4xy \cdot \frac{1}{6xy} \quad \textit{(Use distributive law)}$$
$$= \frac{2x^2y}{6xy} + \frac{3xy^2}{6xy} - \frac{4xy}{6xy}$$
$$= \frac{x}{3} + \frac{y}{2} - \frac{2}{3} \quad \textit{(This is the final answer.)}$$

Classroom Exercises : Perform the following divisions:

(a) $25a^9b^3c^4 \div (10a^7b^5c^6)$

(b) $(4x^2y^3z + 3x^3y^3z - 7xyz) \div (5xyz)$

(c) $(28r^3s^2t - 49r^5s^4t^2 - 2r^2s^2t^2) \div (14r^2st)$

5.5.1 Homework Exercises

Perform the following divisions:

1. $75a^6b^5c^4 \div (20a^7b^4c^6)$
2. $44x^8y^3r^9 \div (26x^3y^9r^{10})$
3. $(4x^2y^3z + 3x^3y^3z - 7xyz) \div (5xyz)$
4. $(8a^2b^5c^3 + 6a^2b^3c^2 - 3abc^2) \div (6ab)$
5. $(3r^4s^3t^2 - 4r^2s^3t^2 + 5rs^4t) \div (12s^2t)$
6. $(18x^3y^4a^5 - 33x^3ya^2 + 12x^3ya) \div (15x^2y)$
7. $(26a^2b^3 + 24a^5b^2c^3 - 8a^3b^2c) \div (13a^2)$

8. $(22x^2 - 18x^2y) \div (6x)$

5.6 Factoring polynomials

When we write $2 \times 3 = 6$, we say that the product of 2 and 3 is 6, and that 2 and 3 are factors of 6. A prime number is a natural number other than 1 whose only factors are itself and 1.

Analogously, when $(x + y)(x - y) = x^2 - y^2$, we say that the product of $(x + y)$ and $(x - y)$ is $x^2 - y^2$, and that $(x + y)$ and $(x - y)$ are factors of $x^2 - y^2$. An irreducible polynomial is a polynomial other than the constant polynomial 1, whose only factors are itself and 1.

To factor a polynomial is to write it as a product of irreducible polynomials. For this we go through several steps. The reader is urged to check every step before moving on to the next one.

5.6.1 The Greatest Common Factor (GCF)

Recall from your previous classes, the concept of the Greatest Common Factor, or GCF, of a list of natural numbers. It is the greatest natural number which is a factor of every number in the list.

Examples :

- GCF(12,33,18) = 3.
- GCF(15,10,20) = 5.
- GCF(44,6,12) = 2.
- GCF(12,18,30)= 6.

Classroom Exercises : What is the GCF in each of the cases?

(a) $\{12, 15, 18\}$ (b) $\{15, 30, 50\}$ (c) $\{12, 25\}$

The **Greatest Common Factor (GCF)** of two or more monomials with integer coefficients is the product of

- the GCF of the coefficients, and
- the highest power of every variable that is a factor of every monomial.

Examples :

- $\text{GCF}(12x^2y^3, 33x^4y^2, 18x^3y^3) = 3x^2y^2$. Note that $\text{GCF}(12, 33, 18) = 3$. Further, x^2 is the highest power of x that is a factor of x^2y^3, x^4y^2, x^3y^3; similarly, y^2 is the highest power of y that is a factor of x^2y^3, x^4y^2, x^3y^3.

- $\text{GCF}(15a^4b^7, 10a^3b^9, 20a^6b^4) = 5a^3b^4$. Again, the $\text{GCF}(15, 10, 20) = 5$, while a^3 is the highest power of a that is a factor of every monomial, and b^4 is the highest power of b that is a factor of every monomial.

- $\text{GCF}(44r^3s^4t^2, 6r^5s^3t^7, 12r^4s^3t^5) = 2r^3s^3t^2$.

Classroom Exercises : What is the GCF in each of the cases?

(a) $\{12a^3b^4c^2, 15a^5b^3c^5, 18a^3b^2c^7\}$ (b) $\{15x^3, 30x^7, 50x^4\}$ (c) $\{12r^9s^4t^4, 25r^4t\}$

Now we are ready to factor. The very first step in factoring any polynomial is to factor out the GCF of its terms using the distributive law of multiplication over addition/subtraction.

Examples : Factor the following:

- $12x^2y^3+33x^4y^2+18x^3y^3 = 3x^2y^2(4y+11x^2+6xy)$. Note that we have used the distributive law of multiplication over addition. To factor $(4y + 11x^2 + 6xy)$, if possible, is beyond the scope of our book.

- $15a^4b^7 - 10a^3b^9 - 20a^6b^4 = 5a^3b^4(3ab^3 - 2ab^5 - 4a^3)$.

- $-44r^3s^4t^2 + 6r^5s^3t^7 - 12r^4s^3t^5 = 2r^3s^3t^2(-22rs + 3r^2t^5 - 6rt^3)$.

- $2x^2 + 10x - 12 = 2(x^2 + 5x - 6)$. In this case, the second factor $(x^2 + 5x - 6)$ can be factored further, which we will see in a short while.

Classroom Exercises : Factor the following:

(a) $12a^3b^4c^2 + 15a^5b^3c^5 - 18a^3b^2c^7$

(b) $-15x^3 + 30x^7 - 50x^4$

(c) $12r^9s^4t^4 + 25r^4t$

(d) $27x^2 - 3y^2$

5.6.2 The Grouping method

This method is typically applied when the polynomial to be factored has four terms. Given any polynomial, we first check whether any GCF can be factored out. Next, if it is a four-term polynomial, we check the grouping method.

Examples :

- $$\begin{aligned} & 2ab+6ac+b^2+3bc && \textit{(The GCF here is } 1\textit{. So we proceed.)} \\ &= \underbrace{2ab+6ac}+\underbrace{b^2+3bc} && \textit{(Group the first two terms, and the last two terms.)} \\ &= 2a(b+3c)+b(b+3c) && \textit{(Factor out the GCF from each group.)} \\ &= (2a+b)(b+3c) && \textit{(Factor out } (b+3c) \textit{ to get the final answer.)} \end{aligned}$$

- $$\begin{aligned} & 18px^2-9x^2q+6xyp-3xyq && \textit{(The GCF here is } 3x\textit{, which we first factor.)} \\ &= 3x(6px-3xq+2yp-yq) && \\ &= 3x(\underbrace{6px-3xq}+\underbrace{2yp-yq}) && \textit{(Group the first two terms, and the last two terms.)} \\ &= 3x\left(3x(2p-q)+y(2p-q)\right) && \textit{(Factor out the GCF from each group.)} \\ &= 3x(3x+y)(2p-q) && \textit{(Factor out } (2p-q) \textit{ to get the final answer.)} \end{aligned}$$

- $$\begin{aligned} & 12x^2ab+9xab-8xyab-6yab && \textit{(The GCF is } ab\textit{, which we factor.)} \\ &= ab(12x^2+9x-8xy-6y) && \\ &= ab(\underbrace{12x^2+9x}\underbrace{-8xy-6y}) && \textit{(Group the relevant terms.)} \\ & && \textit{(Keep track of the negatives.)} \\ &= ab\left(3x(4x+3)-2y(4x+3)\right) && \textit{(Factor out the GCF from each group.)} \\ &= ab(3x-2y)(4x+3) && \textit{(Factor out } (4x+3) \textit{ to get the final answer.)} \end{aligned}$$

- $$\begin{aligned} & 20x^4y^2-70x^3y^2-12x^3y^3+42x^2y^3 && \textit{(The GCF is } 2x^2y^2\textit{, which we factor.)} \\ &= 2x^2y^2(10x^2-35x-6xy+21y) && \\ &= 2x^2y^2(\underbrace{10x^2-35x}\underbrace{-6xy+21y}) && \textit{(Group the relevant terms.)} \\ & && \textit{(Keep track of the negatives.)} \\ &= 2x^2y^2\left(5x(2x-7)-3y(2x-7)\right) && \textit{(Factor out the GCF from each group.)} \\ &= 2x^2y^2(5x-3y)(2x-7) && \textit{(Factor out } (2x-7) \textit{ to get the final answer.)} \end{aligned}$$

In some cases, we will have to rearrange the terms to be able to use this method. The following example illustrates this.

- $28x^2 - 3y + 7xy - 12x$ *(The GCF here is 1. So we proceed.)*
  $= \underbrace{28x^2 - 3y} + \underbrace{7xy - 12x}$ *(Notice that these groups do not lead to a factorization.)*
  (We need to rearrange the terms.)
  $= 28x^2 - 12x + 7xy - 3y$
  $= \underbrace{28x^2 - 12x} + \underbrace{7xy - 3y}$
  $= 4x(7x - 3) + y(7x - 3)$
  $= (4x + y)(7x - 3)$ *(The final answer.)*

The grouping method may not work. The following example illustrates this.

- $28x^2 - 4x + 7xy - 3y$ *(The GCF here is 1. So we proceed.)*
  $= \underbrace{28x^2 - 4x} + \underbrace{7xy - 3y}$
  $= 4x(28x^2 - 4x) + 7xy - 3y$ *(This does not lead us to a factorization.)*
  (Rearranging also does not lead us to a factorization.)

Classroom Exercises : Factor the following by first checking for the GCF, and then using the grouping procedure:

(a) $2pr - ps + 2qr - qs$

(b) $6ac + 18a - 3bc - 9b$

(c) $5x^2 - 15x + 4xy - 12y$

(d) $15r^4s^3 - 45r^3s^2 - 6r^3s^4 + 18r^2s^3$

(e) $16abxy + 24axy - 4b^2xy - 6bxy$

(f) $4xy - 15x + 6x^2 - 10y$ (This will require a rearrangement of the terms).

5.6.3 The Standard Formulae

Given a polynomial, so far, we have learnt to factor out the GCF, and then use the grouping method whenever relevant. Now we use the standard formulae we had encountered earlier in the chaper.

Difference of squares :

$$\begin{aligned}
(a+b)(a-b) &= (a+b)\cdot a-(a+b)\cdot b && \textit{(Use distributive law)}\\
&= a^2+ab-(ab+b^2) && \textit{(Keep track of the negative)}\\
&= a^2+ab-ab-b^2 &&\\
&= a^2-b^2 &&\\
(a+b)(a-b) &= a^2-b^2 && \textit{(The difference of squares)}
\end{aligned}$$

Examples : Factor the following:

- x^2-y^2 *(GCF =1; grouping is not relevant.)*
  (Proceed with standard formulae.)
  $=(x+y)(x-y)$ *(The difference of squares.)*

- $4r^2-9$ *(GCF =1; grouping is not relevant.)*
  (Proceed with standard formulae.)
  $=(2r)^2-(3)^2$
  $=(2r+3)(2r-3)$ *(The difference of squares.)*

- $25x^2y^2-36a^2b^2$ *(GCF =1; grouping is not relevant.)*
  (Proceed with standard formulae.)
  $=(5xy)^2-(6ab)^2$
  $=(5xy+6ab)(5xy-6ab)$ *(The difference of squares.)*

- $12a^3b-27ab^3$ *(Factor out the GCF = 3ab.)*
  $=3ab(4a^2-9b^2)$ *(Grouping is not relevant.)*
  $=3ab\left((2a)^2-(3b)^2\right)$ *(Proceed with standard formulae.)*
  $=3ab(2a+3b)(2a-3b)$ *(The difference of squares.)*

- $4x^4-400y^6$ *(Factor out the GCF = 4.)*
  $=4(x^4-100y^6)$ *(Grouping is not relevant.)*
  $=4\left((x^2)^2-(10y^3)^2\right)$ *(Proceed with standard formulae.)*
  $=4(x^2+10y^3)(x^2-10y^3)$ *(The difference of squares.)*

- $3x^2 - y^2$ *(GCF =1; grouping is not relevant.)* *(Proceed with standard formulae.)*

  $= (\sqrt{3}x)^2 - (y)^2$

  $= (\sqrt{3}x + y)(\sqrt{3}x - y)$ *(The difference of squares.)*

- $7m^2 - 5n^2$ *(GCF =1; grouping is not relevant.)* *(Proceed with standard formulae.)*

  $= (\sqrt{7}m)^2 - (\sqrt{5}n)^2$

  $= (\sqrt{7}m + \sqrt{5}n)(\sqrt{7}m - \sqrt{5}n)$ *(The difference of squares.)*

Classroom Exercises : Factor the following, by first checking for the GCF, then checking for the relevance of grouping method, and lastly, the standard formulae:

(a) $p^2 - 9$

(b) $9m^2 - 16$

(c) $49p^2q^2 - 25r^2t^2$

(d) $8x^2 - 18y^2$

(e) $3a^4 - 12a^2b^2$

(f) $m^2 - 2n^2$

(g) $6p^2 - 10q^2$

Difference of cubes Consider the following product:

$$\begin{aligned}
(a-b)(a^2+ab+b^2) &= (a-b)\cdot a^2 + (a-b)\cdot ab + (a-b)\cdot b^2 && \textit{(Use distributive law)}\\
&= a^3 - a^2b + a^2b - ab^2 + ab^2 - b^3 && \textit{(Use distributive law again)}\\
&= a^3 - b^3 && \textit{(Combine like terms)}\\
(a-b)(a^2+ab+b^2) &= a^3 - b^3 && \textit{(Difference of cubes.)}
\end{aligned}$$

Sum of cubes Consider the following product:

$$
\begin{aligned}
(a+b)(a^2-ab+b^2) &= (a+b)\cdot a^2 - (a+b)\cdot ab + (a+b)\cdot b^2 && \textit{(Use distributive law)}\\
&= a^3 + a^2b - (a^2b + ab^2) + ab^2 + b^3 && \textit{(Use distributive law again)}\\
&= a^3 + a^2b - a^2b - ab^2 + ab^2 + b^3 && \textit{(Keep track of the negative)}\\
&= a^3 + b^3 && \textit{(Combine like terms)}\\
(a+b)(a^2-ab+b^2) &= a^3 + b^3 && \textit{(Sum of cubes.)}
\end{aligned}
$$

Examples :

- $t^3 - 1$ *(GCF=1; grouping is irrelevant;)*
  (This is not a difference of squares.)
  $= t^3 - 1^3$
  $= (t-1)(t^2+t+1)$ *(Difference of cubes.)*

- $t^3 + 1$ *(GCF=1; grouping is irrelevant;)*
  (This is not a difference of squares or cubes.)
  $= t^3 + 1^3$
  $= (t+1)(t^2-t+1)$ *(Sum of cubes.)*

- $2a^3 - 16$ *(GCF=2, factor it; grouping is irrelevant;)*
  $= 2(a^3-8)$ *(This is not a difference of squares.)*
  $= 2(a^3-2^3)$
  $= 2(a-2)(a^2+2a+4)$ *(Difference of cubes.)*

- $x^3 + 27y^3$ *(GCF=1; grouping is irrelevant;)*
  (This is not a difference of squares or cubes.)
  $= x^3 + (3y)^3$
  $= (x+3y)(x^2-3xy+9y^2)$ *(Sum of cubes.)*

Classroom Exercises : Factor, by checking for the GCF, then grouping method, then standard formulae, in that order.

(a) $x^3 - 27$

(b) $a^3 + 8$

(c) $2r^3 - 2$

(d) $2r^3 + 16$

5.6.4 Monic Quadratics in one variable

Recall that a trinomial is a polynomial with three nonzero terms. A quadratic polynomial in one variable x is a trinomial and can be written in the form

$$ax^2 + bx + c \qquad \textit{for real numbers } a, b, c \textit{ with } a \neq 0.$$

When the real number $a = 1$, the trinomial is called monic. That is, a monic quadratic polynomial takes the form

$$x^2 + bx + c \qquad \textit{for real numbers } b, c.$$

In this section we will assume that b and c are integers, and factor them. In some cases the factorization is possible only with complex numbers. We will not address such problems.

Factoring a monic quadratic in one variable $x^2 + bx + c$ requires finding two factors of c, such that their sum is b. That is, we look for numbers whose product is c and their sum is c. We will now see some examples.

The reader is **urged** to provide detailed work. Do not merely write down the final answer. In this method we convert the given trinomial to a four-term polynomial and then use the grouping procedure.

Examples :

- $$\begin{aligned} & x^2 + 13x + 30 && \textit{(GCF=1; standard formulae are irrelevant.)} \\ &= x^2 + 10x + 3x + 30 && \textit{(Grouping method is now used.)} \\ &= \underbrace{x^2 + 10x} + \underbrace{3x + 30} \\ &= x(x + 10) + 3(x + 10) \\ &= (x + 3)(x + 10) && \textit{(The final answer.)} \end{aligned}$$

  $(+10) \quad + \quad (+3) = +13$

  $+30$

Note that the signs of the coefficients are extremely important.

- $$\begin{aligned}
&x^2 + 13x - 30 && \textit{(GCF=1; standard formulae are irrelevant.)}\\
&= x^2 + 15x - 2x - 30 && \textit{(Grouping method is now used.)}\\
&= \underbrace{x^2 + 15x}\underbrace{-2x - 30} &&\\
&= x(x + 15) - 2(x + 15) &&\\
&= (x - 2)(x + 15) && \textit{(The final answer.)}
\end{aligned}$$

  $(+15) \quad + \quad (-2) = +13$

  -30

- $$\begin{aligned}
&x^2 + x - 30 && \textit{(GCF=1; standard formulae are irrelevant.)}\\
&= x^2 + 6x - 5x - 30 && \textit{(Grouping method is now used.)}\\
&= \underbrace{x^2 + 6x}\underbrace{-5x - 30} &&\\
&= x(x + 6) - 5(x + 6) &&\\
&= (x - 5)(x + 6) && \textit{(The final answer.)}
\end{aligned}$$

  $(+6) \quad + \quad (-5) = +1$

  -30

Here is a generalization for factoring a monic trinomial in two variables.

- $$\begin{aligned}
&a^2 + 3ab - 28b^2 && \textit{(GCF=1; standard formulae are irrelevant.)}\\
&= a^2 + 7ab - 4ab - 28b^2 && \textit{(Grouping method is now used.)}\\
&= \underbrace{a^2 + 7ab}\underbrace{-4ab - 28b^2} &&\\
&= a(a + 7b) - 4b(a + 7b) &&\\
&= (a - 4b)(a + 7b) && \textit{(The final answer.)}
\end{aligned}$$

  $(+7) \quad + \quad (-4) = +3$

  -28

- $$\begin{aligned}
&3x^2 - 24x + 21 && \textit{(Factor the GCF} = 3.\textit{)}\\
&= 3(x^2 - 8x + 7) && \textit{(Standard formulae are irrelevant.)}\\
&= 3(x^2 - 7x - 1x + 7) && \textit{(Grouping method is now used.)}\\
&= 3(\underbrace{x^2 - 7x}\underbrace{-1x + 7}) &&\\
&= 3(x(x - 7) - 1(x - 7)) &&\\
&= 3(x - 1)(x - 7) && \textit{(The final answer.)}
\end{aligned}$$

  $(-7) \quad + \quad (-1) = -8$

  $+7$

Here is a generalization for factoring a monic trinomial of higher degree.

- $x^4 - 26x^2 + 25$ *(GCF=1; standard formulae are irrelevant.)*
  $= x^4 - 25x^2 - 1x^2 + 25$ *(Grouping method is now used.)*
  $= \underbrace{x^4 - 25x^2}\underbrace{-1x^2 + 25}$
  $= x^2(x^2 - 25) - 1(x^2 - 25)$
  $= (x^2 - 1)(x^2 - 25)$ *(Now, difference of squares.)*
  $= (x + 1)(x - 1)(x + 5)(x - 5)$ *(The final answer.)*

  $(-25) \; + \; (-1) = -26$

  $+25$

Classroom Exercises : Factor the following trinomials. Provide detailed work.

(a) $x^2 + 2x + 1$

(b) $x^2 - 4x - 21$

(c) $x^2 + 22x + 21$

(d) $p^2 + 4pq - 96q^2$

(e) $2a^2 - 10ab - 72b^2$

(f) $x^4 - 5x^2 + 4$

5.6.5 Non-monic Quadratics in one variable

A non-monic quadratic in one variable, x, can be written in the form

$$ax^2 + bx + c \qquad \textit{for real numbers } a, b, c, a \neq 0, a \neq 1.$$

In this section, we will consider non-monic quadratics of the form

$$ax^2 + bx + c \qquad \textit{for integers } a, b, c, a \neq 0, a \neq 1.$$

To factor a non-monic quadratic, we use a procedure called the ***ac*-method**. Sometimes the coefficients are very large. In such cases, we use the ac-method along with a small trick.
Small coefficients : First we learn the ac-method for small coefficients. We look for factors of ac whose sum is b. The student is **cautioned** against using guessing as a procedure. A carefully detailed procedure is guaranteed to give you correct answers, which guessing cannot.
Examples :

- $2x^2 + 7x + 3$ *(GCF=1; standard formulae are irrelevant.)*

  $a = 2, b = 7, c = 3$ $(ac = 2 \cdot 3 = 6.$ *Factor* $6.)$

  $= 2x^2 + 6x + 1x + 3$ *(Use grouping method.)*

  $= \underbrace{2x^2 + 6x} + \underbrace{1x + 3}$

  $= 2x(x + 3) + 1(x + 3)$

  $= (2x + 1)(x + 3)$ *(The final answer.)*

  $(+6) \quad + \quad (+1) = +7$

  $(+6)$

- $3x^2 - 25x + 8$ *(GCF=1; standard formulae are irrelevant.)*

  $a = 3, b = -25, c = 8$ $(ac = 3 \cdot 8 = 24.$ *Factor* $24.)$

  $= 3x^2 - 24x - 1x + 8$ *(Use grouping method.)*

  $= \underbrace{3x^2 - 24x}\underbrace{-1x + 8}$

  $= 3x(x - 8) - 1(x - 8)$

  $= (3x - 1)(x - 8)$ *(The final answer.)*

  $(-24) \quad + \quad (-1) = -25$

  $(+24)$

Here is a generalization for factoring a trinomial in higher degree.

- $12x^4 + 15x^3 - 18x^2$ *(Factor the GCF=*$3x^2$*; standard formulae are irrelevant.)*

  $= 3x^2(4x^2 + 5x - 6)$ $a = 4, b = 5, c = -6$ $(ac = 4 \cdot (-6) = -24.$ *Factor* $-24.)$

  $= 3x^2(4x^2 + 8x - 3x - 6)$ *(Use grouping method.)*

  $= 3x^2(\underbrace{4x^2 + 8x}\underbrace{-3x - 6})$

  $= 3x^2\left(4x(x + 2) - 3(x + 2)\right)$

  $= 3x^2(4x - 3)(x + 2)$ *(The final answer.)*

  $(+8) \quad + \quad (-3) = +5$

  (-24)

Here is a generalization for factoring a trinomial in two variables.

- $8x^3y - 32x^2y^2 + 30xy^3$ *(Factor out GCF=2xy; standard formulae are irrelevant.)*
  $= 2xy(4x^2 - 16xy + 15y^2)$ $a = 4, b = -16, c = 15$ $(ac = 4 \cdot 15 = 60.$ *Factor* $60.)$
  $= 2xy(4x^2 - 10xy - 6xy + 15y^2)$ *(Use grouping method.)*
  $= 2xy(\underbrace{4x^2 - 10xy}\underbrace{-6xy + 15y^2})$
  $= 2xy\left(2x(2x - 5y) - 3y(2x - 5y)\right)$
  $= 2xy(2x - 3y)(2x - 5y)$ *(The final answer.)*

  $(-10) + (-6) = -16$

  $(+60)$

Here is a generalization of factoring a trinomial in higher degree.

- $36x^7 - 69x^5 + 15x^3$ *(Factor out the GCF=3x³; standard formulae are irrelevant.)*
  $= 3x^3(12x^4 - 23x^2 + 5)$ $a = 12, b = -23, c = 5$ $(ac = 12 \cdot 5 = 60.$ *Factor* $60.)$
  $= 3x^3(12x^4 - 20x^2 - 3x^2 + 5)$ *(Use grouping method.)*
  $= 3x^3(\underbrace{12x^4 - 20x^2}\underbrace{-3x^2 + 5})$
  $= 3x^3\left(4x^2(3x^2 - 5) - 1(3x^2 - 5)\right)$
  $= 3x^2(4x^2 - 1)(3x^2 - 5)$ *(Differences of squares.)*
  $= 3x^2(2x + 1)(2x - 1)(\sqrt{3}x + \sqrt{5})(\sqrt{3}x - \sqrt{5})$
  (The final answer.)

  $(-20) + (-3) = -23$

  $(+60)$

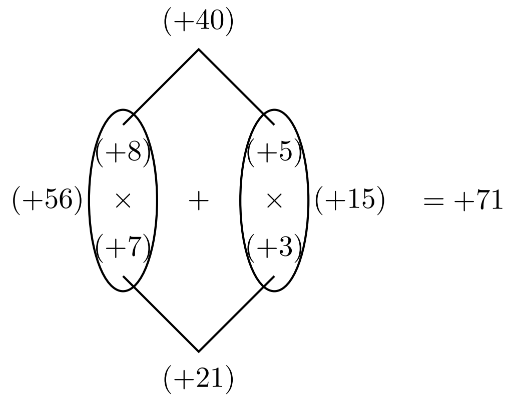
Classroom Exercises : Factor the following trinomials.

1. (a) $2x^2 + x - 6$ (b) $7x^2 + 20x - 3$
2. (a) $15x^2 + x - 2$ (b) $24a^4 + 10a^3 - 4a^2$
3. (a) $18p^4q + 21p^3q^2 - 9p^2q^3$ (b) $5x^9 - 47x^7 + 18x^5$

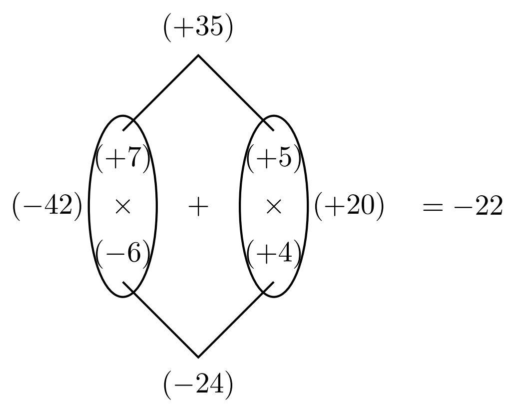
Large coefficients : Sometimes, the coefficients in the trinomials are so large that the ac-method becomes quite difficult. In this case, instead of factoring the product ac, we factor the numbers a and c separately in a meaningful way. It is best explained through these examples.

Examples :

- $40x^2 + 71x + 21$ *(GCF =1; standard formulae are irrelevant.)*
  $= 40x^2 + 56x + 15x + 21$ $a = 40, b = 71, c = 21$. *Factor* 40 *and* 21.
  $= \underbrace{40x^2 + 56x} + \underbrace{15x + 21}$
  $= 8x(5x + 7) + 3(5x + 7)$
  $= (8x + 3)(5x + 7)$
  (The final answer.)



- $35x^2 - 22x - 24$ *(GCF =1; standard formulae are irrelevant.)*
  $= 35x^2 - 42x + 20x - 24$ $a = 35, b = -22, c = -24$. *Factor* 35 *and* -24.
  $= \underbrace{35x^2 - 42x} + \underbrace{20x - 24}$
  $= 7x(5x - 6) + 4(5x - 6)$
  $= (7x + 4)(5x - 6)$
  (The final answer.)



Classroom Exercises : Factor the following trinomials.

(a) $40x^2 + 122x + 33$

(b) $21x^2 - 83x + 72$

5.6.6 Summary

To factorize a polynomial, follow every step:

Step 1 Factor out the GCF.

Step 2 Check whether the grouping method can be used.

Step 3 Check for any of the standard formulae (Difference of squares, difference of cubes, sum of cubes).

Step 4 Convert the monic trinomial to a four-term polynomial and apply the grouping procedure.

Step 5 Convert the non- monic trinomial to a four-term polynomial and apply the grouping procedure.

5.6.7 Homework Exercises

1. What is the GCF in each of the cases?

   (a) $\{14, 49, 56\}$
   (b) $\{30, 45, 60\}$
   (c) $\{18, 24, 36\}$
   (d) $\{14a^5b^7c^4, 49a^3b^4c^9, 56abc\}$
   (e) $\{30x^8, 45x^4, 60x^5\}$
   (f) $\{18t^4s^3, 24ts, 36t^9\}$

2. Factor the following:

   (a) $14a^5b^7c^4 - 49a^3b^4c^9 + 56abc$
   (b) $30x^8 + 45x^4 + 60x^5$
   (c) $-18t^4s^3 - 24ts + 36t^9$
   (d) $12x^5 - 16x^4 - 5x^3$
   (e) $-25x^7 - 30x^9 + 15x^{15}$
   (f) $2pm + 3qm + 2pn + 3qn$
   (g) $40ma + 32m - 60na - 48n$
   (h) $32y^2 + 3x + 4y + 24xy$
   (i) $20y^2 + 4y + 24xy + 3x$

(j) $15a^5b^4 - 35a^4b^4 - 6a^4b^5 + 14a^3b^5$
(k) $18p^2rs + 24prs - 6nprs - 8nrs$
(l) $a^2 - b^2$
(m) $x^2 - 4$
(n) $9r^2 - 1$
(o) $4x^2 - 25y^2$
(p) $100m^2n^2 - 9p^2q^2$
(q) $16p^4 - 9q^8$ (This will require several factoring steps)
(r) $48r^2t^4 - 12$
(s) $7r^2 - 3s^2$
(t) $5x^2 - 1$
(u) $x^3 - 1$
(v) $x^3 + 1$
(w) $8a^3 - 27$
(x) $8a^3 + 27$
(y) $3r^3 - 24$
(z) $3r^3 + 24$

3. Factor the following trinomials:

(a) $x^2 + x - 20$
(b) $x^2 + 10x + 16$
(c) $x^2 - 17x + 16$
(d) $3p^2 + 36pq + 33q^2$
(e) $3a^2 - 21ab - 54b^2$
(f) $2x^2 + 28xy + 48y^2$
(g) $x^4 - 13x^2 + 36$ (This will require several factoring steps)
(h) $3x^2 + 11x + 10$
(i) $2x^2 - 3x - 14$
(j) $6x^2 - 11x + 4$
(k) $36x^4 - 39x^3 + 9x^2$
(l) $12p^4q^2 - 26p^3q^3 + 12p^2q^4$
(m) $3x^{10} - 16x^8 + 16x^6$ (This will require several factoring steps)
(n) $21x^2 + 59x + 40$
(o) $30x^2 + x - 20$

5.7 Solving Quadratic Equations By Factoring

An equation is a mathematical statement involving an equality (=). A quadratic equation in one variable, x, can be written in the form

$$ax^2 + bx + c = 0 \qquad \textit{for real numbers } a, b, c, a \neq 0.$$

To solve a quadratic equation is to find those numbers which, when substituted for x, satisfy the equation.

In this section, we will solve quadratic equations in which the quadratic polynomial can be factored.

First, the **Zero product law:**

$$\textit{When } a \cdot b = 0 \quad \textit{then } a = 0 \textit{ or } b = 0.$$

That is, if the product of two numbers is zero, then one or the other has to be zero. Notice that, this is true only for zero. For instance, if $a \cdot b = 6$, then it does not mean that $a = 6$ or $b = 6$. Indeed, $2 \cdot 3 = 6$. Thus, zero alone has this property.

Using the Zero product law, we proceed.

Examples :

- $x(x-1) \quad = 0$

  $x = 0 \quad$ or $\quad x - 1 = 0 \quad$ *(Zero product law.)*

  $x = 0 \quad$ or $\quad x = 1 \quad$ *(Solve the two linear equations.)*

- $x(x-1) \quad = 6 \quad$ *(We need a zero on one side of the equation.)*

  $x^2 - x \quad = 6 \quad$ *(Subtract 6 to get a zero.)*

  $x^2 - x - 6 \quad = 0 \quad$ *(Factor the left.)*

  $(x-3)(x+2) \quad = 0$

  $x - 3 = 0 \quad$ or $\quad x + 2 = 0 \quad$ *(Zero product law)*

  $x = 3 \quad$ or $\quad x = -2 \quad$ *(Solve the two linear equations.)*

- $$\begin{aligned}
3x(2x-1) &= -2(x-1) && \textit{(We need a zero.)}\\
3x(2x-1)+2(x-1) &= 0 && \textit{(Adding } 2(x-1) \textit{ to get a zero.)}\\
6x^2-3x+2x-2 &= 0 && \textit{(Simplify the left.)}\\
6x^2-x-2 &= 0 && \textit{(Now factor the left .)}\\
(2x+1)(3x-2) &= 0 &&\\
2x+1=0 \quad &\text{or} \quad 3x-2=0 && \textit{(Zero product law)}\\
2x=-1 \quad &\text{or} \quad 3x=2 &&\\
x=-\frac{1}{2} \quad &\text{or} \quad x=\frac{2}{3} &&
\end{aligned}$$

- $$\begin{aligned}
4x^3 &= \frac{15x^2(1-x)}{(3x-1)} && \textit{(First cross-multiply.)}\\
4x^3(3x-1) &= 15x^2(1-x) && \textit{(Get a zero on the right.)}\\
4x^3(3x-1)-15x^2(1-x) &= 0 && \textit{(Simplify the left.)}\\
12x^4-4x^3-15x^2+15x^3 &= 0 &&\\
12x^4+11x^3-15x^2 &= 0 && \textit{(GCF} = x^2.\textit{)}\\
x^2(12x^2+11x-15) &= 0 && \textit{(Factor the left.)}\\
x^2(4x-3)(3x+5) &= 0 &&\\
x^2=0, \quad 4x-3=0, \quad &\text{or} \quad 3x+5=0 &&\\
x=0, \quad \text{or } 4x=3, \quad &\text{or} \quad 3x=-5 &&\\
x=0, \quad \text{or } x=\frac{3}{4}, \quad &\text{or} \quad x=-\frac{5}{3} &&\\
x=0, \quad \text{or } x=\frac{3}{4}, \quad &\text{or} \quad x=-1\frac{2}{3} &&
\end{aligned}$$

Classroom Exercises : Solve the following equations.

(a) $x^2+3x=0$

(b) $x^2 + 3x = 10$

(c) $6x(x + 2) = x - 4$

(d) $12x = \dfrac{x - 3}{x - 1}$

(e) $x^4 = \dfrac{15x^3(x - 1)}{(4x - 1)}$

5.7.1 Homework Exercises

Solve the following equations:

1. $x^2 + 7x = 0$
2. $x^2 + 7x = -12$
3. $x(2x + 25) = 14(x - 1)$
4. $2x(x + 2) = 15(x - 1)$
5. $6x = \dfrac{-(x + 12)}{(x - 3)}$
6. $x^6 = \dfrac{2x^5(2x - 1)}{3(x - 1)}$

5.8 Solving Word Problems Using Quadratic Equations

In this section we use techniques of quadratic equations to address certain word problems.
Examples :

- The difference of two numbers is 11, while their product is -30. What are the two numbers?

  Solution. *Let the smaller number be x.*
  Then, the bigger number is $(x + 11)$
  Their product is -30.
  Thus,

$$\begin{aligned} x(x + 11) &= -30; \\ x^2 + 11x &= -30; \\ x^2 + 11x + 30 &= 0; \\ (x + 5)(x + 6) &= 0. \end{aligned}$$

By the Zero Product law, we have

$$x + 5 = 0 \text{ or } x + 6 = 0;$$
$$x = -5 \text{ or } x = -6.$$

Since the bigger number is 11 more than the smaller number, when the smaller number is -5*, the bigger number is* 6*; and, when the smaller number is* -6*, the bigger number is* 5*.*

- The sum of the squares of two consecutive even integers is 340. Find the two integers.

  Solution. *Let the smaller even integer be* x*.*
  Then, the bigger even integer is $(x+2)$*. (The next even integer will be two more than the previous one).*
  The sum of their squares is 340*.*
  That is,

$$\begin{aligned} x^2 + (x+2)^2 &= 340; \\ x^2 + x^2 + 2x + 4 &= 340; \\ 2x^2 + 4x + 4 &= 340; \\ 2x^2 + 4x + 4 - 340 &= 0; \\ 2x^2 + 4x - 336 &= 0; \\ 2(x^2 + 2x - 168) &= 0; \quad \textit{(Divide both sides by 2)} \\ x^2 + 2x - 168 &= 0; \\ (x+14)(x-12) &= 0 \end{aligned}$$

  By the Zero Product Law, we have

$$x + 14 = 0 \text{ or } x - 12 = 0;$$
$$x = -14 \text{ or } x = 12.$$

  Since the bigger even integer is two more than the smaller integer, when the smaller integer is -14*, the bigger one is* -12*; and, when the smaller integer is* 12*, the bigger one is* 14*.*

- The sum of the squares of two consecutive odd integers is 394. Find the two integers.

  Solution. *Let the smaller odd integer be* x*.*
  Then, the bigger odd integer is $(x+2)$*. (The next odd integer will be two more than the previous one).*

The sum of their squares is 394.
That is,

$$\begin{aligned} x^2 + (x+2)^2 &= 394; \\ x^2 + x^2 + 2x + 4 &= 394; \\ 2x^2 + 4x + 4 &= 394; \\ 2x^2 + 4x + 4 - 394 &= 0; \\ 2x^2 + 4x - 390 &= 0; \end{aligned}$$

$$\begin{aligned} 2(x^2 + 2x - 195) &= 0; \quad \textit{(Divide both sides by 2)} \\ x^2 + 2x - 195 &= 0; \\ (x+15)(x-13) &= 0 \end{aligned}$$

By the Zero Product Law, we have

$$\begin{aligned} x + 15 = 0 \ &\textit{or}\ x - 13 = 0; \\ x = -15 \ &\textit{or}\ x = 13. \end{aligned}$$

Since the bigger odd integer is two more than the smaller integer, when the smaller integer is -15*, the bigger one is* -13*; and, when the smaller integer is* 13*, the bigger one is* 15*.*

- The height of a triangle is 3 inches more than the base. Find the height and the base, if the area of the triangle is 44 square inches.

  Solution. *Let the base of the triangle be* x *inches.*
  Then, the height of the triangle is $(x+3)$ *inches.*
  The area of the triangle is 44 *square inches.*
  Therefore,

$$\begin{aligned} \frac{x(x+3)}{2} &= 44; \quad \textit{Multiply both sides by 2} \\ x(x+3) &= 2 \times 44; \\ x^2 + 3x &= 88; \\ x^2 + 3x - 88 &= 0; \\ (x+11)(x-8) &= 0. \end{aligned}$$

  Hence, by the Zero Product Law, we have

$$\begin{aligned} x + 11 = 0 \ &\textit{or}\ x - 8 = 0; \\ x = -11 \ &\textit{or}\ x = 8. \end{aligned}$$

Note that x denotes the base, and hence cannot be negative. Thus, the base is 8 inches, and the height is 11 inches (since the height is 3 inches more than the base).

- The length of a rectangle is twice its width. Find the length and the width if the area of the rectangle is 128 square centimeters.

  Solution. *Let the width of a rectangle be x cm.*
  Then the length is $2x$ cm.
  The area of the rectangle is 128 sq. cm.
  Therefore,

$$\begin{aligned} 2x \times x &= 128; \\ 2x^2 &= 128; \quad \textit{Divide both sides by } 2; \\ x^2 &= 64; \quad \textit{Take square roots} \\ x &= \pm 8. \end{aligned}$$

  Note that x represents the width of the rectangle, and therefore cannot be a negative number. Hence, the width is 8 cm while the length is 16 cm (since the length is twice the width).

Classroom Exercises :

(a) The difference of two numbers is 8, while their product is -15. What are the two numbers?

(b) The sum of the squares of two consecutive even integers is 244. Find the two integers.

(c) The sum of the squares of two consecutive odd integers is 290. Find the two integers.

(d) The height of a triangle is 3 inches more than the base. Find the height and the base, if the area of the triangle is 35 square inches.

(e) The length of a rectangle is twice its width. Find the length and the width if the area of the rectangle is 98 square centimeters.

5.8.1 Homework Exercises

State clearly what your variables stand for. Provide detailed work.

1. The difference of two numbers is 13, while their product is -42. What are the two numbers?

2. The sum of the squares of two consecutive even integers is 100. Find the two integers.

3. The sum of the squares of two consecutive odd integers is 202. Find the two integers.

4. The height of a triangle is 3 inches more than the base. Find the height and the base, if the area of the triangle is 20 square inches.

5. The length of a rectangle is twice its width. Find the length and the width if the area of the rectangle is 72 square centimeters.

Chapter 6

Radical expressions and Complex numbers

6.1 Roots and Radicals

We were introduced to roots and radicals in Section 1.4. Recall that the symbol $\sqrt{}$ is called the **radical** and the expression inside the $\sqrt{}$ symbol is called the **radicand.** For example,

$$\text{in the expression } \sqrt[3]{\frac{xy}{a+b}}, \text{ the radicand is } \frac{xy}{a+b}.$$

For a natural number n, we say that a is an n**-th root** of a number b if and only if $a^n = b$. We write the phrase, "n-th root of b," in short as $\sqrt[n]{b}$. Therefore,

$$a = \sqrt[n]{b} \textit{ if and only if } a^n = b.$$

The first thing to note is that by definition, $\left(\sqrt[n]{b}\right)^n = b$.

Terminology : The second root of b is called the **square-root** of b; it is denoted by $\sqrt{b}$ instead of $\sqrt[2]{b}$. The third root of b is called the **cube-root** or the **cubic-root** of b.

First we note that there are two square-roots of 25 as $5^2 = 25$ and $(-5)^2 = 25$. Similarly, as $3^4 = 81$, and $(-3)^4 = 81$, there are two fourth roots of 81. The **principal square-root** of a non-negative number is its **positive** square-root. Similarly, the **principal fourth root** of a non-negative number is its **positive** fourth root. For example, the principal square-root of 9 is $+3$. We therefore write

$$\sqrt{25} = 5, \quad -\sqrt{25} = -5; \quad \sqrt[4]{81} = 3, \quad -\sqrt[4]{81} = -3.$$

In general, for an **even** natural number n, $n \geq 2$, the **principal** n**-th root** is the **positive** n-th root.

Note that, the n-th root of a real number need not always be a real number. For example, $\sqrt{-9}$ is not a real number. This is because, the square of a real number is non-negative. For instance,

$$3^2 = 9, \ \ \textit{and}\ (-3)^2 = 9.$$

In other words, there is no real number a, such that $a^2 = -9$. Therefore, $\sqrt{-9}$ is not a real number. In fact, $\sqrt{-9}$ is a complex number, and we will learn about complex numbers in Section 6.3.

Likewise, $\sqrt[4]{-625}$ is not a real number because the fourth power of any real number is non-negative. Again, $\sqrt[4]{-625}$ is a complex number. Note,

$$5^4 = 625, \ \ \textit{and}\ (-5)^4 = 625.$$

You probably have realized the following generalization:
For n an **even natural number**, the n-th root of a **negative real number** is **not a real number.**

This is not to be confused with the following situation: What is square-root of 3? It is the **irrational number** $\sqrt{3}$. There is no simpler way of writing $\sqrt{3}$. This is a real number. Likewise, $\sqrt[3]{10}$, $\sqrt[4]{25}$, or $\sqrt[9]{100}$ are all irrational **real** numbers.

Here are some examples.

$$
\begin{array}{llll}
\sqrt[n]{0} & = & 0 & \textit{because } 0^n = 0 \textit{ for any natural number } n. \\
\sqrt{64} & = & 8 & \textit{because } 8^2 = 64. \\
\sqrt{-64} & \textit{is} & \textit{not a real number} & \textit{because the square of a real number can not be negative.} \\
\sqrt[3]{64} & = & 4 & \textit{because } 4^3 = 64. \\
\sqrt[3]{-64} & = & -4 & \textit{because } (-4)^3 = (-4) \times (-4) \times (-4) = -64. \\
\sqrt[4]{81} & = & 3 & \textit{because } 3^4 = 81. \\
\sqrt[4]{-81} & \textit{is} & \textit{not a real number} & \textit{because the fourth power of a real number cannot be negative.} \\
\sqrt[5]{32} & = & 2 & \textit{because } 2^5 = 32. \\
\sqrt[5]{-32} & = & -2 & \textit{because } (-2)^5 = (-2) \times (-2) \times (-2) \times (-2) \times (-2) = -32. \\
\sqrt[3]{\dfrac{27}{1000}} & = & \dfrac{3}{10} & \textit{because } \left(\dfrac{3}{10}\right)^3 = \dfrac{3}{10} \times \dfrac{3}{10} \times \dfrac{3}{10} = \dfrac{27}{1000}. \\
\sqrt[3]{-\dfrac{27}{1000}} & = & -\dfrac{3}{10} & \textit{because } \left(-\dfrac{3}{10}\right)^3 = \left(-\dfrac{3}{10}\right) \times \left(-\dfrac{3}{10}\right) \times \left(-\dfrac{3}{10}\right) = -\dfrac{27}{1000}. \\
\sqrt{-\dfrac{49}{81}} & \textit{is} & \textit{not a real number} & \textit{because the square of a real number can not be negative.}
\end{array}
$$

Here are some important properties of radicals :

Property 1 *For real numbers* a, b *and natural number* n*, if* $\sqrt[n]{a}$, $\sqrt[n]{b}$ *are real numbers, then*

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

Proof :

$$\text{Suppose } x = \sqrt[n]{a}. \text{ Then } x^n = a. \text{ Supose } y = \sqrt[n]{b}. \text{ Then } y^n = b.$$

$$\text{Note that } \underbrace{x^n}_{a} \cdot \underbrace{y^n}_{b} = (x \cdot y)^n.$$

$$\text{In other words, } a \cdot b = (x \cdot y)^n. \text{ Hence } \sqrt[n]{a \cdot b} = x \cdot y. \text{ That is,}$$

$$\underbrace{\sqrt[n]{a \cdot b}}_{\sqrt[n]{a \cdot b}} = \underbrace{x}_{\sqrt[n]{a}} \cdot \underbrace{y}_{\sqrt[n]{b}}.$$

$$\text{We have therefore proved that } \sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

For example :

- $\sqrt[3]{8 \times 125} = \sqrt[3]{8} \times \sqrt[3]{125} = 2 \times 5 = 10.$
- $\sqrt{900} = \sqrt{9} \times \sqrt{100} = 3 \times 10 = 30.$
- $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}.$ (*This can not be simplified further.*)
- $-\sqrt[4]{162} = -\sqrt[4]{81 \times 2} = -\sqrt[4]{81} \times \sqrt[4]{2} = -3\sqrt[4]{2}.$ (*This can not be simplified further.*)
- $\sqrt[5]{160} = \sqrt[5]{32 \times 5} = \sqrt[5]{32} \times \sqrt[5]{5} = 2\sqrt[5]{5}.$ (*This can not be simplified further.*)

Property 2 *For real numbers* a, b*, with* $b \neq 0$*, and a natural number* n *such that* $\sqrt[n]{a}$, $\sqrt[n]{b}$ *are real numbers, then*

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Proof

$$\text{Suppose } x = \sqrt[n]{a}. \text{ Then } x^n = a. \text{ Supose } y = \sqrt[n]{b}. \text{ Then } y^n = b.$$

$$\text{Note that } \frac{a}{b} = \frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n.$$

$$\text{That is, } \frac{a}{b} = \left(\frac{x}{y}\right)^n.$$

$$\text{Therefore, } \sqrt[n]{\frac{a}{b}} = \frac{x}{y}.$$
$$\text{That is, } \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

For example :

- $\sqrt{\dfrac{49}{81}} = \dfrac{\sqrt{49}}{\sqrt{81}} = \dfrac{7}{9}.$
- $\sqrt[3]{-\dfrac{8}{125}} = \sqrt[3]{\dfrac{-8}{125}} = \dfrac{\sqrt[3]{-8}}{\sqrt[3]{125}} = \dfrac{-2}{5} = -\dfrac{2}{5}.$
- $-\sqrt{\dfrac{100}{63}} = -\dfrac{\sqrt{100}}{\sqrt{63}} = -\dfrac{10}{\sqrt{9\times 7}} = -\dfrac{10}{\sqrt{9}\times\sqrt{7}} = -\dfrac{10}{3\sqrt{7}}.$
  Here, we need to rationalize the denominator.
  $$\begin{aligned} -\frac{10}{3\sqrt{7}} &= -\frac{10\times\sqrt{7}}{3\sqrt{7}\times\sqrt{7}} \\ &= -\frac{10\sqrt{7}}{3\times 7} && \textit{as } \sqrt{7}\times\sqrt{7}=7 \\ &= -\frac{10\sqrt{7}}{21}. && \textit{This can not be simplified further.} \end{aligned}$$
- $\sqrt[3]{\dfrac{27}{100}} = \dfrac{\sqrt[3]{27}}{\sqrt[3]{100}} = \dfrac{3}{\sqrt[3]{100}}$. Here, we need to rationalize the denominator.
  $$\begin{aligned} \frac{3}{\sqrt[3]{100}} &= \frac{3\times\sqrt[3]{10}}{\sqrt[3]{100}\times\sqrt[3]{10}} \\ &= \frac{3\sqrt[3]{10}}{\sqrt[3]{100\times 10}} \\ &= \frac{3\sqrt[3]{10}}{\sqrt[3]{1000}} = \frac{3\sqrt[3]{10}}{10} && \textit{This can not be simplified further.} \end{aligned}$$

Caution : $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ and $\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$.
For example, $\sqrt{4+9} = \sqrt{13}$, while $\sqrt{4}+\sqrt{9} = 2+3 = 5$.
Similarly, $\sqrt{25-9} = \sqrt{16} = 4$, while $\sqrt{25}-\sqrt{9} = 5-3 = 2$.
Recall from the definition and properties of the absolute value of a real number. The **absolute value** of a real number is its distance from 0 on the number line. The absolute value of x is denoted by $|x|$. So, $|3| = 3$, and $|-3| = 3$. This is because, both 3 and -3 are at a distance of 3 from 0 on the number line. Note that $\sqrt{3^2} = \sqrt{9} = 3$, and $\sqrt{(-3)^2} = \sqrt{9} = 3$. In general,

$$\sqrt{x^2} = |x| \textit{ for any real number } x.$$

Here is yet another situation:

$$\sqrt[4]{5^4} = \sqrt[4]{625} = 5.$$
$$\sqrt[4]{(-5)^4} = \sqrt[4]{625} = 5.$$
$$\textit{In general, } \sqrt[4]{x^4} = |x| \textit{ for any real number } x.$$

One more example:

$$\sqrt[6]{10^6} = \sqrt[6]{1000000} = 10.$$
$$\sqrt[6]{(-10)^6} = \sqrt[6]{1000000} = 10.$$
$$\textit{In general, } \sqrt[6]{x^6} = |x| \textit{ for any real number } x.$$

Note that for any **even natural number** n, and any real number x, we get x^n to be a non-negative number. Therefore, the principal n-th root of x^n will be a **non-negative** real number. Therefore,

For an **even** *natural number n, and any real number x, we have $\sqrt[n]{x^n} = |x|$.*

Now consider the following situation:

$$\sqrt[3]{2^3} = \sqrt[3]{8} = 2.$$
$$\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2.$$
$$\textit{In general, } \sqrt[3]{x^3} = x \textit{ for any real number } x.$$

Here is yet another situation:

$$\sqrt[5]{3^5} = \sqrt[5]{243} = 3.$$
$$\sqrt[5]{(-3)^5} = \sqrt[5]{-243} = -3.$$
$$\textit{In general, } \sqrt[5]{x^5} = x \textit{ for any real number } x.$$

We can draw a general conclusion from this:

For an **odd** *natural number n, and any real number x, we have $\sqrt[n]{x^n} = x$.*

Here are some examples on simplifying algebraic expressions. Assume that all the variables are positive.

- $-\sqrt{81x^2} = -9x$. This can not be simplified further.
- $\sqrt{80x^3} = \sqrt{16 \times 5 \times x^2 \times x} = \sqrt{16 \times x^2 \times 5 \times x} = \sqrt{16} \times \sqrt{x^2} \times \sqrt{5 \times x} = 4x\sqrt{5x}$.

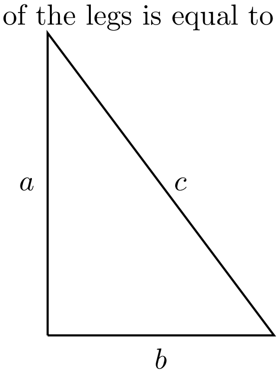
- $\sqrt{\dfrac{12x^3}{y}} = \dfrac{\sqrt{12x^3}}{\sqrt{y}} = \dfrac{\sqrt{4x^2 \times 3x}}{\sqrt{y}} = \dfrac{2x\sqrt{3x}}{\sqrt{y}}$.

  Now rationalize the denominator
  $$\frac{2x\sqrt{3x}}{\sqrt{y}} = \frac{2x\sqrt{3x} \times \sqrt{y}}{\sqrt{y} \times \sqrt{y}} = \frac{2x\sqrt{3xy}}{y}.$$
  There are several ways of arriving at this simplification. Here is another one:
  $$\sqrt{\frac{12x^3}{y}} = \sqrt{\frac{12x^3 \times y}{y \times y}} = \frac{\sqrt{12x^3y}}{\sqrt{y^2}} = \frac{\sqrt{4x^2 \times 3xy}}{\sqrt{y^2}} = \frac{2x\sqrt{3xy}}{y}.$$

- $\sqrt[3]{-\dfrac{40x^4}{y^5}} = \sqrt[3]{\dfrac{-40x^4}{y^5}} = \dfrac{\sqrt[3]{-40x^4}}{\sqrt[3]{y^5}} = \dfrac{\sqrt[3]{-8x^3 \times 5x}}{\sqrt[3]{y^3 \times y^2}} = \dfrac{\sqrt[3]{-8x^3} \times \sqrt[3]{5x}}{\sqrt[3]{y^3} \times \sqrt[3]{y^2}} = \dfrac{-2x\sqrt[3]{5x}}{y\sqrt[3]{y^2}}$.

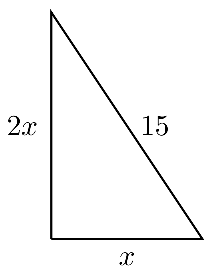
  Here, we need to rationalize the denominator.
  $$\frac{-2x\sqrt[3]{5x}}{y\sqrt[3]{y^2}} = \frac{-2x\sqrt[3]{5x} \times \sqrt[3]{y}}{y\sqrt[3]{y^2} \times \sqrt[3]{y}} = \frac{-2x\sqrt[3]{5xy}}{y\sqrt[3]{y^3}} = \frac{-2x\sqrt[3]{5xy}}{y \times y} = \frac{-2x\sqrt[3]{5xy}}{y^2}.$$
  Here is another way of arriving at the same simplification:
  $$\begin{aligned}\sqrt[3]{-\frac{40x^4}{y^5}} = \sqrt[3]{-\frac{40x^4 \times y}{y^5 \times y}} = \sqrt[3]{-\frac{40x^4y}{y^6}} &= \sqrt[3]{\frac{-40x^4y}{y^6}} \\ &= \frac{\sqrt[3]{-40x^4y}}{\sqrt[3]{y^6}} \\ &= \frac{\sqrt[3]{-8x^3 \times 5xy}}{\sqrt[3]{y^6}} \\ &= \frac{-2x\sqrt[3]{5xy}}{y^2}.\end{aligned}$$

Pythagorean Theorem: Given a right angled triangle, the sum of the squares of the lenghts of the legs is equal to the square of the the length of the hypotenuse. That is,



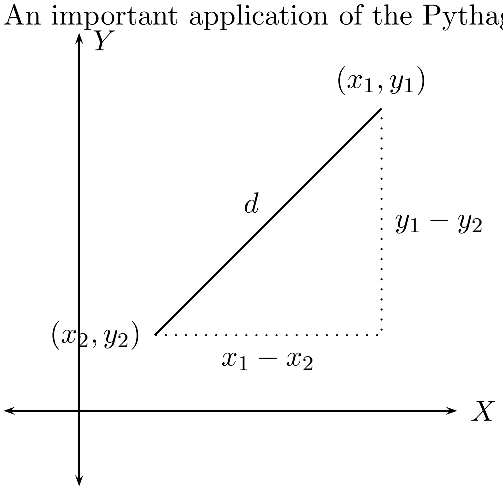
Pythagorean Theorem:
$a^2 + b^2 = c^2$

Example:
Find the value of x (simplify the radical).



$(2x)^2 + x^2 = 15^2$	*By Pythagorean theorem*
$4x^2 + x^2 = 225$	
$5x^2 = 225$	*Divide both sides by* 5
$x^2 = 45$	*Take square-roots of both sides*
$x = \sqrt{45} = 3\sqrt{5}$ units	*We only consider the positve square-root.*

An important application of the Pythagorean theorem is the **distance formula.**



Given two points (x_1, y_1) and (x_2, y_2) on the coordinate plane, let the distance between them be d.
By Pythagorean theorem we have
$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$.
Thus, we have the ***distance formula***

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

For example, to find the distance between points $(2, 3)$ and $(-1, 4)$, we first set $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (-1, 4)$. Then, the distance d is given by

$$d = \sqrt{(2 - (-1))^2 + (3 - 4)^2} = \sqrt{(2 + 3)^2 + (-1)^2} = \sqrt{5^2 + (-1)^2} = \sqrt{25 + 1} = \sqrt{26}.$$

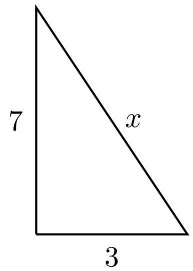
Classroom Exercises : Simplify (rationalize the denominator if needed):

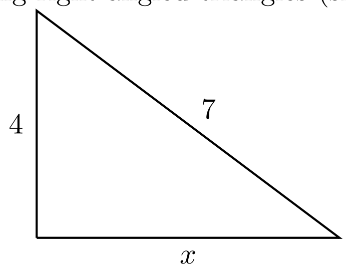
(a) $\sqrt{36}$ (b) $-\sqrt{\frac{225}{256}}$ (c) $\sqrt{\frac{9}{125}}$ (d) $\sqrt{3^2 + 4^2}$ (e) $\sqrt{(-4)^2} - \sqrt{4}$

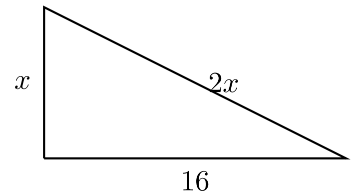
(f) $\sqrt{18} + \sqrt{2}$ (g) $-\sqrt{12} + \sqrt{27}$ (h) $\sqrt{\frac{32}{75}}$ (i) $\sqrt[3]{-\frac{64}{343}}$ (j) $\sqrt[4]{\frac{32y^7}{z^8}}$

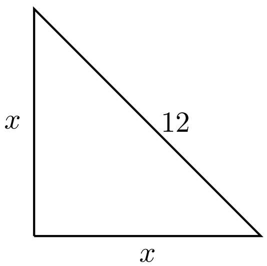
(k) $\sqrt[4]{2^4}$ (l) $\sqrt[4]{(-2)^4}$ (m) $\sqrt[3]{2^3}$ (n) $\sqrt[3]{(-2)^3}$ (o) $\sqrt[3]{\frac{32y^7}{z^6}}$

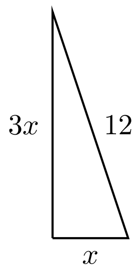
Find x in the following right angled triangles (simplify the radicals):

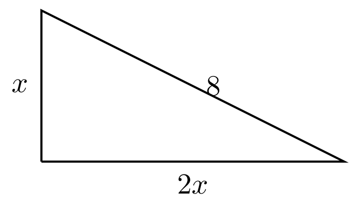












Find the distance between the given pair of points:

(a) $(3, 4)$ and $(-3, -4)$.

(b) $(-1, -1)$ and $(5, 6)$.

(c) $(3, -4)$ and $(-1, -4)$.

(d) $(-1, 6)$ and $(0, 6)$.

6.1.1 Homework Exercises

Simplify and rationalize the denominator whenever necessary. The variables here take positive values only.

1. $\sqrt[6]{(-3)^6}$
2. $\sqrt[3]{(-3)^3}$
3. $\sqrt[5]{243}$
4. $\sqrt[3]{\dfrac{27}{75}}$

5. $\sqrt[4]{\dfrac{32}{243}}$

6. $\sqrt{300}$

7. $\sqrt{\dfrac{49}{60}}$

8. $\sqrt{\dfrac{99}{128}}$

9. $\sqrt{36+225}$

10. $\sqrt{36}+\sqrt{225}$

11. $\sqrt{144-64}$

12. $\sqrt{144}-\sqrt{64}$

13. $\left(\sqrt[6]{10}\right)^6$

14. $\sqrt[6]{(-10)^6}$

15. $\sqrt[8]{x^8}$

16. $\sqrt[7]{x^7}$

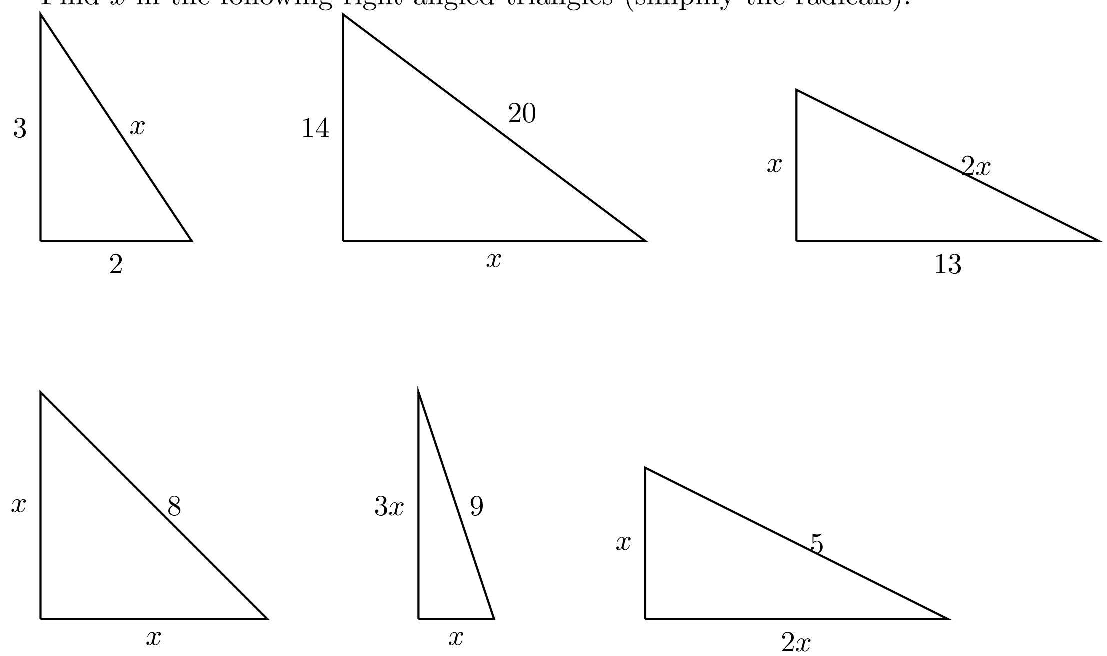
17. $\sqrt[5]{\dfrac{32x^5}{y^5}}$

18. $\sqrt[5]{\dfrac{100000x^7}{y^9}}$

19. $\sqrt{\dfrac{72x^5y^{11}z^{13}}{75a^3}}$

20. $\sqrt[4]{\dfrac{48a^6}{b^8}}$

Find x in the following right angled triangles (simplify the radicals):



Find the distance between the following pairs of points on the coordinate plane.

1. $(3, 8)$ and $(-9, -7)$
2. $(2, 0)$ and $(0, 2)$
3. $(3, -5)$ and $(9, -7)$
4. $(2, 4)$ and $(1, 2)$

6.2 Operations on Radical expressions

In this lesson we continue working with radical expressions following the properties presented in lesson 28. Recall

- For a natural number n, and real numbers a, b, we say $a = \sqrt[n]{b}$ if and only if $a^n = b$. That is, $\left(\sqrt[n]{b}\right)^n = b$.
- On the other hand, $\sqrt[n]{x^n} = |x|$ if n is an **even** natural number, and x is any real number. If x is allowed only positive values, then $\sqrt[n]{x^n} = x$.
- If n is an **odd** natural number, then $\sqrt[n]{x^n} = x$ for any real number x.

- For n an **even** natural number, and a negative, $\sqrt[n]{a}$ is not a real number. But if n is an **odd** natural number, then $\sqrt[n]{a}$ is a real number for any real number a.

- For n any natural number, and a, b any real numbers such that $\sqrt[n]{a}$, $\sqrt[n]{b}$ are real numbers, we have $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

- For n any natural number, and a, b any real numbers such that $b \neq 0$, and $\sqrt[n]{a}$, $\sqrt[n]{b}$ are real numbers, we have $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

- **Caution :** $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ and $\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$.

Now we can proceed with some examples of operations on radical expressions. A strategy for simplifying radicals is to look for perfect n-th power factors of $\sqrt[n]{}$.

1. $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$. (Here, we treat terms involving $\sqrt{2}$ as like terms. For instance, $3x + 5x = 8x$.)

2. $9\sqrt[3]{5} - 12\sqrt[3]{5} = -3\sqrt[3]{5}$. (We treat terms involving $\sqrt[3]{5}$ as like terms. For instance, $9x - 12x = -3x$.)

3. $4\sqrt[6]{10} - 8\sqrt[6]{10} + 22\sqrt[6]{10} = 18\sqrt[6]{10}$.

4. $-5\sqrt{12} + 7\sqrt{3} - \sqrt{75}$. Here we cannot proceed unless we simplify the individual radical terms.

$$\begin{aligned}
-5\sqrt{12} + 7\sqrt{3} - \sqrt{75} &= -5\sqrt{4 \times 3} + 7\sqrt{3} - \sqrt{25 \times 3} \\
&= -5\sqrt{4} \times \sqrt{3} + 7\sqrt{3} - \sqrt{25} \times \sqrt{3} \\
&= -5 \times 2 \times \sqrt{3} + 7\sqrt{3} - 5 \times \sqrt{3} \\
&= -10\sqrt{3} + 7\sqrt{3} - 5\sqrt{3} = -8\sqrt{3}.
\end{aligned}$$

5. $3\sqrt[4]{5} + 7\sqrt[4]{9}$. This cannot be simplified further as $\sqrt[4]{5}$ and $\sqrt[4]{9}$ are unlike radical expressions.

6. $3\sqrt[4]{5} - 8\sqrt[3]{5}$. This cannot be simplified further as $\sqrt[4]{5}$ and $\sqrt[3]{5}$ are unlike radical expressions.

7. $7\sqrt[3]{5} - 8\sqrt[3]{\dfrac{40}{27}} + \dfrac{\sqrt[3]{320}}{2}$. This needs careful analysis.

$$\begin{aligned}
7\sqrt[3]{5} - 8\sqrt[3]{\frac{40}{27}} + \frac{\sqrt[3]{320}}{2} &= 7\sqrt[3]{5} - 8\frac{\sqrt[3]{40}}{\sqrt[3]{27}} + \frac{\sqrt[3]{320}}{2} \\
&= 7\sqrt[3]{5} - 8\frac{\sqrt[3]{8 \times 5}}{\sqrt[3]{27}} + \frac{\sqrt[3]{64 \times 5}}{2} \\
&= 7\sqrt[3]{5} - 8\frac{\sqrt[3]{8} \times \sqrt[3]{5}}{\sqrt[3]{27}} + \frac{\sqrt[3]{64} \times \sqrt[3]{5}}{2} \\
&= 7\sqrt[3]{5} - \frac{8}{1} \times \frac{2\sqrt[3]{5}}{3} + \frac{\cancelto{2}{4}\sqrt[3]{5}}{\cancelto{1}{2}} \\
&= 7\sqrt[3]{5} - \frac{16\sqrt[3]{5}}{3} + 2\sqrt[3]{5} \\
&= \frac{3}{3} \times 7\sqrt[3]{5} - \frac{16\sqrt[3]{5}}{3} + \frac{3}{3} \times 2\sqrt[3]{5} && \textit{The least common denominator is } 3 \\
&= \frac{21\sqrt[3]{5}}{3} - \frac{16\sqrt[3]{5}}{3} + \frac{6\sqrt[3]{5}}{3} \\
&= \frac{21\sqrt[3]{5} - 16\sqrt[3]{5} + 6\sqrt[3]{5}}{3} \\
&= \frac{11\sqrt[3]{5}}{3}
\end{aligned}$$

8. $3\sqrt[4]{32} - \dfrac{9}{\sqrt[4]{8}}$. We first simplify the radical expressions, and rationalize the denominator.

$$\begin{aligned}
3\sqrt[4]{32} - \frac{9}{\sqrt[4]{8}} &= 3\sqrt[4]{16 \times 2} - \frac{9 \times \sqrt[4]{2}}{\sqrt[4]{8} \times \sqrt[4]{2}} \\
&= 3\sqrt[4]{16} \times \sqrt[4]{2} - \frac{9 \times \sqrt[4]{2}}{\sqrt[4]{16}} \\
&= 6\sqrt[4]{2} - \frac{9\sqrt[4]{2}}{2} \\
&= \frac{2}{2} \times 6\sqrt[4]{2} - \frac{9\sqrt[4]{2}}{2} && \textit{The least common denominator is } 2 \\
&= \frac{12\sqrt[4]{2}}{2} - \frac{9\sqrt[4]{2}}{2} \\
&= \frac{12\sqrt[4]{2} - 9\sqrt[4]{2}}{2} = \frac{3\sqrt[4]{2}}{2}
\end{aligned}$$

9. $\sqrt[3]{4x} - \dfrac{7x}{\sqrt[3]{16x^2}}$. Again, first rationalize the denominator.

$$\begin{aligned}
\sqrt[3]{4x} - \frac{7x}{\sqrt[3]{16x^2}} &= \sqrt[3]{4x} - \frac{7x \times \sqrt[3]{4x}}{\sqrt[3]{16x^2} \times \sqrt[3]{4x}} \\
&= \sqrt[3]{4x} - \frac{7x\sqrt[3]{4x}}{\sqrt[3]{64x^3}} \\
&= \sqrt[3]{4x} - \frac{7\cancelto{1}{x}\ \sqrt[3]{4x}}{4\cancelto{1}{x}} \\
&= \frac{4}{4} \times \frac{\sqrt[3]{4x}}{1} - \frac{7\sqrt[3]{4x}}{4} \\
&= \frac{4\sqrt[3]{4x}}{4} - \frac{7\sqrt[3]{4x}}{4} \\
&= \frac{4\sqrt[3]{4x} - 7\sqrt[3]{4x}}{4} = -\frac{3\sqrt[3]{4x}}{4}
\end{aligned}$$

10. $-5\sqrt[3]{7} + 8\sqrt[4]{9} + 9\sqrt[3]{7} - 11\sqrt[4]{9}$. We have two distinct kinds of radical expressions. So, combining like radicals,

$$\begin{aligned}
-5\sqrt[3]{7} + 8\sqrt[4]{9} + 9\sqrt[3]{7} - 11\sqrt[4]{9} &= -5\sqrt[3]{7} + 9\sqrt[3]{7} + 8\sqrt[4]{9} - 11\sqrt[4]{9} \\
&= 4\sqrt[3]{7} - 3\sqrt[4]{9}.
\end{aligned}$$

11. $\sqrt[3]{7} \cdot \sqrt[3]{11} = \sqrt[3]{7 \cdot 11} = \sqrt[3]{77}$ which cannot be simplified further.

12. $\sqrt[3]{5} \cdot \sqrt[4]{11}$. This cannot be simplified any further. Note that the cube-root and the fourth root are distinct kinds of roots.

13. $\sqrt[3]{5} \cdot \sqrt[4]{5}$. This cannot be simplified any further. Note that the cube-root and the fourth root are distinct kinds of roots.

14. $\sqrt[3]{4}\left(\sqrt[3]{5} - 2\sqrt[3]{2}\right)$. Multiplication distributes over subtraction. So,

$$\begin{aligned}
\sqrt[3]{4}\left(\sqrt[3]{5} - 2\sqrt[3]{2}\right) &= \sqrt[3]{4} \times \sqrt[3]{5} - \sqrt[3]{4} \times 2\sqrt[3]{2} \\
&= \sqrt[3]{4 \times 5} - 2\sqrt[3]{4 \times 2} \\
&= \sqrt[3]{20} - 2\sqrt[3]{8} \\
&= \sqrt[3]{20} - 2 \times 2 = \sqrt[3]{20} - 4.
\end{aligned}$$

15. $\left(\sqrt{x}-\sqrt{y}\right)^2$. Assume that the variables are non-negative.

$$\begin{aligned}\left(\sqrt{x}-\sqrt{y}\right)^2 &= \left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)\\ &= \sqrt{x}\sqrt{x}-\sqrt{x}\sqrt{y}-\sqrt{y}\sqrt{x}+\sqrt{y}\sqrt{y}\\ &= x-\sqrt{xy}-\sqrt{yx}+y\\ &= x-\sqrt{xy}-\sqrt{xy}+y = x-2\sqrt{xy}+y\end{aligned}$$

16. Note that $(\sqrt{x})^2-\left(\sqrt{y}\right)^2 = x-y$. Thus, $(\sqrt{x})^2-\left(\sqrt{y}\right)^2 \neq \left(\sqrt{x}-\sqrt{y}\right)^2$.

17. $\left(\sqrt{x}+\sqrt{y}\right)^2$. Assume that the variables are non-negative.

$$\begin{aligned}\left(\sqrt{x}+\sqrt{y}\right)^2 &= \left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)\\ &= \sqrt{x}\sqrt{x}+\sqrt{x}\sqrt{y}+\sqrt{y}\sqrt{x}+\sqrt{y}\sqrt{y}\\ &= x+\sqrt{xy}+\sqrt{yx}+y\\ &= x+\sqrt{xy}+\sqrt{xy}+y = x+2\sqrt{xy}+y\end{aligned}$$

18. Note that $(\sqrt{x})^2+\left(\sqrt{y}\right)^2 = x+y$. Thus, $(\sqrt{x})^2+\left(\sqrt{y}\right)^2 \neq \left(\sqrt{x}+\sqrt{y}\right)^2$.

19. $\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)$. Assume that the variables are non-negative. We follow the rules of polynomial multiplication.

$$\begin{aligned}\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right) &= \sqrt{x}\sqrt{x}+\sqrt{x}\sqrt{y}-\sqrt{y}\sqrt{x}-\sqrt{y}\sqrt{y}\\ &= x+\sqrt{xy}-\sqrt{yx}-y\\ &= x+\sqrt{xy}-\sqrt{xy}-y = x-y\end{aligned}$$

20. $\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}-\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)$ Assume that the variables are non-negative.

$$\begin{aligned}&\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}-\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)\\ &\quad= \sqrt[3]{x}\sqrt[3]{x^2}-\sqrt[3]{x}\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{x}\sqrt[3]{y^2}+\sqrt[3]{y}\sqrt[3]{x^2}-\sqrt[3]{y}\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y}\sqrt[3]{y^2}\\ &\quad= \sqrt[3]{x^3}-\sqrt[3]{x^2y}+\sqrt[3]{xy^2}+\sqrt[3]{yx^2}-\sqrt[3]{yxy}+\sqrt[3]{y^3}\\ &\quad= x-\sqrt{x^2y}+\sqrt[3]{xy^2}+\sqrt[3]{x^2y}-\sqrt[3]{xy^2}+y = x+y.\end{aligned}$$

21. $\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)$ Assume that the variables are non-negative.

$$\begin{aligned}&\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)\\ &\quad= \sqrt[3]{x}\sqrt[3]{x^2}+\sqrt[3]{x}\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{x}\sqrt[3]{y^2}-\sqrt[3]{y}\sqrt[3]{x^2}-\sqrt[3]{y}\sqrt[3]{x}\sqrt[3]{y}-\sqrt[3]{y}\sqrt[3]{y^2}\\ &\quad= \sqrt[3]{x^3}+\sqrt[3]{x^2y}+\sqrt[3]{xy^2}-\sqrt[3]{yx^2}-\sqrt[3]{yxy}-\sqrt[3]{y^3}\\ &\quad= x+\sqrt{x^2y}+\sqrt[3]{xy^2}-\sqrt[3]{x^2y}-\sqrt[3]{xy^2}-y = x-y.\end{aligned}$$

The examples 16, 17, and 18 are the radical versions of certain algebraic formulae. Recall and compare the formulae below.

Name	Recall	Compare
Difference of squares	$(a-b)(a+b)=a^2-b^2$	$\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)=x-y$
Sum of cubes	$(a+b)(a^2-ab+b^2)$ $=a^3+b^3$	$\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}-\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)$ $=x+y$
Difference of cubes	$(a-b)(a^2+ab+b^2)$ $=a^3-b^3$	$\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)$ $=x-y$

These formulae help us to rationalize the denominators of certain complicated radical expressions. Here are some examples:

1.

$$\begin{aligned}\frac{2}{\sqrt{3}-\sqrt{5}} &= \frac{2\times\left(\sqrt{3}+\sqrt{5}\right)}{\left(\sqrt{3}-\sqrt{5}\right)\times\left(\sqrt{3}+\sqrt{5}\right)}\\ &= \frac{2\left(\sqrt{3}+\sqrt{5}\right)}{3-5}\\ &= \frac{\cancelto{1}{2}\left(\sqrt{3}+\sqrt{5}\right)}{-\cancelto{1}{2}} = -\left(\sqrt{3}+\sqrt{5}\right)\end{aligned}$$

2.

$$\begin{aligned}\frac{4}{\sqrt[3]{7}-\sqrt[3]{9}} &= \frac{4\times\left(\sqrt[3]{49}+\sqrt[3]{63}+\sqrt[3]{81}\right)}{\left(\sqrt[3]{7}-\sqrt[3]{9}\right)\times\left(\sqrt[3]{49}+\sqrt[3]{63}+\sqrt[3]{81}\right)}\\ &= \frac{4\left(\sqrt[3]{49}+\sqrt[3]{63}+\sqrt[3]{81}\right)}{7-9}\\ &= \frac{\cancelto{2}{4}\left(\sqrt[3]{49}+\sqrt[3]{63}+\sqrt[3]{81}\right)}{-\cancelto{1}{2}}\\ &= -2\left(\sqrt[3]{49}+\sqrt[3]{63}+\sqrt[3]{81}\right)\end{aligned}$$

3. Here is a general case, which will be relevant when we work with complex numbers:

$$\begin{aligned}\frac{1}{a+b\sqrt{c}} &= \frac{1\times(a-b\sqrt{c})}{(a+b\sqrt{c})\times(a-b\sqrt{c})}\\ &= \frac{a-b\sqrt{c}}{a^2-(b\sqrt{c})^2}\\ &= \frac{a-b\sqrt{c}}{a^2-b^2c}.\end{aligned}$$

Classroom Exercises : Simplify

(a) $\sqrt{5}+11\sqrt{5}$

(b) $-4\sqrt[4]{2}+7\sqrt[4]{2}$

(c) $12\sqrt[5]{2}-7\sqrt[5]{64}$

(d) $\sqrt{32}-\sqrt{60}+\sqrt{162}$

(e) $2\sqrt{45}-5\sqrt{20}+\sqrt{3}$

(f) $3\sqrt{8}+2\sqrt{18}-\sqrt{50}$

(g) $-3\sqrt{12}+2\sqrt{27}-4\sqrt{48}$

(h) $\sqrt{5}(\sqrt{5}+\sqrt{3})$

(i) $\sqrt{2}(\sqrt{8}-\sqrt{2})$

(j) $\sqrt{3}(\sqrt{15}+2\sqrt{3})$

(k) $\sqrt[5]{3}\cdot\sqrt[5]{7}$

(l) $\sqrt[5]{3}\cdot\sqrt[7]{5}$

(m) $\sqrt[4]{3}\left(\sqrt[4]{4}-8\sqrt[4]{5}\right)$

(n) $(\sqrt{3}+\sqrt{5})^2$

(o) $(\sqrt{7}-\sqrt{3})^2$

(p) $\left(\sqrt[3]{5}+\sqrt[4]{7}\right)^2$

(q) $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$

(r) $(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})$

(s) $\left(\sqrt{5}+\sqrt{7}\right)\left(\sqrt{5}-\sqrt{7}\right)$

(t) $\dfrac{1}{\sqrt[5]{16}}$

(u) $2\sqrt[5]{2}-3\sqrt[5]{64}+\dfrac{1}{\sqrt[5]{16}}$

(v) $\sqrt[3]{4m}-\dfrac{2m}{\sqrt[3]{2m^2}}$

Rationalize the denominators :

(a) • $\dfrac{3}{\sqrt{5}}$ • $\dfrac{4}{\sqrt{2}}$ • $-\dfrac{3}{\sqrt{11}}$ • $\dfrac{2}{\sqrt[3]{5}}$ • $\dfrac{4}{\sqrt[3]{2}}$

(b) • $\dfrac{1}{3+\sqrt{2}}$ • $\dfrac{4}{3-\sqrt{2}}$ • $-\dfrac{3}{1+\sqrt{11}}$ • $\dfrac{1}{\sqrt{7}+\sqrt{5}}$ • $\dfrac{1}{\sqrt{7}-\sqrt{5}}$

(c) • $\dfrac{1}{1+\sqrt[3]{2}}$ • $\dfrac{4}{1-\sqrt[3]{2}}$ • $-\dfrac{3}{1+\sqrt{11}}$ • $\dfrac{1}{\sqrt[3]{7}-\sqrt[3]{5}}$ • $\dfrac{1}{\sqrt[3]{7}+\sqrt[3]{5}}$

6.2.1 Homework Exercises

Perform the following operations.

(1) $3\sqrt{5}+7\sqrt{5}$ • $4\sqrt{2}-2\sqrt{2}$ • $-6\sqrt{3}+4\sqrt{3}$

(2) $-4\sqrt{7}-8\sqrt{7}$ • $-\sqrt{5}+7\sqrt{5}$ • $-4\sqrt[3]{7}+12\sqrt[3]{7}$

(3) $8\sqrt[10]{5}-9\sqrt[10]{5}+12\sqrt[10]{5}$ • $\sqrt[3]{7}-\sqrt[10]{7}$ • $\sqrt[3]{7}-\sqrt[10]{7}+21\sqrt[3]{7}+16\sqrt[10]{7}$

(4) $3\sqrt{2}-5\sqrt{32}+2\sqrt{162}$ • $-7\sqrt{5}+11\sqrt{20}-12\sqrt{45}$ • $\dfrac{10}{\sqrt{5}}+\sqrt{5}$

(5) $-\dfrac{4}{\sqrt{2}}+9\sqrt{2}$ • $\dfrac{6}{\sqrt{3}}+11\sqrt{3}$ • $\sqrt{2}\cdot\sqrt{18}$

(6) $\sqrt{20}\cdot\sqrt{5}$ • $\sqrt{5}\left(3+\sqrt{7}-8\sqrt{5}\right)$ • $\sqrt{2}\left(\sqrt{8}+2\sqrt{2}-7\sqrt{50}\right)$

(7) $5\sqrt{3}\left(6\sqrt{3}-11\sqrt{12}+9\sqrt{75}\right)$ • $\left(3+\sqrt{3}\right)^2$ $\left(\sqrt{5}-\sqrt{3}\right)^2$

(8) $\left(3+\sqrt{3}\right)\left(9-\sqrt{5}\right)$ • $\left(-2+\sqrt{5}\right)\left(1+\sqrt{7}-\sqrt{5}\right)$ • $\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)$

(9) $\left(2+\sqrt{5}\right)\left(1-2\sqrt{5}+\sqrt{3}\right)$ • $\left(2-\sqrt{5}\right)\left(1+2\sqrt{5}+\sqrt{2}\right)$

Rationalize the denominators

1. $\dfrac{1}{\sqrt{3}}$; $\dfrac{3}{\sqrt{6}}$; $-\dfrac{4}{\sqrt{2}}$; $\dfrac{1}{\sqrt[3]{3}}$; $\dfrac{7}{\sqrt[5]{4}}$.

2. $-\dfrac{4}{\sqrt{12}}$; $\dfrac{3}{\sqrt[3]{4}}$; $\dfrac{3}{2-\sqrt{5}}$; $\dfrac{3}{3+\sqrt{7}}$.

3. $\dfrac{4}{2-\sqrt[3]{7}}$; $\dfrac{4}{2+\sqrt[3]{7}}$.

6.3 Complex numbers

Recall the argument why $\sqrt{9} = 3$. Note, $3 \times 3 = 9$. Therefore $\sqrt{9} = 3$. Now recall the argument why $\sqrt{-9}$ cannot be a real number. The square of a real number cannot be negative. Therefore, there is no real number which is $\sqrt{-9}$.

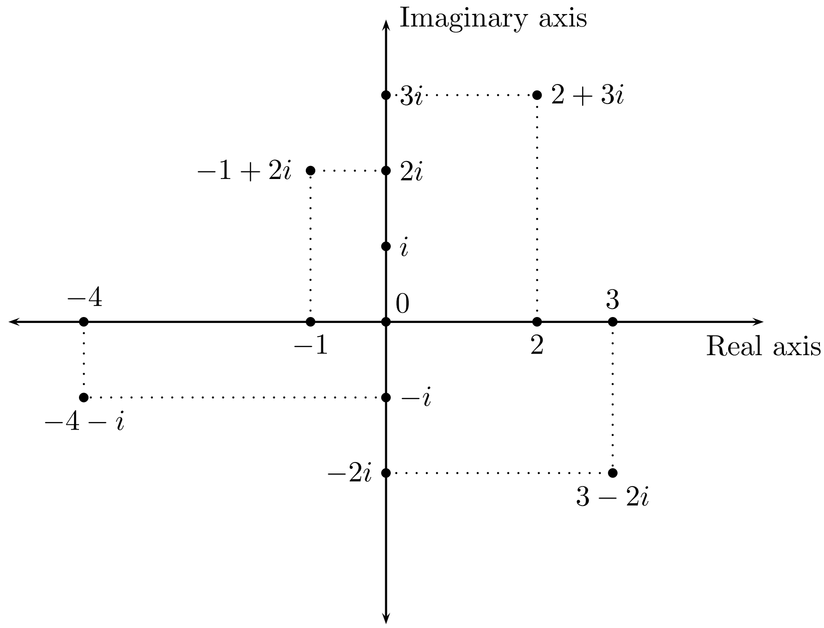
Complex numbers allow us to take square-roots of any real number. We first define

$$i = \sqrt{-1}. \textit{ Therefore, } i^2 = \left(\sqrt{-1}\right)^2 = -1.$$

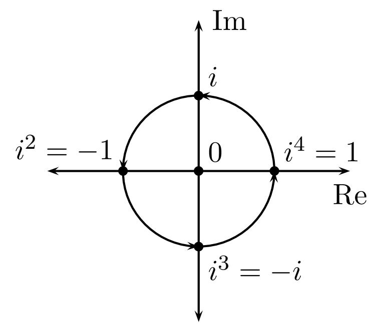
The letter "i" stands for "imaginary."

Every complex number z can be written in the form of $z = a + bi$ where a, b are real numbers. The real number a is called the **real part** of z, and bi is called the **imaginary part** of z. For instance, the real part of the compex number $3 + 4i$ is 3, and its imaginary part is $4i$. Similarly, the real part of the complex number $\left(\frac{1}{2} - \frac{7}{9}i\right)$ is $\frac{1}{2}$ and its imaginary part is $-\frac{7}{9}i$.

Complex numbers are arranged on a plane. The horizontal axis is the real number line and the vertical axis is the imaginary line. These two lines intersect at the complex number $0 = 0 + 0i$. A complex number is then plotted on this plane. The complex number $a + bi$ is positioned at the point (a, b) as in coordinate geometry. Here are some examples :



Power of i	Value
i^1	i
i^2	-1
i^3	$i^2 \times i = -1 \times i = -i$
i^4	$i^2 \times i^2 = -1 \times -1 = 1$



This table allows us to find other powers of i.

- $i^{11} = i^{8+3} = i^8 \times i^3 = 1 \times -i = -i$. Notice that we write $11 = 8 + 3$ because 8 is a multiple of 4. In other words, we separate out as many 4's as possible from the exponent first. Mathematically speaking, we divide 11 by 4, and get a remainder of 3. That is, $11 = 4 \times 2 + 3$. Geometrically, this can be seen by going around 0 **counter-clockwise** starting from 1 in the above diagram, and reading, "$i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1, i^7 = -i, i^8 = 1, i^9 = i, i^{10} = -1, i^{11} = -i$."
- $i^{245} = i^{244+1} = i^{244} \times i^1 = 1 \times i = i$. Here, dividing 245 by 4 gives us a remainder of 1. Check: $245 = 4 \times 61 + 1 = 244 + 1$.
- $i^{98} = i^{96+2} = i^{96} \times i^2 = 1 \times -1 = -1$. Again, dividing 98 by 4 gives us a remainder of 2. Check: $98 = 4 \times 24 + 2$.

- $i^{-1} = \frac{1}{i} = \frac{i^4}{i} = i^3 = -i$. Geometrically, this is obtained by going **clockwise** around 0 starting from 1, and reading, $i^{-1} = -i$.

- $i^{-2} = \frac{1}{i^2} = \frac{i^4}{i^2} = i^2 = -1$.

- $i^{-3} = \frac{1}{i^3} = \frac{i^4}{i^3} = i$.

- $i^{-7} = \frac{1}{i^7} = \frac{1}{i^{4+3}} = \frac{1}{i^4 \times i^3} = \frac{1}{i^3} = \frac{i^4}{i^3} = i$

- $i^{-54} = \frac{1}{i^{54}} = \frac{1}{i^{52+2}} = \frac{1}{i^{52} \times i^2} = \frac{1}{1 \times -1} = \frac{1}{-1} = -1$. Check: When 54 is divided by 4, the remainder is 2. We get $54 = 4 \times 13 + 2 = 52 + 2$.

Addition and subtraction of complex numbers follow the same rules as radical expressions. This is expected because i is the radical expression $\sqrt{-1}$. Here are some examples:

1. $(2 + 3i) + (5 + 4i) = (2 + 5) + (3i + 4i) = 7 + 7i$.

2. $(2 + 3i) - (5 + 4i) = 2 + 3i - 5 - 4i = (2 - 5) + (3i - 4i) = -3 - i$.

3. $(\sqrt{3} + \sqrt[3]{5}i) + (3 + 5i) = (\sqrt{3} + 3) + (\sqrt[3]{5} + 5)i$

4. $(\sqrt{3} + \sqrt[3]{5}i) - (3 + 5i) = \sqrt{3} + \sqrt[3]{5}i - 3 - 5i = (\sqrt{3} - 3) + (\sqrt[3]{5} - 5)i$

5. $$\begin{aligned}
&\left(\frac{2}{5} - 7i\right) + \left(2 + \frac{1}{3}i\right) \\
&= \frac{2}{5} - 7i + 2 + \frac{1}{3}i \\
&= \left(\frac{2}{5} + 2\right) + \left(-7i + \frac{1}{3}i\right) && \textit{Combine the real parts and the imaginary parts} \\
&= \left(\frac{2}{5} + \frac{2 \times 5}{5}\right) + \left(\frac{-7 \times 3}{3}i + \frac{1}{3}i\right) && \textit{Get common denominators} \\
&= \left(\frac{2}{5} + \frac{10}{5}\right) + \left(\frac{-21}{3}i + \frac{1}{3}i\right) \\
&= \left(\frac{2 + 10}{5}\right) + \left(\frac{-21 + 1}{3}i\right) = \frac{12}{5} - \frac{20}{3}i
\end{aligned}$$

6.

$$\begin{aligned}
&\left(\frac{2}{5} - 7i\right) - \left(2 + \frac{1}{3}i\right) \\
&= \frac{2}{5} - 7i - 2 - \frac{1}{3}i && \textit{Distribute the sign} \\
&= \left(\frac{2}{5} - 2\right) + \left(-7i - \frac{1}{3}i\right) && \textit{Combine the real parts and the imaginary parts} \\
&= \left(\frac{2}{5} - \frac{2\times 5}{5}\right) + \left(\frac{-7\times 3}{3}i - \frac{1}{3}i\right) && \textit{Get common denominators} \\
&= \left(\frac{2}{5} - \frac{10}{5}\right) + \left(\frac{-21}{3}i - \frac{1}{3}i\right) \\
&= \left(\frac{2-10}{5}\right) + \left(\frac{-21-1}{3}i\right) = -\frac{8}{5} - \frac{22}{3}i
\end{aligned}$$

Multiplication of complex numbers also follow the same rules as for multiplication of radical expressions. Keep in mind that $i^2 = -1$. Because of this, the product of two complex numbers can be written in the form of $a + bi$, with no higher power of i appearing. Here are some examples:

- $3(4 + 5i) = 12 + 15i$
- $3i(4 - 5i) = 12i - 15\cancel{i^2}^{\times(-1)} = 12i + 15 = 15 + 12i.$
- $\sqrt{-4}\sqrt{-9} = 2i \cdot 3i = 6i^2 = -6.$
- $\sqrt{-5}\sqrt{-11} = \sqrt{5}i \cdot \sqrt{11}i = \sqrt{55}i^2 = -\sqrt{55}.$
- $(2 + 3i)(5 + 4i).$

$$\begin{aligned}
(2+3i)(5+4i) &= 2(5+4i) + 3i(5+4i) && \textit{Distribution} \\
&= 10 + 8i + 15i \underbrace{+12\cancel{i^2}^{\times(-1)}}_{-12} && i^2 \textit{ is equal to } (-1) \\
&= 10 + 8i + 15i - 12 \\
&= (10 - 12) + (8i + 15i) && \textit{Combining real and imaginary parts} \\
&= -2 + 23i.
\end{aligned}$$

- $(\sqrt{3}+\sqrt[3]{5}i)(3+5i)$.

$$\begin{aligned}
(\sqrt{3}+\sqrt[3]{5}i)(3+5i) &= \sqrt{3}(3+5i)+\sqrt[3]{5}i(3+5i) && \textit{Distribute}\\
&= 3\sqrt{3}+5\sqrt{3}i+3\sqrt[3]{5}i+5\sqrt[3]{5}\overset{\times(-1)}{\cancel{i^2}} && i^2 \textit{ is equal to } (-1)\\
&= 3\sqrt{3}+5\sqrt{3}i+3\sqrt[3]{5}i-5\sqrt[3]{5}\\
&= (3\sqrt{3}-5\sqrt[3]{5})+(5\sqrt{3}+3\sqrt[3]{5})i && \textit{This cannot be simplified further.}
\end{aligned}$$

- $\left(\frac{2}{5}-7i\right)\left(2+\frac{1}{3}i\right)$

$$\begin{aligned}
\left(\frac{2}{5}-7i\right)\left(2+\frac{1}{3}i\right) &= \frac{2}{5}\left(2+\frac{1}{3}i\right)-7i\left(2+\frac{1}{3}i\right) && \textit{Distribute}\\
&= \frac{2}{5}\times\frac{2}{1}+\frac{2}{5}\times\frac{1}{3}i-\frac{7}{1}\times\frac{2}{1}i-\frac{7}{1}\times\frac{1}{3}\overset{\times(-1)}{\cancel{i^2}} && i^2 \textit{ is equal to } (-1)\\
&= \frac{4}{5}+\frac{2}{15}i-\frac{14}{1}i+\frac{7}{3}\\
&= \left(\frac{4}{5}+\frac{7}{3}\right)+\left(\frac{2}{15}-\frac{14}{1}\right)i && \textit{Collect the real and imaginary parts}\\
&= \left(\frac{4\times 3}{5\times 3}+\frac{7\times 5}{3\times 5}\right)+\left(\frac{2}{15}-\frac{14\times 15}{1\times 15}\right)i && \textit{Get common denominators}\\
&= \left(\frac{12}{15}+\frac{35}{15}\right)+\left(\frac{2}{15}-\frac{210}{1\times 15}\right)i\\
&= \frac{47}{15}+\frac{-208}{15}i=\frac{47}{15}-\frac{208}{15}i
\end{aligned}$$

Before we can divide complex numbers we need to know the concepts of the **complex conjugate** and the **norm** of a complex number. The **complex conjugate** of the complex number $a+bi$ is $a-bi$. For example

- The complex conjugate of $3+4i$ is $3-4i$
- The complex conjugate of $5-7i$ is $5-(-7i)=5+7i$.
- The complex conjugate of $\sqrt{11}-\sqrt[3]{7}i$ is $\sqrt{11}+\sqrt[3]{7}i$.
- The complex conjugate of $\frac{3}{4}+\frac{2}{11}i$ is $\frac{3}{4}-\frac{2}{11}i$.
- The complex conjugate of 10 is 10. Note, $10=10+0i$ and therefore, its complex conjugate is $10-0i$ which is 10.

- The complex conjugate of $10i$ is $-10i$. Note, $10i = 0 + 10i$ and therefore, its complex conjugate is $0 - 10i$ which is $-10i$.

The **norm** of the complex number $a+bi$ is $\sqrt{(a+bi)(a-bi)} = \sqrt{a^2+b^2}$. The norm of $a+bi$ is denoted by $||a+bi||$. The norm is the the distance of the complex number from 0 in the complex plane.

Examples:

- $||3+4i|| = \sqrt{(3+4i)(3-4i)} = \sqrt{9 - 12i + 12i - 16\cancelto{(-1)}{i^2}} = \sqrt{9+16} = \sqrt{25} = 5.$
- $||5-7i|| = \sqrt{(5-7i)(5+7i)} = \sqrt{25 + 35i - 35i - 49\cancelto{(-1)}{i^2}} = \sqrt{25+49} = \sqrt{74}.$
-
$$\begin{aligned}
||\sqrt{11} - \sqrt[3]{7}i|| &= \sqrt{(\sqrt{11} - \sqrt[3]{7}i)(\sqrt{11} + \sqrt[3]{7}i)} \\
&= \sqrt{(\sqrt{11})^2 + \sqrt{11}\sqrt[3]{7}i - \sqrt{11}\sqrt[3]{7}i - (\sqrt[3]{7})^2\cancelto{(-1)}{i^2}} \\
&= \sqrt{11 + \sqrt[3]{49}}
\end{aligned}$$
-
$$\begin{aligned}
||\frac{3}{4} + \frac{2}{11}i|| &= \sqrt{\left(\frac{3}{4} + \frac{2}{11}i\right)\left(\frac{3}{4} - \frac{2}{11}i\right)} \\
&= \sqrt{\frac{3}{4} \times \frac{3}{4} - \frac{3}{4} \times \frac{2}{11}i + \frac{3}{4} \times \frac{2}{11}i - \frac{2}{11} \times \frac{2}{11}\cancelto{(-1)}{i^2}}
\end{aligned}$$
$$= \sqrt{\frac{9}{16} + \frac{4}{121}} = \sqrt{\frac{1089+64}{1936}} = \sqrt{\frac{1153}{1936}} = \frac{\sqrt{1153}}{44}$$
- $||10|| = \sqrt{10 \times 10} = \sqrt{100} = 10$
- $||10i|| = \sqrt{10i \times -10i} = \sqrt{-100\cancelto{(-1)}{i^2}} = \sqrt{100} = 10.$

Before we learn to divide complex numbers, recall how we rationalized the denominator of $\dfrac{1}{a+b\sqrt{c}}$. Now we are ready to divide complex numbers:

- $3 \div (4+5i)$

$$\begin{aligned}
3 \div (4+5i) &= \frac{3}{(4+5i)} && \textit{Multiply the numerator and denominator} \\
&= \frac{3 \times (4-5i)}{(4+5i)\times(4-5i)} && \textit{by the complex conjugate of the denominator} \\
&= \frac{12-15i}{16-20i+20i-25\cancelto{(-1)}{i^2}} && \textit{Multiply the complex numbers} \\
&= \frac{12-15i}{16+25} \\
&= \frac{12-15i}{41} = \frac{12}{41} - \frac{15}{41}i
\end{aligned}$$

- $3i \div (4-5i)$

$$\begin{aligned}
3i \div (4-5i) &= \frac{3i}{(4-5i)} && \textit{Multiply the numerator and denominator} \\
&= \frac{3i \times (4+5i)}{(4-5i)\times(4+5i)} && \textit{by the complex conjugate of the denominator} \\
&= \frac{12i+15\cancelto{(-1)}{i^2}}{16+20i-20i-25\cancelto{(-1)}{i^2}} && \textit{Multiply the complex numbers} \\
&= \frac{-15+12i}{16+25} \\
&= \frac{-15+12i}{41} = -\frac{15}{41} + \frac{12}{41}i
\end{aligned}$$

-

$$\begin{aligned}
(2+3i) \div (5+4i) &= \frac{2+3i}{(5+4i)} \\
&= \frac{(2+3i)\times(5-4i)}{(5+4i)\times(5-4i)} \\
&= \frac{10-8i+15i-12\cancelto{(-1)}{i^2}}{25-20i+20i-16\cancelto{(-1)}{i^2}} \\
&= \frac{10+12-8i+15i}{25+16} = \frac{22+7i}{41} = \frac{22}{41} + \frac{7}{41}i
\end{aligned}$$

- $(\sqrt{3}+\sqrt[3]{5}i) \div (\sqrt{7}+5i)$

$$
\begin{aligned}
&= \frac{(\sqrt{3}+\sqrt[3]{5}i)}{(\sqrt{7}+5i)} && \textit{Multiply the numerator and denominator}\\
&= \frac{(\sqrt{3}+\sqrt[3]{5}i)\times(\sqrt{7}-5i)}{(\sqrt{7}+5i)\times(\sqrt{7}-5i)} && \textit{by the complex conjugate of the denominator}\\
&= \frac{\sqrt{3}\sqrt{7}-5\sqrt{3}i+\sqrt[3]{5}\sqrt{7}i-5\sqrt[3]{5}\cancelto{(-1)}{i^2}}{\sqrt{7}\sqrt{7}-5\sqrt{7}i+5\sqrt{7}i-25\cancelto{(-1)}{i^2}} && \textit{Multiply the complex numbers}\\
&= \frac{\sqrt{21}+5\sqrt[3]{5}+(-5\sqrt{3}+\sqrt[3]{5}\sqrt{7})i}{7+25}\\
&= \frac{\sqrt{21}+5\sqrt[3]{5}+(-5\sqrt{3}+\sqrt[3]{5}\sqrt{7})i}{32}\\
&= \frac{\sqrt{21}+5\sqrt[3]{5}}{32}+\frac{(-5\sqrt{3}+\sqrt[3]{5}\sqrt{7})}{32}i
\end{aligned}
$$

- $\left(\frac{2}{5}-7i\right) \div \left(2+\frac{1}{3}i\right)$

$$
\begin{aligned}
&= \frac{\left(\frac{2}{5}-7i\right)}{\left(2+\frac{1}{3}i\right)} && \textit{Multiply the numerator and denominator}\\
&= \frac{\left(\frac{2}{5}-7i\right)\times\left(2-\frac{1}{3}i\right)}{\left(2+\frac{1}{3}i\right)\times\left(2-\frac{1}{3}i\right)} && \textit{by the complex conjugate of the denominator}
\end{aligned}
$$

$$= \frac{\frac{2}{5}\times\frac{2}{1}-\frac{2}{5}\times\frac{1}{3}i-\frac{7}{1}\times\frac{2}{1}i+\frac{7}{1}\times\frac{1}{3}\cancelto{(-1)}{i^2}}{4-2\times\frac{1}{3}i+2\times\frac{1}{3}i-\frac{1}{3}\times\frac{1}{3}\cancelto{(-1)}{i^2}} \qquad \textit{Multiply the complex numbers}$$

$$= \frac{\frac{4}{5}-\frac{7}{3}+\left(-\frac{2}{15}-\frac{14}{1}\right)i}{4+\frac{1}{9}}$$

$$= \frac{\frac{4\times 3}{5\times 3}-\frac{7\times 5}{3\times 5}+\left(-\frac{2}{15}-\frac{14\times 15}{15}\right)i}{\frac{4\times 9}{9}+\frac{1}{9}}$$

$$= \frac{\frac{12-35}{15}+\left(\frac{-2-210}{15}\right)i}{\frac{36+1}{9}}$$

$$= \frac{-\frac{\cancelto{9}{27}}{\cancelto{5}{15}}-\frac{212}{15}i}{\frac{37}{9}} = \frac{-\frac{9}{5}-\frac{212}{15}i}{\frac{37}{9}}$$

$$= \left(-\frac{9}{5}-\frac{212}{15}i\right)\frac{9}{37} = -\frac{9\times 9}{5\times 37}-\frac{212\times 9}{15\times 37}i$$

$$= -\frac{81}{187}-\frac{1908}{525}i$$

Classroom Exercises:

1. Write the real and imaginary parts of $\frac{1}{3}+\sqrt{3}i$; $\quad -\sqrt[3]{4}+\frac{2}{7}i$; $\quad -3-7i$; $\quad 4-8i$; $\quad 0$; $\quad 1$; $\quad i$.

2. What is the complex conjugate of each of the following? 7; $\quad 7i$; $\quad 12-3i$; $\quad \frac{2}{7}+\frac{3}{4}i$; $\quad \sqrt{11}+\sqrt[4]{5}i$.

3. Write the following in the form $a+bi$.

   (a) i^2; $\quad i^3$; $\quad i^4$; $\quad i^7$; $\quad i^{98}$; $\quad i^{-6}$; $\quad i^{-43}$.

   (b) $(3-2i)+(-7+12i)$; $\quad (\sqrt{11}+5i)+(8i-\sqrt[3]{7})$; $\quad \left(\frac{2}{7}+\frac{3}{4}i\right)+\left(-\frac{5}{2}+\frac{7}{12}i\right)$.

   (c) $(3-2i)-(-7+12i)$; $\quad (\sqrt{11}+5i)-(8-\sqrt[3]{7}i)$; $\quad \left(\frac{2}{7}+\frac{3}{4}i\right)-\left(-\frac{5}{2}+\frac{7}{12}i\right)$.

(d) $-5(3-4i)$; $\sqrt{-9}\sqrt{-16}$; $\sqrt{-7}\sqrt{-3}$; $-5i(3-4i)$; $(3-2i)(-7+12i)$; $(\sqrt{11}+5i)(8-\sqrt[3]{7}i)$; $\left(\frac{2}{7}+\frac{3}{4}i\right)\left(-\frac{5}{2}+\frac{7}{12}i\right)$.

(e) $\dfrac{-5}{(3-4i)}$; $\dfrac{-5i}{(3-4i)}$; $\dfrac{(3-2i)}{(-7+12i)}$; $\dfrac{(\sqrt{11}+5i)}{(8-\sqrt{7}i)}$; $\dfrac{\left(\frac{2}{7}+\frac{3}{4}i\right)}{\left(-\frac{5}{2}+\frac{7}{12}i\right)}$.

6.3.1 Homework Exercises

1. Write the real and imaginary parts of

   (a) $\frac{2}{9}+3i$; $-\sqrt{5}+4i$; $-8-9i$; $9-11i$.

   (b) 7; $3i$; $-4i$.

2. What is the complex conjugate of each of the following?

   (a) -3; $-3i$; $-9+12i$; $\sqrt[4]{3}-8i$; $\frac{2}{5}+\frac{7}{9}i$.

3. Write the following in the form $a+bi$

   (a) i^3; i^{33}; i^{333}; i^{-7}; i^{-13}; i^{-220}.

   (b) $(-9+12i)+(-2-6i)$; $(\sqrt[4]{3}-8i)+(\sqrt[4]{5}+7i)$; $\left(\frac{2}{5}+\frac{7}{9}i\right)+\left(\frac{1}{3}+\frac{5}{12}i\right)$.

   (c) $(-9+12i)-(-2-6i)$; $(\sqrt[4]{3}-8i)-(\sqrt[4]{5}+7i)$; $\left(\frac{2}{5}+\frac{7}{9}i\right)-\left(\frac{1}{3}+\frac{5}{12}i\right)$.

   (d) $4(3+12i)$; $\sqrt{-4}\sqrt{-25}$; $\sqrt{-3}\sqrt{-2}$; $4i(-3+12i)$; $(-9+12i)(-2-6i)$; $(\sqrt{3}-8i)(\sqrt{5}+7i)$; $\left(\frac{2}{5}+\frac{7}{9}i\right)\left(\frac{1}{3}+\frac{5}{12}i\right)$.

   (e) $\dfrac{4}{(3+12i)}$; $\dfrac{4i}{(-3+12i)}$; $\dfrac{(-9+12i)}{(-2-6i)}$; $\dfrac{(\sqrt{3}-8i)}{(\sqrt{5}+7i)}$; $\dfrac{\left(\frac{2}{5}+\frac{7}{9}i\right)}{\left(\frac{1}{3}+\frac{5}{12}i\right)}$.

Chapter 7

Quadratic equations and some conics

7.1 Completing the square and the quadratic formula

A polynomial of degree 2 is called, a **quadratic polynomial**. A **quadratic equation** is an equation which can be written in the form of $P = 0$ for a quadratic polynomial P. In our course we will be concerned with **quadratic equations with real coefficients in one variable**. Any quadratic equation with real coefficients in one variable, x, can be written in the form

$$ax^2 + bx + c = 0 \quad \text{for } a, b, c \in \mathbb{R}, a \neq 0.$$

Examples of quadratic equations with real coefficients are

$$2x^2 + 3x + 4 = 0, \quad -3x^2 + 4x - 5 = 0, \quad \frac{1}{4}x^2 - \frac{2}{3}x + 7 = 0,$$
$$\sqrt{3}x^2 - \frac{1}{5}x + \sqrt{2} = 0, \quad 3x^2 + 4x = 0, \quad 3x^2 + 2 = 0, \cdots .$$

We derive the complete squares formulae:

$$\begin{aligned}(x+h)^2 &= (x+h)(x+h) \\ &= x^2 + hx + hx + h^2 \\ &= x^2 + 2hx + h^2.\end{aligned}$$

Likewise,

$$\begin{aligned}(x-h)^2 &= (x-h)(x-h) \\ &= x^2 - hx - hx + h^2 \\ &= x^2 - 2hx + h^2.\end{aligned}$$

Every quadratic equation can be written in terms of a **complete square** as follows.

$$a(x-h)^2 = t \quad \text{for } a, h, t \in \mathbb{R}, a \neq 0.$$

The process of converting any quadratic polynomial in one variable to a polynomial in the form of $a(x-h)^2 - t$ complete square is called **completing the square**.

Example 1:

$$x^2 + 4x + 4 = (x+2)^2. \quad \textit{(this is just the complete square formula).}$$

Example 2:

$$\begin{aligned} x^2 + 4x &= \underbrace{x^2 + 4x + 4}{} - 4 \\ &= (x+2)^2 - 4. \end{aligned}$$

Example 3:

$$\begin{aligned} x^2 + 4x + 7 &= \underbrace{x^2 + 4x + 4}{} - 4 + 7 \\ &= (x+2)^2 + 3. \end{aligned}$$

Example 4:

$$\begin{aligned} x^2 - 8x &= \underbrace{x^2 - 8x + (-4)^2}{} - (4)^2 \\ &= \quad (x-4)^2 - 16. \end{aligned}$$

By now the reader must have recognized the pattern.

Example 5:

$$\begin{aligned} x^2 + bx &= \underbrace{x^2 + bx + \left(\frac{b}{2}\right)^2}{} - \left(\frac{b}{2}\right)^2 \\ &= \quad \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \end{aligned}$$

Now consider the general (that is of the form $ax^2 + \cdots$) quadratic polynomial:

Example 6:

$$\begin{aligned} 2x^2 + 5x &= 2\left(x^2 + \frac{5}{2}x\right) \\ &= 2\left(\underbrace{x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2}{} - \left(\frac{5}{4}\right)^2\right) \quad \left(\textit{since } \frac{5}{2} \div 2 = \frac{5}{2} \times \frac{1}{2} = \frac{5}{4}\right); \\ &= \quad 2\left(\left(x + \frac{5}{4}\right)^2 - \frac{25}{16}\right) \end{aligned}$$

$$\begin{aligned}
&= 2\left(x+\frac{5}{4}\right)^2 - 2\times\frac{25}{16} \quad (\textit{distribute multiplication by } 2) \\
&= 2\left(x+\frac{5}{4}\right)^2 - \frac{25}{8}.
\end{aligned}$$

In general, we see the following:
Example 7:

$$\begin{aligned}
ax^2+bx &= a\left(x^2+\frac{b}{a}x\right) \\
&= a\left(\underbrace{x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2}-\left(\frac{b}{2a}\right)^2\right) \quad \left(\textit{since } \frac{b}{a}\div 2=\frac{b}{a}\times\frac{1}{2}=\frac{b}{2a}\right); \\
&= \quad a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right) \\
&= a\left(x+\frac{b}{2a}\right)^2 - a\times\frac{b^2}{4a^2} \quad (\textit{distribute the multiplication by } a) \\
&= a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a}.
\end{aligned}$$

Example 8:

$$\begin{aligned}
ax^2+bx+c &= a\left(x^2+\frac{b}{a}x\right)+c \\
&= a\left(\underbrace{x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2}-\left(\frac{b}{2a}\right)^2\right)+c \\
&= \quad a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right)+c \\
&= a\left(x+\frac{b}{2a}\right)^2 - a\times\frac{b^2}{4a^2}+c \\
&= a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2}{4a}+c \\
&= a\left(x+\frac{b}{2a}\right)^2 - \left(\underbrace{\frac{b^2}{4a}-c}\right) \\
&= a\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a}
\end{aligned}$$

Classroom Exercises : Complete the following as squares:

(a) $x^2 - 6x$ (b) $x^2 + 8x$

(c) $x^2 + 7x$ (d) $x^2 + 9x$

(e) $3x^2 + 6x + 4$ (f) $2x^2 - 4x$

(g) $2x^2 - 7x + 3$ (h) $3x^2 + 7x$

Once we know to complete a quadratic polynomial to a square, we can solve a quadratic equation. To solve a quadartic equation is to find the value of x which satisfies the equation.

Let us consider some of the examples considered above.

Example 1:

$$\begin{aligned}
x^2 + 4x + 4 &= 0 \\
(x+2)^2 &= 0 \quad (\textit{completing of the square}) \\
(x+2) &= \sqrt{0} = 0 \quad (\textit{taking square-roots of both sides}) \\
x &= -2 \quad (\textit{subtract 2 from both sides}).
\end{aligned}$$

Example 3: Here we get **complex solutions**.

$$\begin{aligned}
x^2 + 4x + 7 &= 0 \\
(x+2)^2 + 3 &= 0 \quad (\textit{completing of the square}) \\
(x+2)^2 &= -3 \quad (\textit{subtract 3 from both sides}) \\
(x+2) &= \pm\sqrt{-3} = \pm\sqrt{3}i \quad (\textit{taking square-roots of both sides}) \\
x &= -2 \pm \sqrt{3}i \quad (\textit{subtract 2 from both sides}).
\end{aligned}$$

Example 6:

$$
\begin{aligned}
2x^2 + 5x &= 0 \\
2\left(x + \frac{5}{4}\right)^2 - \frac{25}{8} &= 0 \quad (\textit{completing the square}) \\
2\left(x + \frac{5}{4}\right)^2 &= \frac{25}{8} \quad \left(\textit{add } \frac{25}{8} \textit{ to both sides}\right) \\
\left(x + \frac{5}{4}\right)^2 &= \frac{25}{8} \div 2 = \frac{25}{8} \times \frac{1}{2} \quad (\textit{divide both sides by } 2) \\
\left(x + \frac{5}{4}\right)^2 &= \frac{25}{16} \\
\left(x + \frac{5}{4}\right) &= \pm\sqrt{\frac{25}{16}}
\end{aligned}
$$

$$
\begin{aligned}
x + \frac{5}{4} &= \pm\frac{5}{4} \\
x &= -\frac{5}{4} \pm \frac{5}{4} \\
x &= -\frac{5}{4} + \frac{5}{4} \textit{ or } x = -\frac{5}{4} - \frac{5}{4} \\
x &= 0 \textit{ or } x = -\frac{10}{4} \\
x &= 0 \textit{ or } x = -\frac{5}{2}.
\end{aligned}
$$

A more general example:

$$\begin{aligned}
3x^2 - 7x + 4 &= 0\\
3\left(x^2 - \frac{7}{3}\right) + 4 &= 0\\
3\left(\left(x - \frac{7}{6}\right)^2 - \frac{49}{36}\right) + 4 &= 0 \quad (\textit{check for yourself, the completing of square})\\
3\left(x - \frac{7}{6}\right)^2 - 3 \times \frac{49}{36} + 4 &= 0 \quad (\textit{distribute multiplication by } 3)\\
3\left(x - \frac{7}{6}\right)^2 \underbrace{- \frac{49}{12} + 4} &= 0\\
3\left(x - \frac{7}{6}\right)^2 - \frac{49}{12} + \frac{48}{12} &= 0\\
3\left(x - \frac{7}{6}\right)^2 - \frac{1}{12} &= 0\\
3\left(x - \frac{7}{6}\right)^2 &= \frac{1}{12} \quad \left(\textit{add } \frac{1}{12} \textit{ to both sides}\right)\\
\left(x - \frac{7}{6}\right)^2 &= \frac{1}{12} \div 3 = \frac{1}{12} \times \frac{1}{3}\\
\left(x - \frac{7}{6}\right)^2 &= \frac{1}{36}\\
\left(x - \frac{7}{6}\right) &= \pm\sqrt{\frac{1}{36}}\\
\left(x - \frac{7}{6}\right) &= \pm\frac{1}{6}
\end{aligned}$$

$$\begin{aligned}
x &= \frac{7}{6} \pm \frac{1}{6}\\
x &= \frac{7}{6} + \frac{1}{6} \ \textit{or}\ x = \frac{7}{6} - \frac{1}{6}\\
x &= \frac{8}{6} \ \textit{or}\ x = \frac{6}{6}\\
x &= \frac{4}{3} \ \textit{or}\ x = 1.
\end{aligned}$$

Classroom Exercises : Solve by completing the squares:

(a) $x^2 - 6x = 0$ (b) $x^2 + 8x = 0$

(c) $x^2 + 7x = 0$ (d) $x^2 + 9x = 0$

(e) $3x^2 + 6x + 4 = 0$ (f) $2x^2 - 4x = 0$

(g) $2x^2 - 7x + 3 = 0$ (h) $3x^2 + 7x = 0$

(i) $x^2 + 9x34 = 0$ (j) $2x^2 - 5x + 8 = 0$

We now come to the **quadratic formula**. Consider the general quadratic equation

$$\begin{aligned}
ax^2 + bx + c &= 0 \quad (\textit{ for } a,b,c \ \in \mathbb{R}, a \neq 0) \\
a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} &= 0 \quad (\textit{completing the square}) \\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \quad \left(\textit{add } \frac{b^2 - 4ac}{4a} \textit{ to both sides}\right) \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \div a = \frac{b^2 - 4ac}{4a} \times \frac{1}{a} \quad (\textit{divide both sides by } a) \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
\left(x + \frac{b}{2a}\right) &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Classroom Exercses :

Solve by using the quadratic formula:

(a) $x^2 - 7x = 0$ (here, $a = 1, b = -7, c = 0$ and now substitute in the quadratic formula).

(b) $x^2 + 9x - 3 = 0$.

(c) $2x^2 - 5x + 8 = 0$.

(d) $x^2 - 8 = 0$.

(e) $x^2 + 5x = 0$.

(f) $x^2 - 3x + 4 = 0$.

(g) $3x^2 + 5x - 3 = 0$.

(h) $4x^2 - 7x - 8 = 0$.

7.1.1 Homework Exercises

1. Solve by completing the square:

   (a) $x^2 - 8x + 5 = 0$

   (b) $x^2 + 9x - 4 = 0$

   (c) $3x^2 - x + 5 = 0$

   (d) $2x^2 + 3x = 0$

2. Solve by using the quadratic formula:

   (a) $x^2 - 8x + 5 = 0$

   (b) $x^2 + 9x - 4 = 0$

   (c) $3x^2 - x + 5 = 0$

   (d) $2x^2 + 3x = 0$

7.2 Introduction to Parabolas

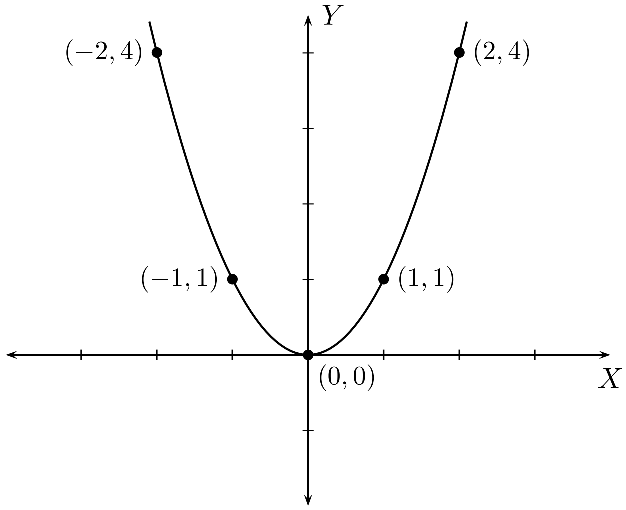
In this section we learn to graph quadratic equations. In other words, we want to plot points (x, y) on the coordinate plane which satisfy the equation

$$y = ax^2 + bx + c \quad \text{for } a, b, c \in \mathbb{R}, a \neq 0.$$

The real number a is called the **leading coefficient** of the quadratic polynomial $ax^2 + bx + c$. We start with the simplest of quadratic equations.

This is the graph of $y = x^2$.

x	$y = x^2$
0	0
-1	1
-2	4
1	1
2	4



The graph of $y = x^2$ is a **parabola**. The point $(0, 0)$ is its **vertex** and this parabola **opens up**. Notice that the parabola is initially **decreasing** and then changes its direction and becomes **increasing**. The number 0 is the **critical number** when the parabola changes its direction. The critical number is the x-coordinate of the vertex for a parabola. The precise definition of a critical number requires knowledge of calculus which is beyond the scope of this course. But we will develop an intuitive idea of the critical number for a parabola by the following examples.

y	Critical number	y	Critical number
$y = (x-3)^2$	$x = 3$	$y = (x+5)^2$	$x = -5$
$y = \left(x - \frac{1}{2}\right)^2$	$x = \frac{1}{2}$	$y = \left(x + \frac{2}{3}\right)^2$	$x = -\frac{2}{3}$
$y = (x - \sqrt{3})^2 + 5$	$x = \sqrt{3}$	$y = (x+4)^2 - 6$	$x = -4$
$y = 2\left(x - \frac{1}{5}\right)^2 - 7$	$x = \frac{1}{5}$	$y = -3\left(x + \frac{2}{5}\right)^2 + 125$	$x = -\frac{2}{5}$

The reader may have noticed the significance of completing of squares for finding the critical number.

Example : Consider $y = 2x^2 + 3x - 4$.

$$\begin{aligned}
y &= 2x^2 + 3x - 4 \\
&= 2\left(x^2 + \frac{3}{2}x\right) - 4 \\
&= 2\left(x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right) - 4 \quad (\textit{Completing the square}) \\
&= 2\left(x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right) - 4 \\
&= 2\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) - 2 \times \frac{9}{16} - 4 \quad (\textit{distribute multiplication by } 2) \\
&= 2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8} - 4 \\
&= 2\left(x + \frac{3}{4}\right)^2 - \frac{41}{8}.
\end{aligned}$$

Hence, the critical number is $x = -\frac{3}{4}$.

y	Critical number	y	Critical number
$y = 3x^2 + 4x - 1$	$x = -\frac{4}{2 \times 3} = -\frac{2}{3}$	$y = -2x^2 + 5x + 12$	$x = -\frac{5}{2 \times -2} = +\frac{5}{4}$
$y = 4x^2 + \frac{2}{3}x - 1$	$x = -\frac{2}{3} \div (2 \times 4) = -\frac{1}{12}$	$y = -2x^2 + 7$	$x = -\frac{0}{2 \times -2} = 0$

In general, the critical number of

$$y = ax^2 + bx + c \text{ is given by } x = -\frac{b}{2a}.$$

Classroom Exercises : Find the critical number of

(a) $y = x^2 + 5x$

(b) $y = 2x^2 - 7$

(c) $y = -3x^2$

(d) $y = -2(x + 4)^2$

(e) $y = (x - \sqrt{5})^2$

(f) $y = 2(x+1)^2 - 17$

(g) $y = 2x^2 - 3x + 5$

(h) $y = 3x^2 + \dfrac{1}{2}x + 7$

(i) $y = \dfrac{2}{3}x^2 - \dfrac{3}{5}x + 5$

Since the critical number is the x-coordinate of the **vertex**, we can then find the vertex by calculating the y-coordinate. The y-coordinate of the vertex is obtained by substituting the critical number for x in the expression for y. The sign of the leading coefficient tells us whether the parabola will **open up** or **open down**. Recall that the leading coefficient of $ax^2 + bx + c$ or of $a(x-h)^2 + k$ is a. If $a > 0$, then the parabola opens up. If $a < 0$, then the parabola opens down. The following table gives examples.

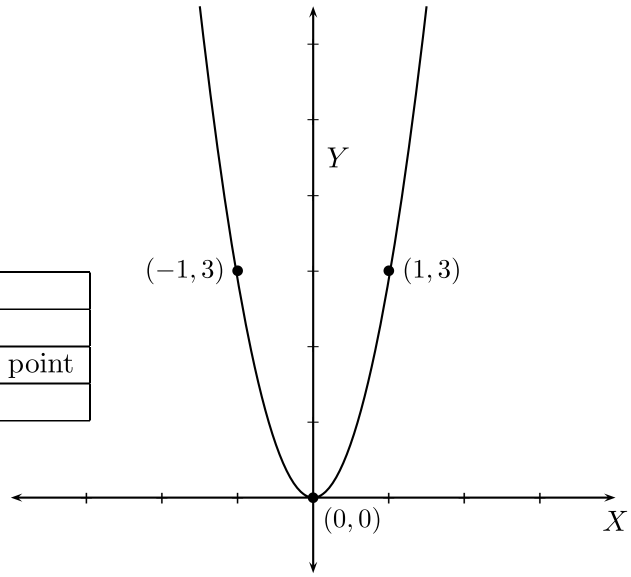
$y = a(x-h)^2 + k$ $y = ax^2 + bx + c$	Critical number h	y-coordinate k	Vertex (h, k)	Parabola opens
$y = 3x^2$	0	$3(0)^2 = 0$	$(0,0)$	up $(3 > 0)$
$y = -4x^2$	0	$-4(0)^2 = 0$	$(0,0)$	down $(-4 < 0)$
$y = 2(x-3)^2$	3	$2(3-3)^2 = 0$	$(3,0)$	up $(2 > 0)$
$y = -5(x+1)^2$	-1	$-5(-1+1)^2 = 0$	$(-1,0)$	down $(-5 < 0)$
$y = 2(x-\sqrt{3})^2 + 1$	$\sqrt{3}$	$2(\sqrt{3}-\sqrt{3})^2 + 1 = 1$	$(\sqrt{3}, 1)$	up $(2 > 0)$
$y = \left(x - \dfrac{1}{2}\right)^2 - 4$	$\dfrac{1}{2}$	$y = \left(\dfrac{1}{2} - \dfrac{1}{2}\right)^2 - 4 = -4$	$\left(\dfrac{1}{2}, -4\right)$	up $(1 > 0)$
$y = 3x^2 + 4x + 5$	$-\dfrac{4}{2 \times 3} = -\dfrac{2}{3}$	$3\left(-\dfrac{2}{3}\right)^2 + 4\left(-\dfrac{2}{3}\right) + 5$ $= 3\dfrac{2}{3}$	$\left(-\dfrac{2}{3}, 3\dfrac{2}{3}\right)$	up $(3 > 0)$
$y = -2x^2 + 5x - 3$	$-\dfrac{5}{2 \times -2} = \dfrac{5}{4}$	$-2\left(\dfrac{5}{4}\right)^2 + 5\left(\dfrac{5}{4}\right) - 3$ $= \dfrac{1}{8}$	$\left(\dfrac{5}{4}, \dfrac{1}{8}\right)$	down $(-2 < 0)$

To graph the parabola, it is convenient to find the vertex, and two points which are **equidistant** from the vertex on the parabola. These two points will then be symmetric about the **axis of symmetry**. To find these two points, we take two numbers on either side of the critical number which are both equidistant from the critical number. Then substitute these values to find the required points. Lastly, we can see the **range** as those y-values attained by the quadratic function. We graph the quadratic equations listed above.

This is the graph of $y = 3x^2$.

x	$y = 3x^2$	Description of point
-1	$3(-1)^2 = 3$	Left of vertex
0	0	Vertex and x,y-intercept point
1	$3(1)^2 = 3$	Right of vertex

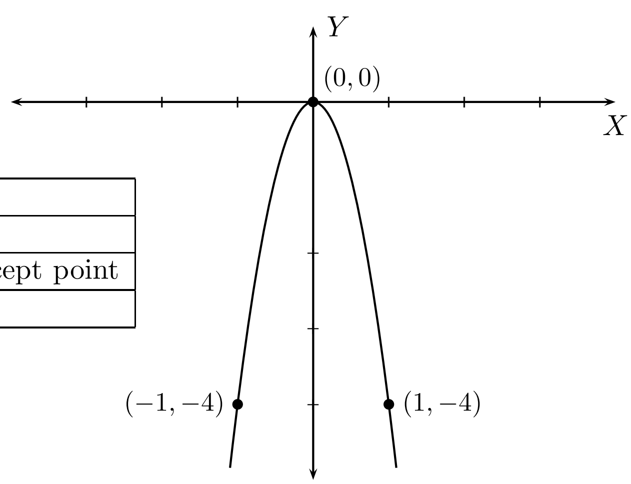
Axis of symmetry is the line
$x = 0$ (the y-axis).
The x-intercept is 0, and the y-intercept is 0.
The range is $[0, \infty)$.



This is the graph of $y = -4x^2$.

x	$y = -4x^2$	Description of point
-1	$-4(-1)^2 = -4$	Left of vertex
0	0	Vertex and x,y-intercept point
1	$-4(1)^2 = -4$	Right of vertex

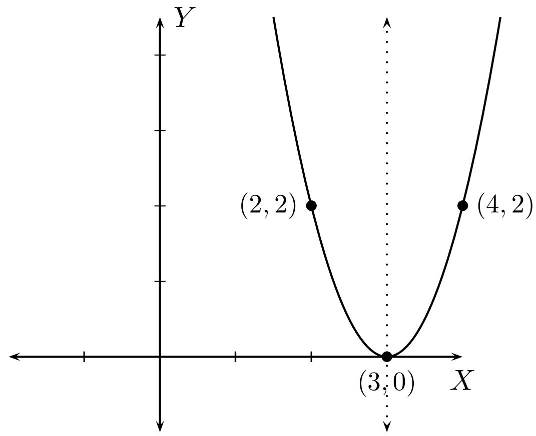
Axis of symmetry is the line
$x = 0$ (the y-axis).
The x-intercept is 0, and the y-intercept is 0.
The range is $(-\infty, 0]$.



This is the graph of $y = 2(x-3)^2$.

x	$y = 2(x-3)^2$	Description of point
2	$2(2-3)^2 = 2$	Left of vertex
3	0	Vertex and x-intercept point
4	$2(4-3)^2 = 2$	Right of vertex
0	$2(0-3)^2 = 18$	y-intercept point

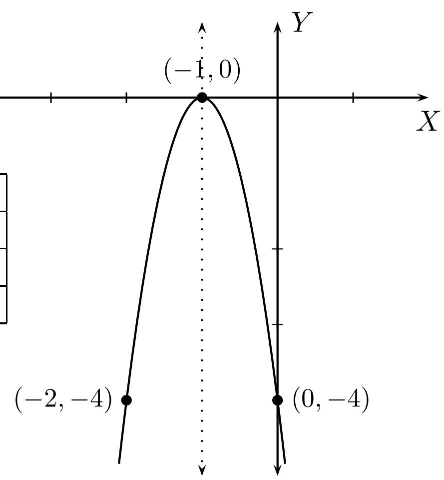
Axis of symmetry is the line $x = 3$.
The x-intercept is 3, and the y-intercept is 18.
The range is $[0, \infty)$.



This is the graph of $y = -4(x+1)^2$.

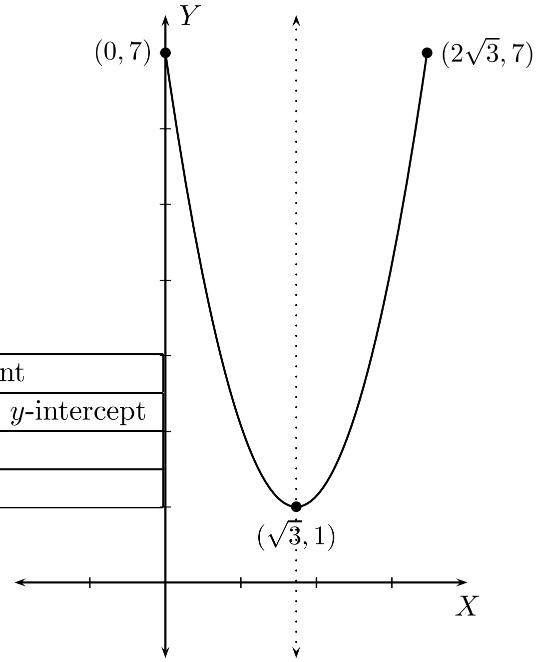
x	$y = -4(x+1)^2$	Description of point
-2	$-4(-2+1)^2 = -4$	Left of vertex
-1	0	Vertex and x-intercept point
0	$-4(0+1)^2 = -4$	Right of vertex and y-intercept

Axis of symmetry is the line $x = -1$.
The x-intercept is -1, and the y-intercept is -4.
The range is $(-\infty, 0]$.



This is the graph of $y = 2(x - \sqrt{3})^2 + 1$.

x	$y = 2(x - \sqrt{3})^2 + 1$	Description of point
0	$2(0 - \sqrt{3})^2 + 1 = 7$	Left of vertex and y-intercept
$\sqrt{3}$	1	Vertex
$2\sqrt{3}$	$2(2\sqrt{3} - \sqrt{3})^2 + 1 = 7$	Right of vertex

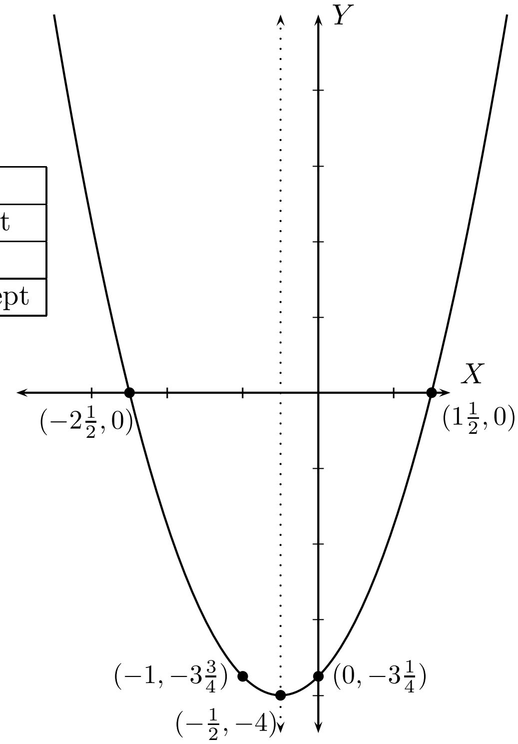


Axis of symmetry is the line $x = \sqrt{3}$.
There is no x-intercept, and the y-intercept is 7.
The range is $[1, \infty)$.

This is the graph of $y = \left(x + \dfrac{1}{2}\right)^2 - 4$.

x	$y = (x + \frac{1}{2})^2 - 4$	Description of point
-1	$(-1 + \frac{1}{2})^2 - 4 = -3\frac{3}{4}$	Left of vertex and y-intercept
$-\frac{1}{2}$	-4	Vertex
0	$3(0 + \frac{1}{2})^2 - 4 = -3\frac{3}{4}$	Right of vertex and y-intercept

Axis of symmetry is the line $x = -\dfrac{1}{2}$.
The y-intercept is $-3\dfrac{3}{4}$.
To find the x-intercept, set $y = 0$ and solve for x.
The x-intercepts are $1\dfrac{1}{2}$ and $-2\dfrac{1}{2}$.
The range is $[-4, \infty)$.



This is the graph of $y = 3x^2 + 4x + 5$.

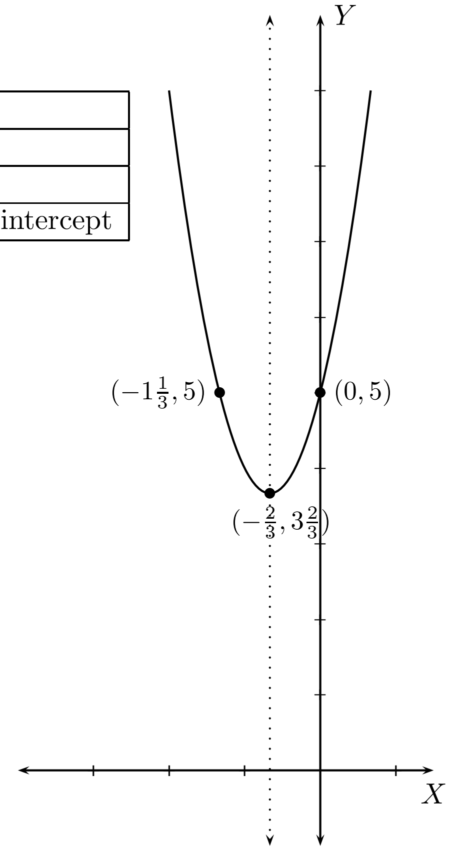
x	$y = 3x^2 + 4x + 5$	Description of point
$-1\frac{1}{3}$	$3(-1\frac{1}{3})^2 + 4(-1\frac{1}{3}) + 5 = 5$	Left of vertex
$-\frac{2}{3}$	$3\frac{2}{3}$	Vertex
0	$3(0)^2 + 4(0) + 5 = 5$	Right of vertex and y-intercept

Axis of symmetry is the line $x = -\dfrac{2}{3}$.

The y-intercept is 5.

There is no x-intercept.

The range is $[3\frac{2}{3}, \infty)$.



This is the graph of $y = -2x^2 + 5x - 3$.
Find the x-intercepts by setting $y = 0$
and solving for x.

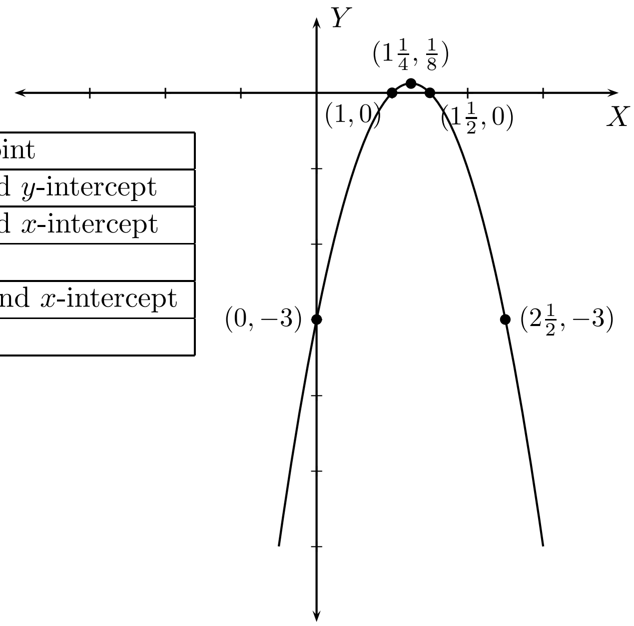
x	$y = -2x^2 + 5x - 3$	Description of point
0	-3	Left of vertex and y-intercept
1	0	Left of vertex and x-intercept
$1\frac{1}{4}$	$\frac{1}{8}$	Vertex
$1\frac{1}{2}$	0	Right of vertex and x-intercept
$2\frac{1}{2}$	-3	Right of vertex

Axis of symmetry is the line $x = 1\dfrac{1}{4}$.

The x-intercepts are 1 and $1\dfrac{1}{2}$

The y-intercept is -3.

The range is $(-\infty, \frac{1}{8}]$.



Classroom Exercises: For each of the following quadratic equations find the critical number, vertex, x, y-intercepts, if any, axis of symmetry, and two points on the parabola symmetric about the axis of symmetry. Tell whether the parabola opens up or down, and explain why. Graph the parabola and find the range.

(a) $y = -x^2$

(b) $y = 3(x - 1)^2$

(c) $y = -2(x + 1)^2$

(d) $y = (x - 3)^2 + 2$

(e) $y = -3\left(x + \dfrac{1}{2}\right)^2 + 4$

(f) $y = x^2 + 6x$

(g) $y = x^2 + 6$

(h) $y = x^2 - 4x$

(i) $y = x^2 - 4$

(j) $y = x^2 + 5x + 4$

(k) $y = x^2 + 5x$

(l) $y = x^2 - 7x$

(m) $y = 2x^2 - 4x + 5$

(n) $y = 3x^2 - 6x + 5$

(o) $y = 3x^2 + 2x + 2$

7.2.1 Homework Exercises

For each of the following quadratic equations find the critical number, vertex, x, y-intercepts, if any, axis of symmetry, and two points on the parabola symmetric about the axis of symmetry. Tell whether the parabola opens up or down, and explain why. Graph the parabola and find the range.

1. $y = -3x^2$
2. $y = 2(x - 1)^2$
3. $y = -3(x + 2)^2$
4. $y = (x - 2)^2 + 1$
5. $y = -2\left(x + \dfrac{1}{3}\right)^2 + 8$
6. $y = x^2 + 8x$
7. $y = x^2 - 6x$
8. $y = x^2 + 8$
9. $y = x^2 - 6$
10. $y = 3x^2 + 6x - 2$
11. $y = 2x^2 - 12x + 4$
12. $y = x^2 + 3x - 4$
13. $y = 3x^2 + 3x + 1$

Appendix A

Scientific Notation

In the the study of astronomy we encounter very large numbers. For instance, distance between planets is measured in light years, where one light-year is the distance that light travels in a vacuum in one Julian year. One light year is about $5,878,625,373,183.6$ miles. On the other hand, we encounter very small numbers in other areas of sciences. For example, the mass of a dust particle is 0.000000000753 kg.

A.1 Terminology

Scientific notation helps us understand large and small numbers. In Scientific Notation, numbers are written as a product of a number with absolute value greater than or equal to 1 but less than 10, called the coefficient, and a power of 10. Every number can be written uniquely this way. The following table illustrates this notation and the accompanying terminology.

Number	Scientific Notation	Coefficient	Base	Exponent
$352,000,000$	3.52×10^8	3.52	10	8
0.000000003235	3.235×10^{-10}	3.235	10	-9
$5,878,625,373,183.6$	$5.8786253731836 \times 10^{12}$	5.8786253731836	10	12
0.000000000753	7.53×10^{-10}	7.53	10	-10
9.4578	9.4578×10^0	9.4578	10	0
-0.000453987	-4.53987×10^{-4}	-4.53987	10	-4

Note that the absolute value of the coefficient is greater than or equal to 1 and less than 10.

Classroom Exercises: Write each of the given numbers in scientific notation; in each case identify the coefficient, the base, and the exponent.

(a) $365,554,612,457$ (b) $8,979,751,000$ (c) $100,201,343$ (d) 123.4567
(e) 0.0000001 (f) 0.00000000212 (g) 0.00000085023 (h) -245.7802

A.1.1 Homework Exercises

Write each of the given numbers in scientific notation; in each case identify the coefficient, the base, and the exponent.

(1) $73,456,789,001$	(2) $123,456,789,012$	(3) $20,340,000,000$	(4) 234.56008
(5) 0.000324012	(6) 0.0000000124	(7) 0.0000009876	(8) -3478.901

A.2 Addition and Subtraction of numbers written in scientific notation

Additions and subtractions are routine matters unless the exponents of 10 are different. We illustrate this in the following examples.

Example 1:

$\left(3.24 \times 10^7\right) + \left(1.38 \times 10^7\right)$ *The exponents are equal*;
$=4.62 \times 10^7$ *The final answer (note that* $3.24 + 1.38 = 4.62$*).*

Example 2:

$\left(3.24 \times 10^6\right) + \left(1.38 \times 10^7\right)$ *The exponents are not equal*;
$=\left(3.24 \times 10^6\right) + \left(1.38 \times 10^1 \times 10^6\right)$
$=\left(3.24 \times 10^6\right) + \left(13.8 \times 10^6\right)$ *Now the exponents are equal;*
$=17.04 \times 10^6$ *Note that* $3.24 + 13.8 = 17.04$*;*
$=1.704 \times 10^1 \times 10^6$ *Conversion to scientific notation;*
$=1.704 \times 10^7$ *The final answer.*

Example 3:

$\left(3.24 \times 10^7\right) - \left(1.38 \times 10^7\right)$ *The exponents are equal*;
$=1.86 \times 10^7$ *The final answer (note that* $3.24 - 1.38 = 1.86$*).*

Example 4:

$\left(3.24 \times 10^6\right) - \left(1.38 \times 10^7\right)$ *The exponents are not equal*;
$=\left(3.24 \times 10^6\right) - \left(1.38 \times 10^1 \times 10^6\right)$ *Get smaller exponent*;
$=\left(3.24 \times 10^6\right) - \left(13.8 \times 10^6\right)$ *Now the exponents are equal;*
$=-10.56 \times 10^6$ *Note that* $3.24 - 13.8 = -10.56$
(recall subtraction of numbers);
$=-1.056 \times 10^1 \times 10^6$ *Conversion to scientific notation;*
$=-1.056 \times 10^7$ *The final answer.*

Example 5:

$\left(2.14 \times 10^{-6}\right) - \left(1.38 \times 10^{-7}\right)$	*The exponents are not equal;*
$=\left(2.14 \times 10^{1} \times 10^{-7}\right) - \left(1.38 \times 10^{-7}\right)$	*Get smaller exponent;*
$=\left(21.4 \times 10^{-7}\right) - \left(1.38 \times 10^{-7}\right)$	*Note that* $2.114 \times 10^1 = 21.4$*;*
	Now the exponents are equal;
$=20.02 \times 10^{-7}$	*Note that* $21.4 - 1.38 = 20.02$
	(recall subtraction of numbers);
$=2.002 \times 10^{1} \times 10^{-7}$	*Conversion to scientific notation;*
$=2.002 \times 10^{-6}$	*The final answer.*

Classroom Exercises: Find the following sums or differences. Write your results in scientific notation.

(a) $\left(2.56 \times 10^{9}\right) + \left(6.38 \times 10^{9}\right)$ (b) $\left(2.56 \times 10^{9}\right) - \left(6.38 \times 10^{9}\right)$
(c) $\left(2.56 \times 10^{9}\right) + \left(6.38 \times 10^{8}\right)$ (d) $\left(2.56 \times 10^{8}\right) - \left(6.38 \times 10^{9}\right)$
(e) $\left(7.56 \times 10^{-9}\right) + \left(4.38 \times 10^{-9}\right)$ (f) $\left(7.56 \times 10^{-9}\right) - \left(4.38 \times 10^{-9}\right)$
(g) $\left(7.56 \times 10^{-9}\right) + \left(4.38 \times 10^{-8}\right)$ (h) $\left(7.56 \times 10^{-8}\right) - \left(4.38 \times 10^{-9}\right)$

A.2.1 Homework Exercises

Find the following sums or differences. Write your results in scientific notation.

(1) $\left(2.56 \times 10^{12}\right) + \left(6.38 \times 10^{12}\right)$ (2) $\left(2.56 \times 10^{12}\right) - \left(6.38 \times 10^{12}\right)$
(3) $\left(2.56 \times 10^{12}\right) + \left(6.38 \times 10^{13}\right)$ (4) $\left(2.56 \times 10^{13}\right) - \left(6.38 \times 10^{12}\right)$
(5) $\left(7.56 \times 10^{-12}\right) + \left(4.38 \times 10^{-13}\right)$ (6) $\left(7.56 \times 10^{-13}\right) - \left(4.38 \times 10^{-12}\right)$
(7) $\left(7.56 \times 10^{-12}\right) + \left(4.38 \times 10^{-13}\right)$ (8) $\left(7.56 \times 10^{-13}\right) - \left(4.38 \times 10^{-12}\right)$

A.3 Multiplication and Division of numbers written in scientific notation

The reader is urged to review multiplication and divisions of decimal numbers from their previous courses. We will need that knowledge here. We now begin with our examples.

Example 1:

$\left(3 \times 10^{4}\right) \times \left(2 \times 10^{9}\right)$	
$=3 \times 2 \times 10^{4} \times 10^{9}$	*Multiplication is commutative;*
$=6 \times 10^{13}$	*The final answer (recall that* $10^4 \times 10^9 = 10^{13}$*).*

Example 2:

$\left(4 \times 10^{-4}\right) \times \left(8 \times 10^{9}\right)$	
$=4 \times 8 \times 10^{-4} \times 10^{9}$	*Multiplication is commutative;*
$=32 \times 10^{5}$	*This is not in scientific notation (recall that* $10^{-4} \times 10^9 = 10^5$*);*
$=3.2 \times 10^{1} \times 10^{5}$	*Converting* 32 *to scientific notation;*
$=3.2 \times 10^{6}$	*The final answer.*

Example 3:

$\left(3.24 \times 10^{7}\right) \times \left(9.38 \times 10^{11}\right)$	
$=3.24 \times 9.38 \times 10^{7} \times 10^{11}$	*Multiplication is commutative;*
$=30.3912 \times 10^{18}$	*This is not in scientific notation;*
$=3.03912 \times 10^{1} \times 10^{18}$	*Converting* 30.3912 *to scientific notation;*
$=3.03912 \times 10^{19}$	*The final answer.*

Example 4:

$\left(8.124 \times 10^{-5}\right) \times \left(9.238 \times 10^{-21}\right)$	
$=8.124 \times 9.238 \times 10^{-5} \times 10^{-21}$	*Multiplication is commutative;*
$=75.049512 \times 10^{-26}$	*This is not in scientific notation;*
$=7.5049512 \times 10^{1} \times 10^{-26}$	*Converting* 75.049512 *to scientific notation;*
$=7.5049512 \times 10^{-25}$	*The final answer.*

Classroom Exercises: Find the following products. Write the results in scientific notation.

(a) $\left(4 \times 10^{9}\right) \times \left(2 \times 10^{5}\right)$ (b) $\left(3 \times 10^{-7}\right) \times \left(2 \times 10^{6}\right)$
(c) $\left(4 \times 10^{8}\right) \times \left(5 \times 10^{5}\right)$ (d) $\left(9 \times 10^{7}\right) \times \left(8 \times 10^{-6}\right)$
(e) $\left(2.3 \times 10^{-9}\right) \times \left(8.1 \times 10^{5}\right)$ (f) $\left(3.5 \times 10^{-7}\right) \times \left(7.2 \times 10^{-6}\right)$
(g) $\left(4.43 \times 10^{8}\right) \times \left(2.32 \times 10^{6}\right)$ (h) $\left(3.12 \times 10^{9}\right) \times \left(4.92 \times 10^{-8}\right)$

Using similar ideas we divide numbers written in scientific notation.

Example 1:

$\left(6 \times 10^{7}\right) \div \left(2 \times 10^{3}\right)$	
$=\dfrac{6 \times 10^{7}}{2 \times 10^{3}}$	*Recall that* $\dfrac{10^{7}}{10^{3}} = 10^{4}$.
$=3 \times 10^{4}$	*The final answer.*

Example 2:

$\left(6 \times 10^{5}\right) \div \left(3 \times 10^{9}\right)$	
$=\dfrac{6 \times 10^{5}}{3 \times 10^{9}}$	
$=2 \times 10^{-4}$	*The final answer.*

Example 3:

$\left(6 \times 10^{12}\right) \div \left(24 \times 10^{4}\right)$	
$=\dfrac{6 \times 10^{12}}{24 \times 10^{4}}$	*Note that* $\dfrac{6}{24} = \dfrac{1}{4} = 0.25$;
$=0.25 \times 10^{8}$	*This is not in scientific notation;*
$=2.5 \times 10^{-1} \times 10^{8}$	*Converting* 0.25 *to scientific notation;*
$=2.5 \times 10^{7}$	*The final answer.*

Example 4:

$\left(2.76 \times 10^2\right) \div \left(2.3 \times 10^4\right)$

$= \dfrac{2.76 \times 10^2}{2.3 \times 10^8}$ *Note that* $2.76 \div 2.3 = 1.2$*;*

$= 1.2 \times 10^{-6}$ *The final answer (recall that* $\dfrac{10^2}{10^8} = 10^{-6}$*)*

Example 5:

$\left(1.12 \times 10^{-2}\right) \div \left(3.2 \times 10^5\right)$

$= \dfrac{1.12 \times 10^{-2}}{3.2 \times 10^5}$ *Note that* $1.12 \div 3.2 = 0.35$*;*

$= 0.35 \times 10^{-7}$ *This is not in scientific notation (recall that* $\dfrac{10^{-2}}{10^5} = 10^{-7}$*);*

$= 3.5 \times 10^{-1} \times 10^{-7}$ *Converting* 0.35 *to scientific notation;*

$= 3.5 \times 10^{-8}$ *The final answer.*

Classroom Exercises: Find the following quotients, and write the results in scientific notation:

(a) $\left(8 \times 10^9\right) \div \left(2 \times 10^5\right)$ (b) $\left(3 \times 10^{-7}\right) \div \left(2 \times 10^6\right)$
(c) $\left(4 \times 10^8\right) \div \left(5 \times 10^5\right)$ (d) $\left(9 \times 10^7\right) \div \left(8 \times 10^{-6}\right)$
(e) $\left(9.72 \times 10^{-9}\right) \div \left(2.7 \times 10^5\right)$ (f) $\left(2.232 \times 10^{-7}\right) \div \left(1.2 \times 10^{-6}\right)$
(g) $\left(7.2 \times 10^8\right) \div \left(1.6 \times 10^6\right)$ (h) $\left(1.159 \times 10^9\right) \div \left(1.9 \times 10^{-8}\right)$

A.3.1 Homework Exercises

1. Find the following products. Write the results in scientific notation.

   (a) $\left(3 \times 10^7\right) \times \left(3 \times 10^4\right)$ (b) $\left(1 \times 10^{-9}\right) \times \left(4 \times 10^5\right)$
   (c) $\left(7 \times 10^9\right) \times \left(5 \times 10^8\right)$ (d) $\left(9 \times 10^{12}\right) \times \left(6 \times 10^{-8}\right)$
   (e) $\left(5.3 \times 10^8\right) \times \left(9.1 \times 10^7\right)$ (f) $\left(8.5 \times 10^{-9}\right) \times \left(4.2 \times 10^{-8}\right)$
   (g) $\left(7.43 \times 10^{-6}\right) \times \left(9.13 \times 10^{21}\right)$ (h) $\left(5.12 \times 10^{-7}\right) \times \left(8.92 \times 10^{-12}\right)$

2. Find the following quotients, and write the results in scientific notation:

   (a) $\left(8 \times 10^5\right) \div \left(4 \times 10^2\right)$ (b) $\left(7 \times 10^{-3}\right) \div \left(4 \times 10^6\right)$
   (c) $\left(6 \times 10^3\right) \div \left(5 \times 10^5\right)$ (d) $\left(6 \times 10^7\right) \div \left(8 \times 10^{-9}\right)$
   (e) $\left(4.9 \times 10^{-10}\right) \div \left(1.4 \times 10^8\right)$ (f) $\left(6.93 \times 10^{-7}\right) \div \left(2.1 \times 10^{-6}\right)$
   (g) $\left(1.183 \times 10^8\right) \div \left(1.3 \times 10^6\right)$ (h) $\left(7.14 \times 10^9\right) \div \left(1.7 \times 10^{-8}\right)$

Appendix B

Percents and Fractional parts

By Anthony Weaver.

B.1 Percents

A *percent* is a fraction in which the denominator is 100. The word "percent" comes from the Latin phrase *per centum* meaning "out of 100," and is symbolized by %. For example,

$$97\% \quad \text{means} \quad \frac{97}{100} \quad \text{or} \quad 0.97.$$

Percents need not be whole numbers, and they need not represent proper fractions. For example,

$$150\% = \frac{150}{100} = 1.5$$
$$0.5\% = \frac{0.5}{100} = 0.005$$

Fractions, decimals, and percents represent the same quantities in different ways, and we need to know how to convert one to another.

A decimal can be converted to a percent by moving the decimal point two places to the right and adjoining the % symbol. This is the same as multiplying the decimal fraction by 100, which exactly cancels the denominator, leaving just the numerator (the percent). Thus

$$0.68 = 68\%$$
$$2.05 = 205\%$$
$$0.708 = 70.8\%$$
$$1.4 = 140\%$$
$$0.0067 = 0.67\%$$

To convert a percent back into a decimal, we simply divide by 100, which, as we know, is equivalent to moving the decimal point two places to the left. Thus

$$\begin{aligned} 92\% &= 0.92 \\ 0.2\% &= 0.002 \\ 138\% &= 1.38 \\ 71.02\% &= 0.7102 \end{aligned}$$

To convert a percent to a fraction (or mixed number), first convert the percent to a decimal, as above, then express the decimal as a fraction (with a visible denominator), and finally, reduce the fraction to lowest terms, if needed.
Example: Convert 10.8% into a fraction in lowest terms.

Solution. We first write 10.8% as a decimal.

$$10.8\% = 0.108.$$

Then, we write the decimal as a fraction with a visible denominator. In this case, because there are 3 decimal places, the denominator is 1000.

$$0.108 = \frac{108}{1000}.$$

Finally, the GCF of the numerator and denominator is 4, so we cancel 4 from both of these numbers, obtaining a fraction in lowest terms:

$$\frac{108}{1000} = \frac{\cancel{108}^{27}}{\cancel{1000}_{250}} = \frac{27}{250}.$$

□

Here are some common percents with their decimal and fractional equivalents (in lowest terms):

$$10\% = 0.1 = \frac{1}{10}, \quad 20\% = 0.2 = \frac{1}{5}, \quad 25\% = 0.25 = \frac{1}{4}, \quad 50\% = 0.5 = \frac{1}{2}.$$

To convert a fraction to a decimal, we just perform long division.
Example: Convert $\frac{1}{4}$ to a decimal.

Solution. Performing the division $1 \div 4$, we obtain

$$
\begin{array}{r|l}
 & \;\;0\,.\,2\,5 \\ \cline{2-2}
4 & \;\;1\,.\,0\,0 \\
 & -\;\;\;8 \\ \cline{2-2}
 & \;\;\;\;\;\;2\,0 \\
 & \;\;\;-2\,0 \\ \cline{2-2}
 & \;\;\;\;\;\;\;\;0
\end{array}
$$

Thus $\frac{1}{4} = 0.25$. □

To convert a fraction to a percent, first convert it to a decimal, and then convert the decimal to a percent.

Example: Convert $\frac{1}{40}$ to a percent.

Solution. Performing the division $1 \div 40$, we obtain 0.025 (verify this – it is quite similar to the last example). Then, we convert 0.025 to a percent by multiplying by 100 (equivalently, by moving the decimal point two places to the right). We obtain

$$\frac{1}{40} = 2.5\%.$$

□

B.1.1 Homework Exercises

1. Convert the following percents to decimals.
   (a) 43%
   (b) 608%
   (c) 56.04%
   (d) 4.09%
2. Convert the following decimals to percents.
   (a) 14.09
   (b) 0.00679
   (c) 1.384
   (d) 0.384

3. Convert the following decimals or percents to fractions (or mixed numbers) in lowest terms.
   (a) 44%
   (b) 2%
   (c) 0.15
   (d) 0.25
   (e) 40%
   (f) 5%
   (g) 98%
   (h) 7.2%
   (i) 18%
4. Convert the following fractions, first to decimals, then to percents. Round percents to the nearest tenth of a percent.
   (a) $\frac{1}{8}$
   (b) $\frac{1}{6}$
   (c) $\frac{2}{5}$
   (d) $\frac{3}{8}$
   (e) $\frac{3}{4}$
   (f) $\frac{1}{13}$
   (g) $\frac{1}{12}$
   (h) $\frac{5}{12}$

B.2 Fractional parts of numbers

We now have several ways of indicating a fractional part of a number. For example, the phrases

$$\begin{array}{ccc} \frac{1}{4} & \text{of} & 90 \\ 0.25 & \text{of} & 90 \\ 25\% & \text{of} & 90 \end{array}$$

are all different ways of describing the number $\frac{1}{4} \times 90 = 22.5$. The word "of" indicates multiplication. In the case of percent, the multiplication is done after first converting the percent to a fraction or a decimal.

The fractional part taken need not be a *proper* fractional part. That is, we could end up with more than we started with.

Example: Find 125% of 500.

Solution. Converting the percent to the decimal 1.25, and then multiplying by 500, we get

$$1.25 \times 500 = 625.$$

□

Example: Sales tax in New York State is $8\frac{1}{4}\%$. What is the sales tax on a shirt priced at \$25?

Solution. $8\frac{1}{4} = 8.25$, and 8.25%, as a decimal, is .0825. So the sales tax on a \$25 shirt is

$$.0825 \times 25 = 2.0625 \approx \$2.06.$$

Note that we rounded off to the nearest cent. □

Example: Josh spends $\frac{2}{5}$ of his income on rent. If he earns \$1250 per month, how much does he have left over, after paying his rent?

Solution. The left over part is $\frac{3}{5}$, since $1 - \frac{2}{5} = \frac{5}{5} - \frac{2}{5} = \frac{3}{5}$. So he has

$$\frac{3}{5} \times 1250 = \frac{3}{\cancel{5}_{\,1}} \times \frac{\cancel{1250}^{\,250}}{1} = \$750$$

left over after paying rent. □

B.2.1 Homework Exercises

1. Find 16% of 75
2. Find $\frac{3}{8}$ of 60
3. Find .05 of 280
4. Find 150% of 105
5. Find $\frac{1}{2}\%$ of 248
6. A \$500 television is being sold at a 15% discount. What is the sale price?
7. Angela gets a 5% raise. Her original salary was \$36,000 per year. What is her new salary?
8. A car loses $\frac{2}{5}$ of its value over a period of years. If the car originally sold for \$12,500, what it would it sell for now?
9. On a test, Jose answers $\frac{7}{8}$ of the problems correctly. If there were 24 problems on the test, how many did he get wrong?
10. Medical expenses can be deducted from a person's income tax if they exceed 2% of total income. If Maribel's medical expenses were \$550, and her total income was \$28,000, can she deduct her medical expenses?

Appendix C

Ratio and Proportion

By Anthony Weaver and some exercises by Sharon Persinger.

In this chapter we develop another interpretation of fractions, as comparisons between two quantities.

C.1 Ratio

If a team wins ten games and loses five, we say that the **ratio** of wins to losses is $2:1$, or "2 to 1." Where did the numbers $2:1$ come from? We simply made a fraction whose numerator is the number of games the team won, and whose denominator is the number of games they lost, and reduced it to lowest terms: $\frac{\text{wins}}{\text{losses}} = \frac{10}{5} = \frac{2}{1}$.

We can do the same for any two quantities, a and b, as long as $b \neq 0$.

> The ratio of a to b ($b \neq 0$) is the fraction $\dfrac{a}{b}$, reduced to lowest terms.

Example: Find (a) the ratio of 9 to 18, (b) the ratio of 21 to 12, (c) the ratio of 64 to 4.

Solution. (a) The ratio of 9 to 18 is

$$\frac{9}{18} = \frac{1}{2} \quad \text{or} \quad 1:2.$$

(b) The ratio of 21 to 12 is

$$\frac{21}{12} = \frac{7}{4} \quad \text{or} \quad 7:4.$$

(c) The ratio of 64 to 4 is

$$\frac{64}{4} = \frac{16}{1} \quad \text{or} \quad 16 : 1.$$

□

Notice that in (c) we left the denominator 1 (rather than just writing 16) to maintain the idea of a *comparison* between two numbers.

We can form ratios of non-whole numbers. When the ratio is expressed in lowest terms, however, it is always the ratio of two whole numbers, as small as possible. This is the main point of a ratio comparison: if the given two numbers were *small whole numbers*, how would they compare?

Example: What is the ratio of $3\frac{3}{4}$ to $1\frac{1}{2}$?

Solution. Converting the mixed numbers to improper fractions, and rewriting the division as multiplication by the reciprocal, we obtain

$$3\frac{3}{4} \div 1\frac{1}{2} = \frac{15}{4} \times \frac{2}{3} = \frac{\cancelto{5}{15}}{\cancelto{2}{4}} \times \frac{\cancelto{1}{2}}{\cancelto{1}{3}} = \frac{5}{2}.$$

The ratio is 5 : 2. □

Example: What is the ratio of 22.5 to 15?

Solution. We could perform the decimal division $22.5 \div 15$, but it is easier to simplify the equivalent fraction

$$\frac{225}{150} = \frac{3}{2}.$$

The ratio is 3 : 2. □

When finding the ratio of two measurable quantities, we must be sure the quantities are expressed in the *same units*. Otherwise the ratio will be a skewed or false comparison.

Example: Find the ratio of 15 dollars to 30 cents.

Solution. If we form the ratio $\frac{15}{30} = \frac{1}{2}$, we imply that a person walking around with \$15 in his pocket has half as much money as a person walking around with only 30 cents! The proper comparison is obtained by converting both numbers to the same units. In this case, we convert dollars to cents. The correct ratio is

$$\frac{1500 \text{ cents}}{30 \text{ cents}} = \frac{50}{1}.$$

□

Example: Manuel gets a five minute break for every hour he works. What is the ratio of work time to break time?

Solution. Since 1 hour = 60 minutes, the ratio of work time to break time is

$$\frac{55 \text{ minutes}}{5 \text{ minutes}} = \frac{11}{1}.$$

$\square$

Recall that percent (%) means "per hundred." So a percent can be viewed as a ratio whose second term (denominator) is 100.

Example: At a certain community college, 55% of the students are female. Find (a) the ratio of female students to the total number of students, and (b) the ratio of female students to male students.

Solution. 55% indicates the fraction

$$\frac{55}{100} = \frac{11}{20},$$

so the ratio of female students to the total number of students is $11 : 20$. For (b), we first deduce that 45% of the students are male (since $45\% = 100\% - 55\%$). So the ratio of female students to male students is

$$\frac{55}{45} = \frac{11}{9} \quad \text{or} \quad 11 : 9.$$

$\square$

Example: A printed page $8\frac{1}{2}$ inches in width has a margin $\frac{5}{8}$ inches wide on either side. Text is printed between the margins. What is the ratio of the width of the printed text to the total margin width?

Solution. The total margin width is $\frac{5}{8} + \frac{5}{8} = \frac{5}{4} = 1\frac{1}{4}$ inches (taking into account both left and right margins.) The width of the printed text is the difference

$$\text{total page width} - \text{total margin width} = 8\frac{1}{2} - 1\frac{1}{4} = 7\frac{1}{4} \text{ inches.}$$

The ratio of the width of the printed text to total margin width is

$$7\frac{1}{4} \div 1\frac{1}{4} = \frac{29}{4} \div \frac{5}{4} = \frac{29}{4} \cdot \frac{4}{5} = \frac{29}{5},$$

or $29 : 5$. $\square$

C.1.1 Homework Exercises

Find the ratios.

1. 14 to 4
2. 30 to 32
3. 56 to 21
4. $1\frac{5}{8}$ to $3\frac{1}{4}$
5. $2\frac{1}{12}$ to $1\frac{1}{4}$
6. 14.4 to 5.4
7. 1.69 to 2.6
8. 3 hours to 40 minutes
9. 8 inches to $5\frac{1}{2}$ feet
10. In the late afternoon, a 35 foot tree casts an 84 foot shadow. What is the ratio of the tree's height to the shadow's length?

C.2 Proportions

A **proportion** is a statement that two ratios are equal. Thus

$$\frac{40}{20} = \frac{10}{5}$$

is a proportion, because both ratios are equivalent to the ratio $2:1$ (= the fraction $\frac{2}{1}$).

A proportion is a statement of the form

$$\frac{a}{b} = \frac{c}{d}$$

where $b, d \neq 0$.

C.2.1 The cross-product property

There is a very useful fact about proportions: If the proportion $\frac{a}{b} = \frac{c}{d}$ is true, then the "cross-products," ad and bc, are equal, and, conversely, if the cross-products are equal, then the proportion must be true.

Example: Verify that the cross-products are equal in the (true) proportion

$$\frac{40}{20} = \frac{10}{5}.$$

Solution. $40 \times 5 = 10 \times 20 = 200$. □

The general cross-product property is stated below for reference:

$$\boxed{\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc.}$$

To prove the cross-product property, we need to say a little bit about **equations**. An equation is a mathematical statement of the form $X = Y$. If the equation $X = Y$ is true, and N is any nonzero number, then the following equations are also true, and have exactly the same solution(s):

$$N \times X = N \times Y \qquad \text{and} \qquad \frac{X}{N} = \frac{Y}{N}.$$

Now go back to the proportion $\dfrac{a}{b} = \dfrac{c}{d}$, and multiply both sides by the nonzero number bd. We get

$$bd\frac{a}{b} = bd\frac{c}{d}.$$

Cancelling b on the left and d on the right yields the cross-product property, $ad = bc$. Conversely, if we have four numbers a, b, c, d, $(b,\ d \neq 0)$, and we know that $ad = bc$, then, dividing both sides by the nonzero number bd gives us the proportion $\frac{a}{b} = \frac{c}{d}$.

Example: Decide if the given proportions are true or false, using the cross-product property.
(a) $\dfrac{3}{11} = \dfrac{2}{7}$; (b) $\dfrac{4}{10} = \dfrac{6}{15}$.

Solution. (a) The cross-products are $3 \cdot 7 = 21$ and $2 \cdot 11 = 22$. They are not equal, so the proportion is false. (b) The cross-products are $4 \cdot 15$ and $10 \cdot 6$, both equal to 60. So the proportion is true. □

C.2.2 Solving a proportion

If one of the four terms of a proportion is missing or unknown, it can be found using the cross-product property. This procedure is called **solving** the proportion. In the proportion below, x represents an unknown term (any other letter would do).

$$\frac{x}{3} = \frac{34}{51}.$$

There is a unique x which makes the proportion true, namely, the one which makes the cross-products, $51x$ and $34(3)$, equal. The equation

$$51x = 34(3) = 102$$

can be divided by 51 (the number which multiplies x) on both sides, giving

$$\frac{51x}{51} = \frac{102}{51}.$$

Cancellation yields

$$\frac{\cancelto{1}{51}x}{\cancelto{1}{51}} = \frac{\cancelto{2}{102}}{\cancelto{1}{51}} = 2.$$

It follows that

$$x = 2$$

which is the solution of the proportion.

It doesn't matter which of the four terms is missing; the proportion can always be solved by a similar procedure.

Example: Solve the proportion

$$\frac{9}{100} = \frac{36}{y}$$

for the unknown term y.

Solution. The cross-products must be equal, so

$$9y = 3600.$$

Dividing both sides of the equation by 9 (the number which multiplies y), and cancelling, yields

$$\frac{\cancelto{1}{9}y}{\cancelto{1}{9}} = \frac{\cancelto{400}{3600}}{\cancelto{1}{9}}$$

$$y = 400.$$

The unknown term is 400. We check our solution by verifying that the cross-products

$$9(400) = 36(100) = 3600$$

are equal. □

Neither the terms nor the solution of a proportion are necessarily whole numbers.

Example: Solve the proportion

$$\frac{10}{B} = \frac{15}{4}$$

for B and check the solution.

Solution. Setting the cross products equal,

$$40 = 15B.$$

Dividing both sides of the equation by 15 (the number multiplying B), and simplifying, yields

$$\frac{\cancel{40}^{\,8}}{\cancel{15}^{\,3}} = \frac{15B}{15}$$
$$\frac{8}{3} = B.$$

The unknown term is $\frac{8}{3}$ or $2\frac{2}{3}$.

Substituting $2\frac{2}{3}$ for B in the original proportion, we check that the cross-products are equal.

$$10(4) \stackrel{?}{=} \left(2\frac{2}{3}\right)(15)$$
$$40 \stackrel{?}{=} \frac{8}{\cancel{3}_{\,1}}\,\frac{\cancel{15}^{\,5}}{1}$$
$$40 = 40.$$

□

Example: Solve the proportion

$$\frac{\left(\frac{2}{3}\right)}{5} = \frac{x}{15}.$$

Solution. Set the cross-products equal, and solve for x:

$$\begin{aligned} \frac{2}{3} \cdot 15 &= 5x \\ \frac{2}{\cancel{3}_1} \cdot \cancel{15}^{5} &= 5x \\ 10 &= 5x \\ 2 &= x. \end{aligned}$$

□

Example: Solve the proportion

$$\frac{42}{70} = \frac{x}{1.5}.$$

Solution. We could immediately set the cross-products equal, but it is simpler to first reduce the fraction $\frac{42}{70}$ to its lowest terms, using – by the way– a proportion: $\frac{42}{70} = \frac{3}{5}$. Stringing two proportions together

$$\frac{3}{5} = \frac{42}{70} = \frac{x}{1.5}$$

lets us skip over the middle fraction. It is evident that the solution to our original proportion is the solution to the simpler proportion

$$\frac{3}{5} = \frac{x}{1.5}.$$

Setting the cross-products equal

$$4.5 = 5x,$$

we obtain the solution $x = \frac{4.5}{5} = 0.9$.

□

We summarize the procedure for solving a proportion.

> To solve a proportion,
>
> 1. Reduce the numerical ratio (not containing the unknown) to lowest terms, if necessary;
> 2. Set the cross-products equal;
> 3. Divide both sides of the resulting equation by the number multiplying the unknown term.
>
> To check the solution,
>
> 1. In the original proportion, replace the unknown term with the solution you obtained;
> 2. Verify that the cross-products are equal.

Equivalence of fractions, and hence, most of fraction arithmetic, is based on proportion. To add

$$\frac{1}{2} + \frac{1}{3}$$

for example, we first solve the two proportions

$$\frac{1}{2} = \frac{x}{6} \qquad \text{and} \qquad \frac{1}{3} = \frac{y}{6}$$

(a task we have performed up to now without much comment) so that the fractions can be written with the LCD (6). Then

$$\frac{1}{2} + \frac{1}{3} = \frac{x+y}{6}.$$

Practice: solve the two proportions for x and y.

C.2.3 Homework Exercises

Solve the following proportions.

1. $\dfrac{1}{5} = \dfrac{3}{x}$
2. $\dfrac{15}{y} = \dfrac{2}{3}$

3. $\dfrac{100}{5} = \dfrac{20}{y}$

4. $\dfrac{A}{9} = \dfrac{5}{3}$

5. $\dfrac{11}{B} = \dfrac{1}{2}$

6. $\dfrac{5}{3} = \dfrac{c}{6}$

7. $\dfrac{s}{3} = \dfrac{4}{13}$

8. $\dfrac{1.2}{7} = \dfrac{A}{0.84}$

9. $\dfrac{P}{100} = \dfrac{75}{125}$

10. $\dfrac{3\frac{1}{5}}{x} = \dfrac{4}{2\frac{1}{2}}$

C.3 Percent problems

Any problem involving percent can be stated (or restated) in the form

"A is P percent of B"

where one of the numbers A, B or P is unknown. We can make this into a mathematical equation by making the following "translations:"

$$\begin{aligned} \text{“is”} \quad &\longleftrightarrow \quad = \\ \text{“}P\text{ percent”} \quad &\longleftrightarrow \quad \frac{P}{100} \\ \text{“of”} \quad &\longleftrightarrow \quad \times \end{aligned}$$

This gives us the equation

$$A = \frac{P}{100} \times B.$$

If we divide both sides of the equation by B, we obtain the proportion in the box below.

The statement "A is P percent of B" is equivalent to the proportion

$$\frac{A}{B} = \frac{P}{100}.$$

The letter B is used to suggest "base amount" – that is, B is the amount *from which* (or *of which*) a percentage is taken. Notice that B follows "of" in the verbal statement. It could be that A (the percentage taken) is greater than B, but only if the percentage P is greater than 100.

In the following examples, the phrase involving "what," as in "what number?" or "what percent?" helps determine which of A, B of P is the unknown.

Example: 6 is what percent of 300?

Solution. The percent is unknown, and the base amount B is 300 (since it follows the word "of"). We can restate the question as: "6 is P percent of 300." The corresponding proportion is

$$\frac{6}{300} = \frac{P}{100}$$

and the solution (verify it!) is $P = 2$. Therefore, 6 is 2% of 300. $\square$

Example: What number is 8% of 150?

Solution. The base amount B is 150, since it follows the word "of," the percentage is $P = 8$, and therefore A is the unknown amount. The question can be restated as: "A is 8% of 150." The corresponding proportion is

$$\frac{A}{150} = \frac{8}{100}$$

and the solution is $A = 12$ (verify!). It follows that 12 is 8% of 150.

(Note that this result could have been obtained without using a proportion, since 8% of 150 simply means $.08 \times 150 = 12$.) $\square$

Example: 58% of what number is 290?

Solution. The base amount B is unknown, and A is 290. The question can be restated as "58% of B is 290." The corresponding proportion is

$$\frac{290}{B} = \frac{58}{100}$$

and the solution is $\frac{29000}{58} = 500$. So 58% of 500 is 290. $\square$

Everyday questions involving percent are not always as straightforward as in the previous examples. But with a little thought, they can be converted into such statements.

Example: Approximately 55% of the students enrolled at BCC are female. If there are 2970 female students, what is the total enrollment at BCC?

Solution. According to the given information, 2970 is 55% of the total enrollment, which is the base amount, B. The question can be restated as "2970 is 55% of B" and the corresponding proportion is therefore

$$\frac{2970}{B} = \frac{55}{100}.$$

Using lowest terms of the fraction $\frac{55}{100} = \frac{11}{20}$, we solve

$$\begin{aligned} \frac{2970}{B} &= \frac{11}{20} \\ 59400 &= 11B \\ B &= \frac{59400}{11} = 5400. \end{aligned}$$

The total enrollment is 5400 students. To check, verify that 55% of 5400 is 2970. □

Example: Full-time tuition at a university increased from \$2,850 to \$3,000. What was the percent increase in the tuition? (Round to the nearest tenth of a percent.)

Solution. The tuition increased by $\$3000 - 2850 = \$150$. The percent increase is the ratio

$$\frac{\text{amount of increase}}{\text{original tuition}} = \frac{150}{2850},$$

expressed as a percent. Equivalently, we want to answer the question "\$150 is what percent of \$2850?" We solve the proportion

$$\begin{aligned} \frac{150}{2850} &= \frac{P}{100} \qquad \text{(C.1)} \\ \frac{3}{57} &= \frac{P}{100} \\ 57P &= 300 \\ P &\approx 5.26 \end{aligned}$$

Rounded to the nearest tenth of a percent, the tuition increase was approximately 5.3%. □

C.3.1 Exercises

1. 12 is 20% of what number?
2. 12 is 30% of what number?
3. 12 is 40% of what number?
4. 90 is what percent of 225?
5. 90 is what percent of 300?
6. 90 is what percent of 375?
7. What is 125% of 600?
8. What is 175% of 600?
9. 250 is what percent of 325? (Round to the nearest tenth of a percent.)
10. 108 is 80% of what number?
11. A baseball team won 93 games, or 62% of the games it played. How many games did the team play?
12. New York State sales tax is 8.25%. If the sales tax on a DVD player is \$16.50, what is the (before-tax) price of the player?
13. Marina's annual salary last year was \$56,000. This year she received a raise of \$4,480. By what percent did her salary increase?
14. A town's population decreased from 13,000 to 12,220. By what percent did the population decrease?

C.4 Rates

There are lots of real-world quantities which compare in a fixed ratio. For example, for any given car, the ratio of miles driven to gallons of gas used,

$$\frac{\text{miles}}{\text{gallon}} \qquad \text{or} \qquad \text{“miles per gallon”}$$

is essentially unchanging, or fixed. If we know, say, that 5 gallons of gas was needed to drive 150 miles, we can predict the amount that will be needed to drive any other distance, by solving a simple proportion.

Example: Maya used 5 gallons of gas to drive 150 miles. How many gallons will she need to drive 225 miles?

Solution. We solve the proportion

$$\frac{150 \text{ miles}}{5 \text{ gallon}} = \frac{225 \text{ miles}}{x \text{ gallons}},$$

where x represents the number of gallons she will need. Before setting the cross-products equal, reduce the fraction on the left side to $\dfrac{30}{1}$. Then

$$\begin{aligned} 30x &= 225 \qquad \text{(C.2)} \\ x &= \frac{225}{30} = 7\frac{1}{2}. \end{aligned}$$

She will need $7\frac{1}{2}$ gallons of gas to drive 225 miles. □

Miles per gallon is an example of a **rate**, or comparison of unlike quantities by means of a ratio. In the example above, the miles per gallon rate for Maya's car was

$$\frac{150 \text{ miles}}{5 \text{ gallons}} = \frac{150}{5} = \frac{30}{1} \qquad \text{or 30 miles per gallon.}$$

Other examples of rates are: dollars per hour (pay rate for an hourly worker), dollars per item (price of an item for sale), calories per minute (energy use by an athlete). You can undoubtedly think of many others.

Example: A runner burns 375 calories running 3.5 miles. (a) How many calories will she burn in running a marathon (approximately 26 miles)? (b) What is her rate of energy use (calories per mile)? Round off to the nearest whole unit.

Solution. (a) The ratio

$$\frac{\text{calories}}{\text{miles}}$$

is assumed fixed and, from the given information, is equal to

$$\frac{375}{3.5}.$$

Let x denote the number of calories the runner burns in running a marathon. Then

$$\begin{aligned} \frac{375}{3.5} &= \frac{x \text{ calories}}{26 \text{ miles}} \qquad \text{(setting cross-products equal)} \\ (375)(26) &= 3.5x \\ x &= \frac{(375)(26)}{3.5} \approx 2786. \end{aligned}$$

She will burn approximately 2786 calories running the marathon.

(b) Her rate of energy use is $\dfrac{375 \text{ calories}}{3.5 \text{ miles}} \approx 107$ calories per mile.

□

Example: A painting crew can paint three apartments in a week. If a building contains 40 equal size apartments, how long will it take the crew to paint all the apartments?

Solution. The ratio

$$\frac{\text{apartments painted}}{\text{time in weeks}}$$

is assumed fixed at $\frac{3}{1}$. If y is the number of weeks needed to paint 40 apartments, then

$$\frac{3}{1} = \frac{40}{y}, \quad \textit{and} \quad y = \frac{40}{3} = 13\frac{1}{3} \quad \textit{weeks.}$$

□

C.4.1 Homework Exercises

Use proportions to solve the following problems.

1. On a map, $\frac{3}{4}$ inch represents 14 miles. If two cities are 42 miles apart, how far apart are they on the map?

2. A truck burns $2\frac{1}{2}$ quarts of oil on an 1800 mile trip. How many quarts will be burned on a cross-country trip of 3240 miles?

3. An investment of $\$2,000$ earns $\$48$ in interest over a year. How much would need to be invested to earn $\$200$ in interest? (Round to the nearest dollar.)

4. In a sample of 600 bottles, 11 were found to be leaking. Approximately how many bottles would you expect to be leaking in a sample of $20,000$ bottles?

5. The ratio of the weight of lead to the weight of an equal volume of aluminum is $21 : 5$. If a bar of aluminum weighs 15 pounds, how much would a bar of lead of the same size weigh?

6. If a car gets 30 miles per gallon of gas, how many gallons of gas are needed to travel 345 miles?

7. On a map, one inch represents 50 kilometers. How many inches represent 160 kilometers?

8. Four workers take one month to repair 23 miles of road. How many miles of road can be repaired by ten workers in one month?

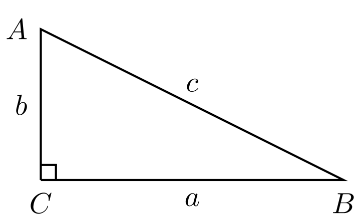
9. Suppose three birds can eat four pounds of grain. What amount of grains can five birds eat?

Appendix D

Right triangles and the Pythagorean Theorem

By Sharon Persinger.

A right triangle is a triangle with one right angle, that is, one angle that measures 90^o. By convention we usually call a right triangle $\triangle ABC$, and give the angles and sides the labels shown in this picture. The vertex of the right angle is labeled with the capital letter C.



The other vertices are labeled A and B.
The right angle is often marked with a small square.
The lowercase letters c, a, and b are used to represent the lengths of the sides of the right triangle.
The label of a vertex and the label of the opposite side correspond.
So the side opposite the right angle C is labeled c;
this special side is called the **hypotenuse** of the right triangle.

The side opposite angle A is labeled a; the side opposite angle B is labeled b. These other two sides (not the hypotenuse) are called **the legs** of the right triangle.

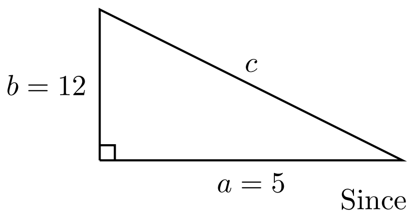
The **Pythagorean Theorem** gives an important relationship among the sides of a right triangle. This Theorem can be used to find the third side of a right triangle when two sides are known.

Pythagorean Theorem: Suppose $\triangle ABC$ is a right triangle with right angle C. Suppose c represents the length of the hypotenuse, and a and b are the lengths of the legs. Then

$$a^2 + b^2 = c^2.$$

Also, if $a^2 + b^2 = c^2$. for any triangle $\triangle ABC$, then the triangle is a right triangle with right angle C.

Example 1: Find the hypotenuse of a right triangle whose legs have lengths 5 inches and 12 inches.



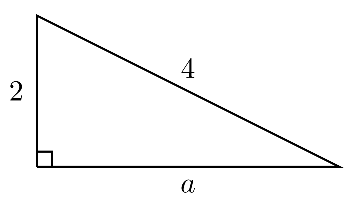
If a picture isnt given, draw one and
label what you know.
You want to find the hypotenuse, which is labeled c.
Use the Pythagorean Theorem.
Substitute the values you know and solve for c.
In the square root step, we take only the positive value.
Since c is a length, it is positive, and hence $\sqrt{c^2} = c$.

Solution.

$$
\begin{aligned}
c^2 &= a^2 + b^2;\\
c^2 &= 5^2 + 12^2;\\
c^2 &= 25 + 144 = 169;\\
c = \sqrt{c^2} &= \sqrt{169} = 13 \text{ inches.}
\end{aligned}
$$

□

Example 2: Example 2: Find the missing side of this triangle.



The unknown side is one of the legs, say a.
Use the Pythagorean Theorem and substitute.
Solve for a. Again, since we know a is a
length, we take only the positive square root.
Be sure to simplify the radical.

Solution.

$$
\begin{aligned}
c^2 &= a^2 + b^2, c = 4 \text{ and } b = 2;\\
4^2 &= a^2 + 2^2;\\
16 &= a^2 + 4;\\
16 - 4 = 12 &= a^2;\\
a &= \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3} \text{ units.}
\end{aligned}
$$

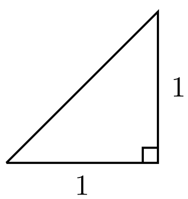
□

D.0.2 Exercises:

In exercises 1-10, find the missing side or sides of the right triangle.

1. $a = 6$ feet, $b = 8$ feet.
2. $b = 8$ meters, $c = 17$ meters.
3. $a = 10$ inches, $c = 15$ inches.
4. $a = 10$ feet, $b = 20$ feet.
5. $b = 10$ feet, $c = 20$ feet.

6.



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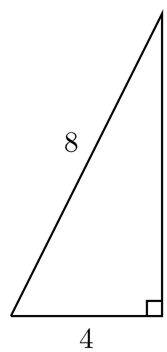

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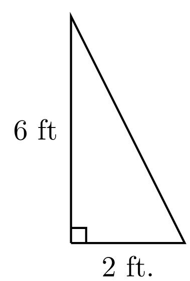

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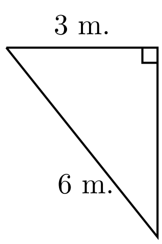

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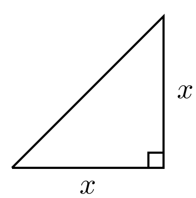

11. If the legs of a right triangle have the same length, what is the length of the hypotenuse? (Hint: Choose a letter to represent the length of a leg.)
12. Suppose a triangle has sides $a = 3$ inches, $b = 4$ inches and $c = 5$ inches. Is angle C a right angle?
13. Suppose a triangle has sides $a = 4$ inches, $b = 5$ inches and $c = 6$ inches. Is angle C a right angle?

Answers to odd exercises: 1. 10 feet; 3. $5\sqrt{5}$ inches; 5. $10\sqrt{3}$ feet; 7. $4\sqrt{3}$; 9. $8\sqrt{3}$; 11. $\sqrt{2}x$; 13. no, since $c^2 \neq a^2 + b^2$; $36 \neq 16 + 25$.