MTH 06. Basic Concepts of Mathematics II

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To my parents and teachers

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Chapter 1

Real Numbers

1.1 Introduction

Let us quickly recall some basic terminology. The reader is informed that some authors and teachers may use different terminology.

The set of **natural numbers** is

$$\mathbb{N} = \{1, 2, 3, \cdots\}.$$

The set of natural numbers is **closed** under addition. That is, given any two natural numbers, their sum is also a natural number. Addition is a **commutative** operation. That is, $a+b = b+a$ for any natural numbers a, b. The set $\mathbb{N}$ is also closed under multiplication, and multiplication is also commutative. That is, $a \cdot b = b \cdot a$ for any natural numbers a, b. Further, it contains the multiplicative identity 1. That is, $1 \cdot a = a = a \cdot 1$ for any natural number a. But $\mathbb{N}$ does not contain the **additive identity**.

The set of **whole numbers** is

$$\mathbb{W} = \{0, 1, 2, 3, \cdots\}.$$

Notice that $\mathbb{W}$ contains just one more element than $\mathbb{N}$, the number 0. The number 0 is the additive identity. That is, $a + 0 = a = 0 + a$ for any natural number a. Since every natural number is also a whole number, we say that $\mathbb{N}$ is a **subset** of $\mathbb{W}$, written mathematically as $\mathbb{N} \subset \mathbb{W}$. Thinking in non-technical terms, we could say, " $\mathbb{N}$ is contained in $\mathbb{W}$."

Notice that while $\mathbb{W}$ is closed under addition and multiplication, it is **not closed** under subtraction. For example, $3 - 7$ is not a whole number. This brings us to the next number system. The set of **integers** is

$$\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}.$$

The set $\mathbb{Z}$ is closed under addition, multiplication, and subtraction, and has the additive identity. Further, $\mathbb{W}$ is a subset of $\mathbb{Z}$. So we have $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z}$.

Notice that $\mathbb{Z}$ is not closed under division. For example, $7 \div 3$ is not an integer. Therefore, a new number system is needed which would contain $\mathbb{Z}$ and be closed under addition, subtraction, multiplication, and division. The set of **rational numbers** is denoted by $\mathbb{Q}$. It is difficult to list all the numbers of $\mathbb{Q}$. We therefore use **set-builder notation**.

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

such that

The set of

Here, $\in$ *is read as "elements of," and* $\neq$ *is read as "not equal to.". Thus,* $\mathbb{Q}$ *is equal to the set of all fractions* $\frac{a}{b}$ *such that* a, b *are integers, and* b *is not equal to* 0.

The set $\mathbb{Q}$ is closed under addition, subtraction, multiplication, and division by nonzero elements, and contains 0 (the additive identity) and 1 (the multiplicative identity). Addition and multiplication are commutative and associative. Moreover, multiplication satisfies the **distributive law** over addition. That is,

$$a \cdot (b + c) = a \cdot b + a \cdot c \textit{ for all rational numbers } a, b, c.$$

For an integer n, we can view n as a rational number as $\frac{n}{1}$. For example, $5 \in \mathbb{Z}$ can be viewed as $\frac{5}{1} \in \mathbb{Q}$. Therefore, $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q}$. Here are some more examples of rational numbers.

$-8 = \frac{-8}{1} = \frac{8}{-1} = -\frac{8}{1}$	$0 = \frac{0}{1}$
$2.3 = \frac{23}{10}$	$-4.5 = -\frac{45}{10} = \frac{-45}{10} = \frac{45}{-10}$
$3.456 = \frac{3456}{1000}$	$0.12 = \frac{12}{100}$
$5\frac{1}{2} = \frac{11}{2}$	$-7\frac{1}{3} = -\frac{22}{3}$
$0.5555\cdots = 0.\overline{5} = \frac{5}{9}$	$0.646464\cdots = 0.\overline{64} = \frac{64}{99}$

The rational numbers can be arranged in a line, but they leave infinitely many holes. These holes get filled by **irrational numbers**. That is, an irrational number cannot be written in the form of a fraction of two integers. Examples of irrational numbers are

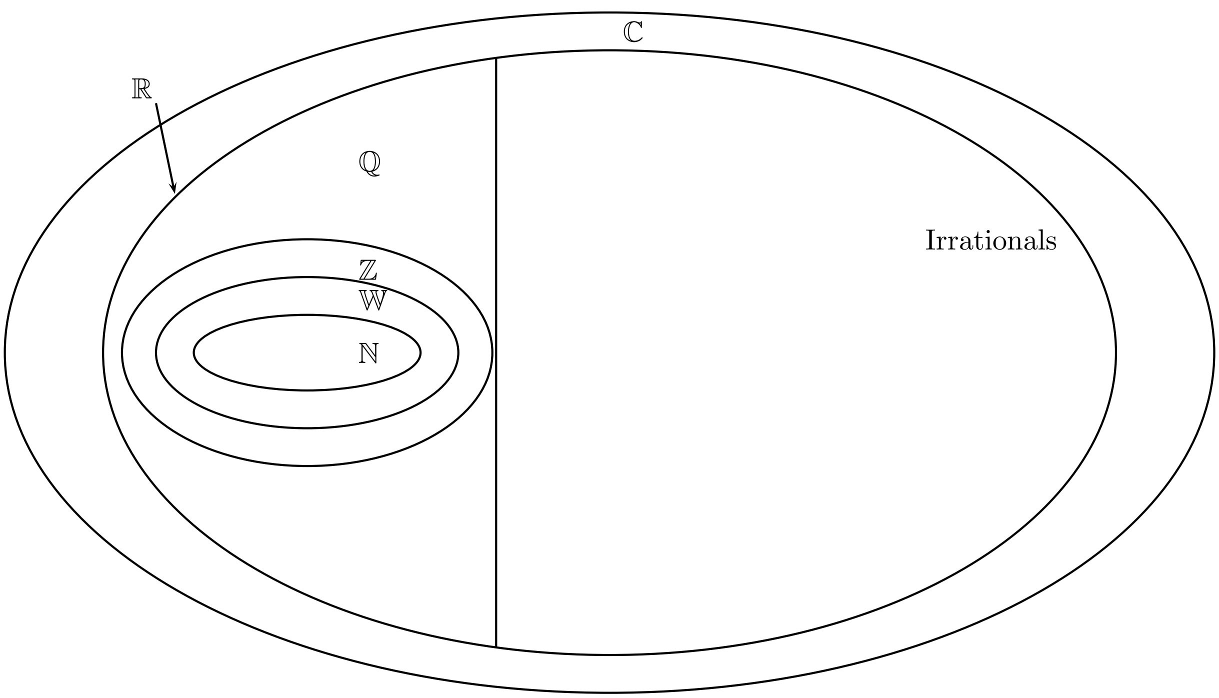
$$\sqrt{2}, \sqrt{3}, \pi \quad (pi), e \quad (\textit{the Euler number}).$$

Irrational numbers are extremely useful, and we will encounter them throughout our course. Numbers which are rational or irrational, are called the **real numbers**. The set of real numbers is denoted by $\mathbb{R}$. There is a bigger number system containing $\mathbb{R}$ on which addition and multiplication are commutative and associative, which is closed under subtraction and division by non-zero numbers, and which contains 0 and 1. That number system is the set of **complex numbers** denoted by $\mathbb{C}$. Let $i = \sqrt{-1}$. It turns out that every element of $\mathbb{C}$ can be written in the form of $a + bi$ where a, b are real numbers. That is,

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$

We will see more on complex numbers towards the end of this course. So we will focus only on the real numbers.

The following figure is a Venn diagram representation of the number systems introduced so far.



In your previous courses you learnt how to add, subtract, multiply, and divide various kinds of numbers. Here we start with the notion of exponents, roots, and absolute values.

1.2 Simple exponents, roots, and absolute values

1.2.1 Exponents and Radicals

Exponents are used to describe several self-multiplications. Instead of writing $2 \times 2 \times 2 \times 2 \times 2$ we write 2^5. For any natural number n, and any real number a, we write

$a^n = a \times a \times \cdots \times a$; *here, a appears n times. Further, $a^0 = 1$ for $a \neq 0$, and 0^0 is undefined.*

We say that a is the **base** and n is the **exponent.** Some examples:

- 2^4: Here, the base is 2 and the exponent is 4. The value of $2^4 = 2 \times 2 \times 2 \times 2 = 16$.
- 3^2: Here, the base is 3 and the exponent is 2. The value of $3^2 = 3 \times 3 = 9$.
- 4^0: Here, the base is 4 and the exponent is 0. The value of $4^0 = 1$.
- $(-5)^2$: Here, the base is -5 and the exponent is 2. The value of $(-5)^2 = (-5) \times (-5) = 25$.
- -5^2: Here, the base is 5 and the exponent is 2. The value of $-5^2 = -5 \times 5 = -25$.
- $\left(\frac{2}{5}\right)^3$: Here, the base is $\left(\frac{2}{5}\right)$ and the exponent is 3.

  The value of $\left(\frac{2}{5}\right)^3 = \left(\frac{2}{5}\right) \times \left(\frac{2}{5}\right) \times \left(\frac{2}{5}\right) = \frac{8}{125}$.

The **inverse** operation of taking exponents is the operation of extracting a root. That is,

$\sqrt{9} = 3$ *because* $3^2 = 9$.
$\sqrt{25} = 5$ *because* $5^2 = 25$.
$\sqrt{1} = 1$ *because* $1^2 = 1$.
$\sqrt{0} = 0$ *because* $0^2 = 0$.
$\sqrt[3]{8} = 2$ *because* $2^3 = 8$.
$\sqrt[3]{27} = 3$ *because* $3^3 = 27$.
$\sqrt[3]{64} = 4$ *because* $4^3 = 64$.
$\sqrt[3]{1} = 1$ *because* $1^3 = 1$.
$\sqrt[3]{0} = 0$ *because* $0^3 = 0$.
$\sqrt[3]{-8} = -2$ *because* $(-2)^3 = -8$.
$\sqrt[3]{-27} = -3$ *because* $(-3)^3 = -27$.

The symbol $\sqrt{}$ is called the **radical.** In an expression $\sqrt{a}$, the number a is called the **radicand.**
a^2 *is read "a -squared;"* $\sqrt{a}$ *is read "square-root of a"*
a^3 *is read "a -cubed;"* $\sqrt[3]{a}$ *is read "cube-root of a"*

Properties of exponents and radicals:

- $a^n \times a^m = a^{n+m}$. For example, $3^2 \times 3^4 = (3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3^6$.
- $(a^n)^m = a^{nm}$. For example, $(5^2)^3 = (5^2) \times (5^2) \times (5^2) = (5 \times 5) \times (5 \times 5) \times (5 \times 5) = 5^6$.
- $(a \times b)^n = a^n \times b^n$. For example, $(4 \times 5)^2 = 4 \times 5 \times 4 \times 5 = 4 \times 4 \times 5 \times 5 = 4^2 \times 5^2$.
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ for $b \neq 0$. For example, $\left(\frac{4}{5}\right)^2 = \frac{4}{5} \times \frac{4}{5} = \frac{4 \times 4}{5 \times 5} = \frac{4^2}{5^2}$.
- $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$ for $a, b \geq 0$. For example, $\sqrt{4 \times 9} = \sqrt{36} = 6 = 2 \times 3 = \sqrt{4} \times \sqrt{9}$.
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ for $a, b \geq 0, b \neq 0$. For example, $\sqrt{\frac{4}{9}} = \frac{2}{3} = \frac{\sqrt{4}}{\sqrt{9}}$.
- $\sqrt[3]{a \times b} = \sqrt[3]{a} \times \sqrt[3]{b}$. For example, $\sqrt[3]{8 \times 27} = \sqrt{216} = 6 = 2 \times 3 = \sqrt[3]{8} \times \sqrt[3]{27}$.
- $\sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$ for $b \neq 0$. For example, $\sqrt[3]{\frac{8}{27}} = \frac{2}{3} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$.

Remark :

- $(a+b)^2 \neq a^2 + b^2$. Check with an example. Let $a = 1$ and $b = 2$. Then

$$\begin{array}{ll} (a+b)^2 & a^2+b^2 \\ = (1+2)^2 & = 1^2+2^2 \\ = 3^2 = 9. & = 1+4 = 5. \end{array}$$

  Therefore, $(1+2)^2 \neq 1^2 + 2^2$.

- $(a-b)^2 \neq a^2 - b^2$. Check with an example. Let $a = 1$ and $b = 2$. Then

$$\begin{array}{ll} (a-b)^2 & a^2-b^2 \\ = (1-2)^2 & = 1^2-2^2 \\ = (-1)^2 = 1. & = 1-4 = -3. \end{array}$$

  Therefore, $(1-2)^2 \neq 1^2 - 2^2$.

- $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. Check with an example. Let $a = 1$ and $b = 4$. Then

$$\begin{array}{ll} \sqrt{a+b} & \sqrt{a}+\sqrt{b} \\ = \sqrt{1+4} & = \sqrt{1}+\sqrt{4} \\ = \sqrt{5} = 2.236067\cdots & = 1+2 = 3. \end{array}$$

Therefore, $\sqrt{1+2} \neq \sqrt{1}+\sqrt{2}$.

- You can check that $\sqrt{a-b} \neq \sqrt{a}-\sqrt{b}$
- You can check that $\sqrt[3]{a+b} \neq \sqrt[3]{a}+\sqrt[3]{b}$
- You can check that $\sqrt[3]{a-b} \neq \sqrt[3]{a}-\sqrt[3]{b}$

1.2.2 Simplifying expressions involving exponents and radicals

Now we are ready to simplify expressions involving exponents and radicals.
Example 1: Simplify $-(-3)^4$.

$$-(-3)^4 = -(-3)(-3)(-3)(-3) = -81 \textit{ There are five negatives.}$$

Example 2: Simplify $(-1)^5$.

$$(-1)^5 = (-1)(-1)(-1)(-1)(-1) = -1.$$

Example 3: Simplify $(-1)^{98}$. Note, $(-1)^{98} = \underbrace{(-1)(-1)\cdots(-1)}_{98 \textit{ times}} = +1$.
Example 4: Simplify $-\sqrt{49}$. Note, $-\sqrt{49} = -7$.
Example 5: Simplify $\sqrt{98}$. Note, $\sqrt{98} = \sqrt{49 \times 2} = \sqrt{49}\sqrt{2} = 7\sqrt{2}$.
Example 6: Simplify $\sqrt{180}$

$$\sqrt{180} = \sqrt{4 \times 45} = \sqrt{4 \times 9 \times 5} = \sqrt{4}\sqrt{9}\sqrt{5} = 2 \times 3 \times \sqrt{5} = 6\sqrt{5}.$$

Example 7: Simplify $\sqrt{(-3)^2}$.

$$\sqrt{(-3)^2} = \sqrt{9} = 3.$$

Example 8: Simplify $-\left(\sqrt{(-5)^2}\right)^3$. Start simplifying from the innermost parentheses.

$$-\left(\sqrt{(-5)^2}\right)^3 = -\left(\sqrt{25}\right)^3 = -(5)^3 = -125.$$

Example 9: Simplify $\left(\sqrt{5}\right)^2$.

$$\left(\sqrt{5}\right)^2 = 5. \quad \textit{This is the meaning of } \sqrt{5}.$$

Yet another way of understanding the above problem is

$$\left(\sqrt{5}\right)^2 = \sqrt{5}\sqrt{5} = \sqrt{25} = 5.$$

Example 10: Simplify $\left(\sqrt[3]{\frac{64}{27}}\right)^2$. Note, $\left(\sqrt[3]{\frac{64}{27}}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$.

Classroom Exercises : Find the value of

(a) $(-1)^{101}$

(b) $-(-2)^6$

(c) $(-3)^4$

(d) $\left(-\sqrt{81}\right)^3$

(e) $\sqrt{(-3)^2}$

(f) $\sqrt{300}$

(g) $\left(\sqrt{7}\right)^2$

1.2.3 Simplifying expressions involving exponents, radicals, and $\times, \div$

Suppose we want to simplify the expression, $2^3 \times 5$. This is equal to $2 \times 2 \times 2 \times 5 = 8 \times 5 = 40$. In other words, when we encounter multiplication and exponents in the same expression, the exponents get performed first. This motivates the following rule:

To simplify an expression involving exponents, radicals, multiplications, and divisions, first perform exponents and radicals, and then perform multiplications and divisions.

Example 1: Simplify $\sqrt{9} \times (4)^3 \div \sqrt[3]{125}$.

$$\sqrt{9} \times (4)^3 \div \sqrt[3]{125} = 3 \times 64 \div 5 = \frac{192}{5}.$$

Example 2: Simplify $\sqrt{72} \div \sqrt{144} \times (-2)^3$

$$\sqrt{72} \div \sqrt{144} \times (-2)^3 = \sqrt{36 \times 2} \div 12 \times (-8) = 6\sqrt{2} \div 12 \times (-8)$$

$$= \frac{6\sqrt{2}}{12} \times (-8) = \frac{-48\sqrt{2}}{12} = -\frac{\cancelto{4}{48}\sqrt{2}}{\cancelto{1}{12}} = -4\sqrt{2}.$$

Example 3: Simplify $\frac{1}{\sqrt{5}}$. (That is, **rationalize the denominator.** This means, the denominator should be free of the radical symbol.)

$$\frac{1}{\sqrt{5}} = \frac{1 \times \sqrt{5}}{\sqrt{5}\sqrt{5}} = \frac{\sqrt{5}}{5}; \textit{ recall that } \sqrt{5}\sqrt{5} = 5.$$

Example 4: Simplify $\sqrt[3]{-343} \div \sqrt{49} \times \sqrt{\frac{25}{12}}$.

$$
\begin{aligned}
&\sqrt[3]{-343} \div \sqrt{49} \times \sqrt{\frac{25}{12}} \\
=&(-7) \div 7 \times \frac{\sqrt{25}}{\sqrt{12}} && \textit{Simplify radicals} \\
=&(-7) \div 7 \times \frac{5}{\sqrt{4 \times 3}} \\
=&\frac{(-7)}{1} \times \frac{1}{7} \times \frac{5}{2\sqrt{3}} && \textit{Multiplication and division now} \\
=&\frac{-\cancelto{1}{7} \times 5}{\cancelto{1}{7} \times 2\sqrt{3}} \\
=&-\frac{5}{2\sqrt{3}} \\
=&-\frac{5\sqrt{3}}{2\sqrt{3}\sqrt{3}} && \textit{Rationalize the denominator} \\
=&-\frac{5\sqrt{3}}{2 \times 3} = -\frac{5\sqrt{3}}{6}. && \textit{The final answer.}
\end{aligned}
$$

As a rule we always rationalize the denominator to obtain the final answer.

Example 5: Simplify $\sqrt{169} \times \sqrt[3]{8} \div 3^3 \div \sqrt{20}$.

$$
\begin{aligned}
&\sqrt{169} \times \sqrt[3]{8} \div 3^3 \div \sqrt{20} \\
=&13 \times 2 \div 27 \div \sqrt{4 \times 5} && \textit{Simplify exponents and radicals} \\
=&13 \times 2 \div 27 \div 2\sqrt{5} \\
=&\frac{13}{1} \times \frac{2}{1} \times \frac{1}{27} \times \frac{1}{2\sqrt{5}} && \textit{Division is multiplication by reciprocal} \\
=&\frac{13 \times \cancelto{1}{2}}{27 \times \cancelto{1}{2}\sqrt{5}} \\
=&\frac{13\sqrt{5}}{27\sqrt{5}\sqrt{5}} && \textit{Rationalize the denominator} \\
=&\frac{13\sqrt{5}}{27 \times 5} = \frac{13\sqrt{5}}{135}. && \textit{The final answer.}
\end{aligned}
$$

Many a time, one has to simplify the radicand (in a radical expression) or the base (in an exponential expression) first.

Example 6: Simplify $\sqrt{36 \times 100 \div 49 \div 81}$.

$$\sqrt{36 \times 100 \div 49 \div 81} = \sqrt{\frac{36}{1} \times \frac{100}{1} \times \frac{1}{49} \times \frac{1}{81}} = \sqrt{\frac{36 \times 100}{49 \times 81}} = \frac{\overset{2}{\cancel{6}} \times 10}{7 \times \underset{3}{\cancel{9}}} = \frac{20}{21}.$$

Example 7: Simplify $(90 \div 9 \div 12 \div 2)^2$

$$(90 \div 9 \div 12 \div 2)^2 = \left(\frac{90}{1} \times \frac{1}{9} \times \frac{1}{12} \times \frac{1}{2}\right)^2 = \left(\frac{\overset{10}{\cancel{90}} \times 1 \times 1 \times 1}{1 \times \underset{1}{\cancel{9}} \times 12 \times 2}\right)^2 = \left(\frac{\overset{5}{\cancel{10}}}{12 \times \underset{1}{\cancel{2}}}\right)^2 = \frac{25}{144}.$$

Classroom Exercises : Simplify the following (Do not forget to rationalize the denominator whenever necessary) :

1. (a) $\sqrt{8}$ (b) $\sqrt{18}$ (c) $\sqrt{32}$ (d) $\sqrt{50}$
2. (a) $\sqrt{12}$ (b) $\sqrt{27}$ (c) $\sqrt{48}$ (d) $\sqrt{75}$
3. (a) $\sqrt{20}$ (b) $\sqrt{45}$ (c) $\sqrt{80}$ (d) $\sqrt{125}$
4. (a) $\sqrt{24}$ (b) $\sqrt{54}$ (c) $\sqrt{28}$ (d) $\sqrt{63}$
5. (a) $\sqrt{\frac{1}{8}}$ (b) $\sqrt{\frac{1}{18}}$ (c) $\sqrt{\frac{1}{32}}$ (d) $\sqrt{\frac{1}{50}}$
6. (a) $\sqrt{\frac{1}{12}}$ (b) $\sqrt{\frac{1}{27}}$ (c) $\sqrt{\frac{1}{48}}$ (d) $\sqrt{\frac{1}{75}}$
7. (a) $\sqrt{\frac{1}{20}}$ (b) $\sqrt{\frac{1}{45}}$ (c) $\sqrt{\frac{1}{80}}$ (d) $\sqrt{\frac{1}{125}}$
8. (a) $\sqrt{\frac{1}{24}}$ (b) $\sqrt{\frac{1}{54}}$ (c) $\sqrt{\frac{1}{28}}$ (d) $\sqrt{\frac{1}{63}}$
9. $\sqrt{8} + \sqrt{18} - 3\sqrt{32} - \sqrt{50}$
10. $\sqrt{12} - 2\sqrt{27} + 4\sqrt{48} + \sqrt{75}$
11. $\sqrt{20} - 5\sqrt{45} + 8\sqrt{80} + \sqrt{125}$
12. $\sqrt{24} + 2\sqrt{54} - 3\sqrt{28} + \sqrt{63}$
13. $\sqrt{121} \div \sqrt[3]{125} \times \left(\sqrt{5}\right)^2$
14. $-\sqrt{289} \div \sqrt{150} \div \sqrt{9}$

15. $\sqrt{\dfrac{400}{45}}$

16. $(7 \div 3 \div 2 \times 5 \div 2)^3$

17. $(4^2 \div 2^4)^0$

1.2.4 Simplifying expressions involving exponents, radicals, and $\times, \div, +, -$

Recall, to simplify an expression involving $\times, \div, +, -$, we first perform $\times, \div$ (left to right) and then perform $+, -$ (left to right).
Therefore, to simplify an expression involving exponents, radicals, $\times, \div, +$, and $-$ we perform the operations in the following order:

1. Exponents and Radicals are computed first.
2. Multiplications and Divisions are computed second (left to right).
3. Additions and Subtractions come at the end (left to right).

Example 1: Simplify $5^2 \div \sqrt{36} + \sqrt[3]{27} \times 2^3$

$$\begin{aligned}
& 5^2 \div \sqrt{36} + \sqrt[3]{27} \times 2^3 \\
=& 25 \div 6 + 3 \times 8 && \textit{Exponents and radicals first} \\
=& \frac{25}{6} + 3 \times 8 && \textit{Multiplications and divisions next (left to right)} \\
=& \frac{25}{6} + 24 \\
=& \frac{25}{6} + \frac{24 \times 6}{6} && \textit{The least common denominator is } 6 \\
=& \frac{25}{6} + \frac{144}{6} = \frac{25 + 144}{6} = \frac{169}{6}.
\end{aligned}$$

Example 2: Simplify $4^2 + \sqrt{80} \times 3 - \sqrt{125}$.

$$\begin{aligned}
& 4^2 + \sqrt{80} \times 3 - \sqrt{125} \\
=& 16 + \sqrt{16 \times 5} \times 3 - \sqrt{25 \times 5} && \textit{Simplify exponents and radicals first} \\
=& 16 + 4 \times \sqrt{5} \times 3 - 5\sqrt{5} \\
=& 16 + 4 \times 3 \times \sqrt{5} - 5\sqrt{5} && \textit{Multiplication is commutative} \\
=& 16 + 12\sqrt{5} - 5\sqrt{5} \\
=& 16 + 7\sqrt{5} && \textit{Additions and subtractions last.}
\end{aligned}$$

Example 3: Simplify $\sqrt{(-3)^2 + (-9)^2}$.

$$\sqrt{(-3)^2 + (-9)^2} = \sqrt{9 + 81} = \sqrt{90} = \sqrt{9 \times 10} = \sqrt{9}\sqrt{10} = 3\sqrt{10}.$$

Notice in this case that $\sqrt{9 + 81} \neq \sqrt{9} + \sqrt{81}$.
Classroom Exercises : Simplify

(a) $\dfrac{3^2}{\sqrt{5}} - \dfrac{4\sqrt[3]{27}}{\sqrt{5}} \div \dfrac{\sqrt{20}}{12}$

(b) $\dfrac{\sqrt{6 \times 4}}{5} + \dfrac{\sqrt{2}}{5} \div \dfrac{\sqrt{3}}{4}$

(c) $\dfrac{\sqrt{8}}{3} + \dfrac{\sqrt{18}}{4} - \dfrac{5\sqrt{2}}{3}$

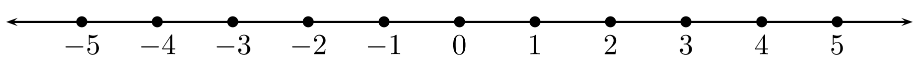
(d) $\sqrt{5^2 + 4^2}$

(e) $\sqrt{12} + \sqrt{75} - \sqrt{27}$

(f) $3\sqrt{8} - 4\sqrt{50} + 2\sqrt{18}$

1.2.5 Absolute value and simplifications involving absolute values

The **absolute value** of a real number is its distance from 0 on the real number line. Absolute value of number a is denoted by $|a|$.



- The distance between 4 and 0 on the number line is 4. Hence, $|4| = 4$.
- The distance between -4 and 0 on the number line is 4. Hence, $|-4| = 4$.
- The distance between 0 and 0 on the number line is 0. Hence, $|0| = 0$.

The absolute value is closely related to exponents and radicals. Pay careful attention to the numbers in the following three examples.

- $\sqrt{4^2} = \sqrt{16} = 4$.
- $\sqrt{(-4)^2} = \sqrt{16} = 4$.
- $\sqrt{0^2} = \sqrt{0} = 0$.

In other words, $\sqrt{a^2} = |a|$ *for any real number* a.

Classroom Exercises :

(a) $\sqrt{5^2} =$

(b) $\sqrt{(-5)^2} =$

(c) $\sqrt{7^2} =$

(d) $\sqrt{(-7)^2} =$

(e) $-\sqrt{7^2} =$

(f) $-3\sqrt{2^2} =$

Mind you, $\sqrt[3]{a^3} \neq |a|$. For instance, $\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2$ which is not the same as $|-2|$.
Properties of the absolute value:

- $|a \times b| = |a| \times |b|$ for any real numbers a, b. For example, $|(-2) \times 3| = |-6| = 6 = |-2| \times |3|$.
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ for any real numbers a, b with $b \neq 0$. For example, $\left|\frac{-2}{3}\right| = \left|-\frac{2}{3}\right| = \frac{2}{3} = \frac{|-2|}{|3|}$.

Remark:

- $|a + b| \neq |a| + |b|$. Check for $a = (-2)$ and $b = 1$.
  $|(-2) + 1| = |-1| = 1$. Whereas, $|-2| + |1| = 2 + 1 = 3$.
- $|a - b| \neq |a| - |b|$. Check for $a = (-2)$ and $b = 1$.
  $|(-2) - 1| = |-3| = 3$. Whereas, $|-2| - |1| = 2 - 1 = 1$.

Classroom Exercises : Find the values of

(a) $\left|\frac{-9}{10}\right|$

(b) $|2 \times (-7)|$

(c) $\sqrt{\left(\frac{-9}{10}\right)^2}$

(d) $\sqrt{(2 \times (-7))^2}$

1.2.6 Simplifying expressions involving absolute values

When we want to simplify an expression involving exponents, radicals, $|\ \ |, \times, \div, +, -$, the absolute value takes the same priority as exponents and radicals. That is,

1. Exponents, Radicals, and Absolute Values are computed first.

2. Multiplications and Divisions are computed second.

3. Additions and Subtractions come at the end.

As always, if exponents, radicals, and absolute values themselves have some complicated expressions within them, then the inner complications have to be simplified. In other words, exponents, radicals, and absolute values can themselves be **grouping** mechanisms.

Example 1: Simplify $|-10| \times 3^2 - \sqrt{9}$.

$$\begin{aligned} & |-10| \times 3^2 - \sqrt{9} && \textit{Absolute value, exponent and radical first} \\ =& 10 \times 9 - 3 && \textit{Multiplication next} \\ =& 90 - 3 = 87. && \textit{Subtraction last.} \end{aligned}$$

Example 2: Simplify $2^3 \div |-5| \times \sqrt{49} + 8$.

$$\begin{aligned} & 2^3 \div |-5| \times \sqrt{49} + 8 && \textit{Exponent, radical, absolute value first} \\ =& 8 \div 5 \times 7 + 8 && \textit{Multiplication and division next} \\ =& \frac{8}{5} \times 7 + 8 = \frac{8}{5} \times \frac{7}{1} + 8 && \textit{Addition last; the least common denominator is } 5 \\ =& \frac{56}{5} + \frac{40}{5} = \frac{96}{5}. \end{aligned}$$

Example 3: Simplify $\left|\dfrac{\sqrt[3]{125}\times|-2|+12^2\div|-8|}{-\left|-3+5\right|^2}\right|+4$. In this case, the outermost absolute value is to be performed after the expression inside is simplified.

$$\left|\frac{\sqrt[3]{125}\times|-2|+12^2\div|-8|}{-\left|-3+5\right|^2}\right|+4$$

The inner exponents, radicals and absolute values are to be simplifed first

$$=\left|\frac{5\times 2+144\div 8}{-\left|2\right|^2}\right|+4$$

$$=\left|\frac{5\times 2+144\div 8}{-(2)^2}\right|+4$$

$$=\left|\frac{10+18}{-4}\right|+4$$

Simplify multiplications and divisions inside

$$=\left|\frac{28}{-4}\right|+4=\left|-\frac{28}{4}\right|+4=\frac{28}{4}+4=7+4=11.$$

Classroom Exercises:

(a) $|-4|$

(b) $-|4|$

(c) $-|-4|$

(d) $|3-8\times 2|$

(e) $|(-1)^{81}|$

(f) $|(-12)\div 6\times 2|$

(g) $|(-1)^3+3\div 5-4|$

(h) $(-2)^3+3\div|2-7|+1$

(i) $\left|\dfrac{\sqrt[3]{27}-(-2)^4+3}{8\times(-2)}\right|$

1.2.7 Homework Exercises

Do not forget to rationalize the denominator whenever necessary.
Simplify (exponents and radicals):

1. $(-1)^{2001}$
2. $-(-5)^3$
3. $-(-(-2)^4)$
4. $-\sqrt{1000}$
5. $\sqrt{128}$
6. $\sqrt{243}$
7. $\sqrt{150}$
8. $\left(\sqrt{15}\right)^2$
9. $\sqrt[3]{\dfrac{1}{64}}$
10. $\left(\sqrt[3]{7}\right)^3$

Simplify (exponents, radicals, multiplications, and divisions)

11. $\sqrt{196} \times \sqrt[3]{1000}$
12. $\sqrt{\dfrac{1}{128}}$
13. $\sqrt[3]{64} \div \sqrt{12}$
14. $\sqrt{12} \div \sqrt{98} \times \sqrt[3]{8}$
15. $\sqrt{30} \times \sqrt{2} \times \sqrt{3}$
16. $\sqrt[3]{8^2 \times 5^2 \div 20^2 \div 3^3}$
17. $6^3 \div \sqrt{108} \div \sqrt{162}$
18. $\sqrt{1 \div 8 \div 18 \div 25}$
19. $(80 \div \sqrt{400} \times \sqrt[3]{7})^3$

20. $\left(\sqrt{(-5)^2 \div (-4)^2}\right)^3$

Simplify (exponents, radicals, multiplications, divisions, additions, and subtractions)

21. $\sqrt{3} + \sqrt{48} - \sqrt{75}$
22. $\sqrt{\frac{9}{2}} - \sqrt[3]{5 \div 5} \div \frac{\sqrt{50}}{3^2}$
23. $\left(\frac{3}{2}\right)^3 + \sqrt{\frac{25}{64}} \div \sqrt[3]{\frac{1}{10}}$
24. $\sqrt{24} - 4\sqrt{6} + \sqrt{150}$
25. $\sqrt{8} - \sqrt{18} + \sqrt{50}$
26. $\frac{5^3 \times 2^3 - \sqrt{8}\sqrt{72}}{\sqrt{25} + \sqrt{3}\sqrt{75}}$
27. $\sqrt{\frac{\sqrt[3]{125} + 2^2 5^3 - 1}{(-4)^2\sqrt{9} + (-2)^3}}$

Simplify (absolute values, exponents, radicals, multiplications, divisions, additions, and subtractions)

28. $\left|(-1)^{2009}\right|$
29. $\left|(-7)^3\right|$
30. $\left|-(-(-3)^4)\right|$
31. $\left|-\sqrt{24}\right|$
32. $\sqrt{|-12|} \div \sqrt{|-98|} \times |\sqrt[3]{-8}|$
33. $3 - 5 + |7 - 12| \times |(-2)^3 - 12| \div 3$
34. $|3 - 5 + 7 \div 2 \times 3|$
35. $|(-2)^{2011}|^0$
36. $\left|-\sqrt{150} + \sqrt{2} \times \sqrt{3}\right|$
37. $\left|2^3 \times 5^2 - 20^2 \div 3\right|$
38. $-80 \div \sqrt{|-400|} \times \left|\sqrt[3]{-2})^3\right|$

39. $\left|-\sqrt{|25| \div (-4)^2}\right|^3$

40. $\left|\dfrac{2^3 - 5^0 - \sqrt{8}\sqrt{72}}{\sqrt{25} + \sqrt{3}\sqrt{75}}\right|$

1.3 Roots and Radicals

We were introduced to roots and radicals in Section 1.4. Recall that the symbol $\sqrt{\ }$ is called the **radical** and the expression inside the $\sqrt{\ }$ symbol is called the **radicand.** For example,

$$\text{in the expression } \sqrt[3]{\frac{xy}{a+b}}, \text{ the radicand is } \frac{xy}{a+b}.$$

For a natural number n, we say that a is an **n-th root** of a number b if and only if $a^n = b$. We write the phrase, "n-th root of b," in short as $\sqrt[n]{b}$. Therefore,

$$a = \sqrt[n]{b} \textit{ if and only if } a^n = b.$$

The first thing to note is that by definition, $\left(\sqrt[n]{b}\right)^n = b$.

Terminology : The second root of b is called the **square-root** of b; it is denoted by $\sqrt{b}$ instead of $\sqrt[2]{b}$. The third root of b is called the **cube-root** or the **cubic-root** of b.

First we note that there are two square-roots of 25 as $5^2 = 25$ and $(-5)^2 = 25$. Similarly, as $3^4 = 81$, and $(-3)^4 = 81$, there are two fourth roots of 81. The **principal square-root** of a non-negative number is its **positive** square-root. Similarly, the **principal fourth root** of a non-negative number is its **positive** fourth root. For example, the principal square-root of 9 is $+3$. We therefore write

$$\sqrt{25} = 5, \quad -\sqrt{25} = -5; \quad \sqrt[4]{81} = 3, \quad -\sqrt[4]{81} = -3.$$

In general, for an **even** natural number n, $n \geq 2$, the **principal n-th root** is the **positive** n-th root.

Note that, the n-th root of a real number need not always be a real number. For example, $\sqrt{-9}$ is not a real number. This is because, the square of a real number is non-negative. For instance,

$$3^2 = 9, \textit{ and } (-3)^2 = 9.$$

In other words, there is no real number a, such that $a^2 = -9$. Therefore, $\sqrt{-9}$ is not a real number. In fact, $\sqrt{-9}$ is a complex number, and we will learn about complex numbers in Section 6.3.

Likewise, $\sqrt[4]{-625}$ is not a real number because the fourth power of any real number is non-negative. Again, $\sqrt[4]{-625}$ is a complex number. Note,

$$5^4 = 625, \ \textit{and} \ (-5)^4 = 625.$$

You probably have realized the following generalization:
For n an **even natural number**, the n-th root of a **negative real number** is **not a real number.**

This is not to be confused with the following situation: What is square-root of 3? It is the **irrational number** $\sqrt{3}$. There is no simpler way of writing $\sqrt{3}$. This is a real number. Likewise, $\sqrt[3]{10}$, $\sqrt[4]{25}$, or $\sqrt[9]{100}$ are all irrational **real** numbers.
Here are some examples.

$\sqrt[n]{0}$	$=$	0	*because* $0^n = 0$ *for any natural number* n.
$\sqrt{64}$	$=$	8	*because* $8^2 = 64$.
$\sqrt{-64}$	*is*	*not a real number*	*because the square of a real number can not be negative.*
$\sqrt[3]{64}$	$=$	4	*because* $4^3 = 64$.
$\sqrt[3]{-64}$	$=$	-4	*because* $(-4)^3 = (-4) \times (-4) \times (-4) = -64$.
$\sqrt[4]{81}$	$=$	3	*because* $3^4 = 81$.
$\sqrt[4]{-81}$	*is*	*not a real number*	*because the fourth power of a real number cannot be negative.*
$\sqrt[5]{32}$	$=$	2	*because* $2^5 = 32$.
$\sqrt[5]{-32}$	$=$	-2	*because* $(-2)^5 = (-2) \times (-2) \times (-2) \times (-2) \times (-2) = -32$.
$\sqrt[3]{\frac{27}{1000}}$	$=$	$\frac{3}{10}$	*because* $\left(\frac{3}{10}\right)^3 = \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} = \frac{27}{1000}$.
$\sqrt[3]{-\frac{27}{1000}}$	$=$	$-\frac{3}{10}$	*because* $\left(-\frac{3}{10}\right)^3 = \left(-\frac{3}{10}\right) \times \left(-\frac{3}{10}\right) \times \left(-\frac{3}{10}\right) = -\frac{27}{1000}$.
$\sqrt{-\frac{49}{81}}$	*is*	*not a real number*	*because the square of a real number can not be negative.*

Here are some important properties of radicals :

Property 1 *For real numbers* a, b *and natural number* n*, if* $\sqrt[n]{a}$, $\sqrt[n]{b}$ *are real numbers, then*

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

Proof :

Suppose $x = \sqrt[n]{a}$. Then $x^n = a$. Supose $y = \sqrt[n]{b}$. Then $y^n = b$.

Note that $\underbrace{x^n}_{a} \cdot \underbrace{y^n}_{b} = (x \cdot y)^n$.

In other words, $a \cdot b = (x \cdot y)^n$. Hence $\sqrt[n]{a \cdot b} = x \cdot y$. That is,

$$\underbrace{\sqrt[n]{a \cdot b}}_{\sqrt[n]{a \cdot b}} = \underbrace{x}_{\sqrt[n]{a}} \cdot \underbrace{y}_{\sqrt[n]{b}}.$$

We have therefore proved that $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

For example :

- $\sqrt[3]{8 \times 125} = \sqrt[3]{8} \times \sqrt[3]{125} = 2 \times 5 = 10.$
- $\sqrt{900} = \sqrt{9} \times \sqrt{100} = 3 \times 10 = 30.$
- $\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$. (*This can not be simplified further.*)
- $-\sqrt[4]{162} = -\sqrt[4]{81 \times 2} = -\sqrt[4]{81} \times \sqrt[4]{2} = -3\sqrt[4]{2}$. (*This can not be simplified further.*)
- $\sqrt[5]{160} = \sqrt[5]{32 \times 5} = \sqrt[5]{32} \times \sqrt[5]{5} = 2\sqrt[5]{5}$. (*This can not be simplified further.*)

Property 2 *For real numbers* a, b*, with* $b \neq 0$*, and a natural number* n *such that* $\sqrt[n]{a}$, $\sqrt[n]{b}$ *are real numbers, then*

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Proof

Suppose $x = \sqrt[n]{a}$. Then $x^n = a$. Supose $y = \sqrt[n]{b}$. Then $y^n = b$.

Note that $\dfrac{a}{b} = \dfrac{x^n}{y^n} = \left(\dfrac{x}{y}\right)^n$.

That is, $\dfrac{a}{b} = \left(\dfrac{x}{y}\right)^n$.

$$\text{Therefore, } \sqrt[n]{\frac{a}{b}} = \frac{x}{y}.$$

$$\text{That is, } \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

For example :

- $\sqrt{\frac{49}{81}} = \frac{\sqrt{49}}{\sqrt{81}} = \frac{7}{9}.$
- $\sqrt[3]{-\frac{8}{125}} = \sqrt[3]{\frac{-8}{125}} = \frac{\sqrt[3]{-8}}{\sqrt[3]{125}} = \frac{-2}{5} = -\frac{2}{5}.$
- $-\sqrt{\frac{100}{63}} = -\frac{\sqrt{100}}{\sqrt{63}} = -\frac{10}{\sqrt{9 \times 7}} = -\frac{10}{\sqrt{9} \times \sqrt{7}} = -\frac{10}{3\sqrt{7}}.$
  Here, we need to rationalize the denominator.

$$\begin{aligned} -\frac{10}{3\sqrt{7}} &= -\frac{10 \times \sqrt{7}}{3\sqrt{7} \times \sqrt{7}} \\ &= -\frac{10\sqrt{7}}{3 \times 7} && \textit{as } \sqrt{7} \times \sqrt{7} = 7 \\ &= -\frac{10\sqrt{7}}{21}. && \textit{This can not be simplified further.} \end{aligned}$$

- $\sqrt[3]{\frac{27}{100}} = \frac{\sqrt[3]{27}}{\sqrt[3]{100}} = \frac{3}{\sqrt[3]{100}}$. Here, we need to rationalize the denominator.

$$\begin{aligned} \frac{3}{\sqrt[3]{100}} &= \frac{3 \times \sqrt[3]{10}}{\sqrt[3]{100} \times \sqrt[3]{10}} \\ &= \frac{3\sqrt[3]{10}}{\sqrt[3]{100 \times 10}} \\ &= \frac{3\sqrt[3]{10}}{\sqrt[3]{1000}} = \frac{3\sqrt[3]{10}}{10} && \textit{This can not be simplified further.} \end{aligned}$$

Caution : $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ and $\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$.
For example, $\sqrt{4+9} = \sqrt{13}$, while $\sqrt{4} + \sqrt{9} = 2 + 3 = 5$.
Similarly, $\sqrt{25-9} = \sqrt{16} = 4$, while $\sqrt{25} - \sqrt{9} = 5 - 3 = 2$.
Recall from the definition and properties of the absolute value of a real number. The **absolute value** of a real number is its distance from 0 on the number line. The absolute value of x is denoted by $|x|$. So, $|3| = 3$, and $|-3| = 3$. This is because, both 3 and -3 are at a distance of 3 from 0 on the number line. Note that $\sqrt{3^2} = \sqrt{9} = 3$, and $\sqrt{(-3)^2} = \sqrt{9} = 3$. In general,

$$\sqrt{x^2} = |x| \textit{ for any real number } x.$$

Here is yet another situation:

$$\sqrt[4]{5^4} = \sqrt[4]{625} = 5.$$
$$\sqrt[4]{(-5)^4} = \sqrt[4]{625} = 5.$$
$$\textit{In general, } \sqrt[4]{x^4} = |x| \textit{ for any real number } x.$$

One more example:

$$\sqrt[6]{10^6} = \sqrt[6]{1000000} = 10.$$
$$\sqrt[6]{(-10)^6} = \sqrt[6]{1000000} = 10.$$
$$\textit{In general, } \sqrt[6]{x^6} = |x| \textit{ for any real number } x.$$

Note that for any **even natural number** n, and any real number x, we get x^n to be a non-negative number. Therefore, the principal n-th root of x^n will be a **non-negative** real number. Therefore,

For an **even** *natural number n, and any real number x, we have $\sqrt[n]{x^n} = |x|$.*

Now consider the following situation:

$$\sqrt[3]{2^3} = \sqrt[3]{8} = 2.$$
$$\sqrt[3]{(-2)^3} = \sqrt[3]{-8} = -2.$$
$$\textit{In general, } \sqrt[3]{x^3} = x \textit{ for any real number } x.$$

Here is yet another situation:

$$\sqrt[5]{3^5} = \sqrt[5]{243} = 3.$$
$$\sqrt[5]{(-3)^5} = \sqrt[5]{-243} = -3.$$
$$\textit{In general, } \sqrt[5]{x^5} = x \textit{ for any real number } x.$$

We can draw a general conclusion from this:

For an **odd** *natural number n, and any real number x, we have $\sqrt[n]{x^n} = x$.*

Here are some examples on simplifying algebraic expressions. Assume that all the variables are positive.

- $-\sqrt{81x^2} = -9x$. This can not be simplified further.
- $\sqrt{80x^3} = \sqrt{16 \times 5 \times x^2 \times x} = \sqrt{16 \times x^2 \times 5 \times x} = \sqrt{16} \times \sqrt{x^2} \times \sqrt{5 \times x} = 4x\sqrt{5x}$.

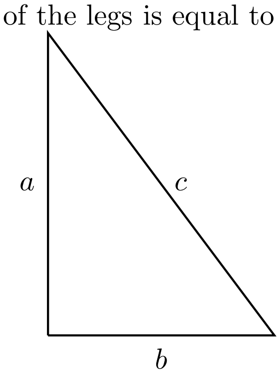
- $\sqrt{\dfrac{12x^3}{y}} = \dfrac{\sqrt{12x^3}}{\sqrt{y}} = \dfrac{\sqrt{4x^2 \times 3x}}{\sqrt{y}} = \dfrac{2x\sqrt{3x}}{\sqrt{y}}.$

  Now rationalize the denominator
  $$\frac{2x\sqrt{3x}}{\sqrt{y}} = \frac{2x\sqrt{3x} \times \sqrt{y}}{\sqrt{y} \times \sqrt{y}} = \frac{2x\sqrt{3xy}}{y}.$$
  There are several ways of arriving at this simplification. Here is another one:
  $$\sqrt{\frac{12x^3}{y}} = \sqrt{\frac{12x^3 \times y}{y \times y}} = \frac{\sqrt{12x^3y}}{\sqrt{y^2}} = \frac{\sqrt{4x^2 \times 3xy}}{\sqrt{y^2}} = \frac{2x\sqrt{3xy}}{y}.$$

- $\sqrt[3]{-\dfrac{40x^4}{y^5}} = \sqrt[3]{\dfrac{-40x^4}{y^5}} = \dfrac{\sqrt[3]{-40x^4}}{\sqrt[3]{y^5}} = \dfrac{\sqrt[3]{-8x^3 \times 5x}}{\sqrt[3]{y^3 \times y^2}} = \dfrac{\sqrt[3]{-8x^3} \times \sqrt[3]{5x}}{\sqrt[3]{y^3} \times \sqrt[3]{y^2}} = \dfrac{-2x\sqrt[3]{5x}}{y\sqrt[3]{y^2}}.$

  Here, we need to rationalize the denominator.
  $$\frac{-2x\sqrt[3]{5x}}{y\sqrt[3]{y^2}} = \frac{-2x\sqrt[3]{5x} \times \sqrt[3]{y}}{y\sqrt[3]{y^2} \times \sqrt[3]{y}} = \frac{-2x\sqrt[3]{5xy}}{y\sqrt[3]{y^3}} = \frac{-2x\sqrt[3]{5xy}}{y \times y} = \frac{-2x\sqrt[3]{5xy}}{y^2}.$$
  Here is another way of arriving at the same simplification:
  $$\begin{aligned}\sqrt[3]{-\frac{40x^4}{y^5}} = \sqrt[3]{-\frac{40x^4 \times y}{y^5 \times y}} = \sqrt[3]{-\frac{40x^4y}{y^6}} &= \sqrt[3]{\frac{-40x^4y}{y^6}} \\ &= \frac{\sqrt[3]{-40x^4y}}{\sqrt[3]{y^6}} \\ &= \frac{\sqrt[3]{-8x^3 \times 5xy}}{\sqrt[3]{y^6}} \\ &= \frac{-2x\sqrt[3]{5xy}}{y^2}.\end{aligned}$$

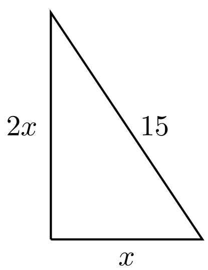
Pythagorean Theorem: Given a right angled triangle, the sum of the squares of the lenghts of the legs is equal to the square of the the length of the hypotenuse. That is,



Pythagorean Theorem:
$a^2 + b^2 = c^2$

Example:

Find the value of x (simplify the radical).



$(2x)^2 + x^2 = 15^2$ *By Pythagorean theorem*

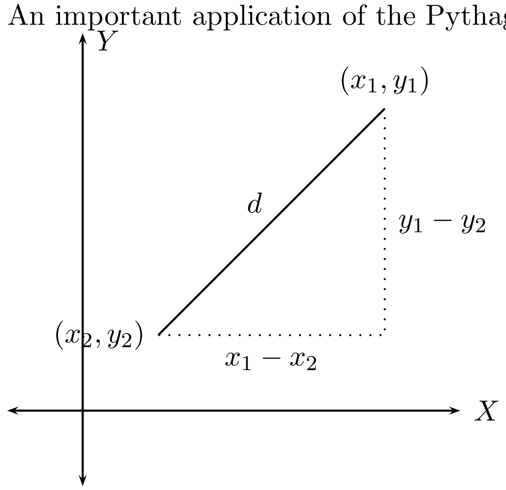
$4x^2 + x^2 = 225$

$5x^2 = 225$ *Divide both sides by* 5

$x^2 = 45$ *Take square-roots of both sides*

$x = \sqrt{45} = 3\sqrt{5}$ units *We only consider the positve square-root.*

An important application of the Pythagorean theorem is the **distance formula.**



Given two points (x_1, y_1) and (x_2, y_2) on the coordinate plane, let the distance between them be d.
By Pythagorean theorem we have
$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$.
*Thus, we have the **distance formula***

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

For example, to find the distance between points $(2, 3)$ and $(-1, 4)$, we first set $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (-1, 4)$. Then, the distance d is given by

$$d = \sqrt{(2-(-1))^2 + (3-4)^2} = \sqrt{(2+3)^2 + (-1)^2} = \sqrt{5^2 + (-1)^2} = \sqrt{25+1} = \sqrt{26}.$$

Classroom Exercises :

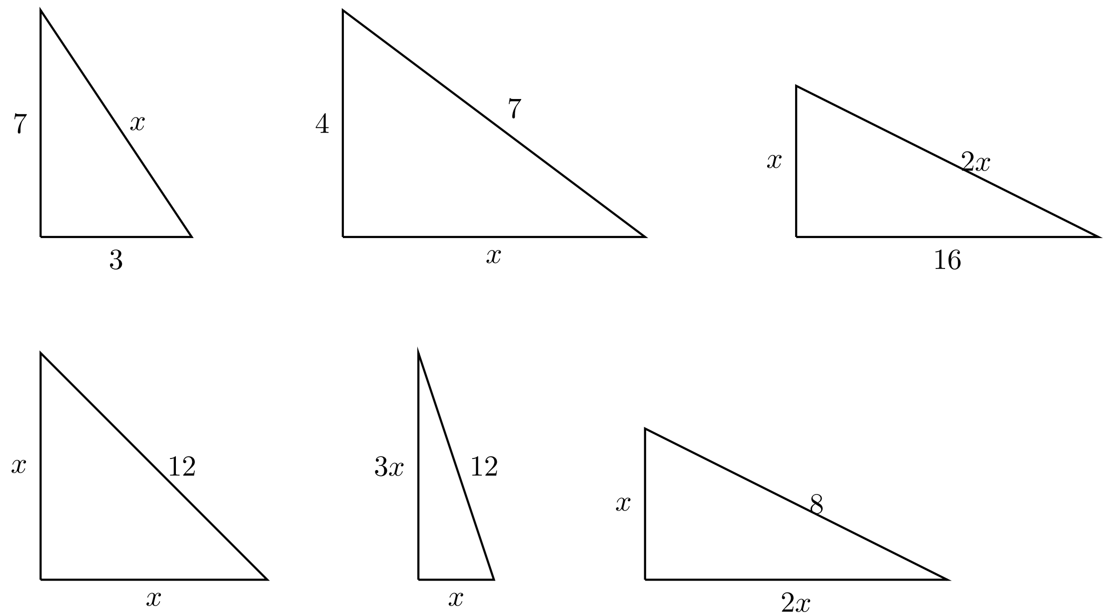
1. Simplify (rationalize the denominator if needed):

(a) $\sqrt{36}$ (b) $-\sqrt{\dfrac{225}{256}}$ (c) $\sqrt{\dfrac{9}{125}}$ (d) $\sqrt{3^2+4^2}$ (e) $\sqrt{(-4)^2} - \sqrt{4}$

(f) $\sqrt{18} + \sqrt{2}$ (g) $-\sqrt{12} + \sqrt{27}$ (h) $\sqrt{\dfrac{32}{75}}$ (i) $\sqrt[3]{-\dfrac{64}{343}}$ (j) $\sqrt[4]{\dfrac{32y^7}{z^8}}$

(k) $\sqrt[4]{2^4}$ (l) $\sqrt[4]{(-2)^4}$ (m) $\sqrt[3]{2^3}$ (n) $\sqrt[3]{(-2)^3}$ (o) $\sqrt[3]{\dfrac{32y^7}{z^6}}$

2. Find x in the following right angled triangles (simplify the radicals):



3. Find the distance between the given pair of points:

   (a) $(3, 4)$ and $(-3, -4)$.

   (b) $(-1, -1)$ and $(5, 6)$.

   (c) $(3, -4)$ and $(-1, -4)$.

   (d) $(-1, 6)$ and $(0, 6)$.

1.3.1 Homework Exercises

1. Simplify and rationalize the denominator whenever necessary. The variables here take positive values only.

   (a) $\sqrt[6]{(-3)^6}$

   (b) $\sqrt[3]{(-3)^3}$

   (c) $\sqrt[5]{243}$

   (d) $\sqrt[3]{\dfrac{27}{75}}$

(e) $\sqrt[4]{\dfrac{32}{243}}$

(f) $\sqrt{300}$

(g) $\sqrt{\dfrac{49}{60}}$

(h) $\sqrt{\dfrac{99}{128}}$

(i) $\sqrt{36+225}$

(j) $\sqrt{36}+\sqrt{225}$

(k) $\sqrt{144-64}$

(l) $\sqrt{144}-\sqrt{64}$

(m) $\left(\sqrt[6]{10}\right)^6$

(n) $\sqrt[6]{(-10)^6}$

(o) $\sqrt[8]{x^8}$

(p) $\sqrt[7]{x^7}$

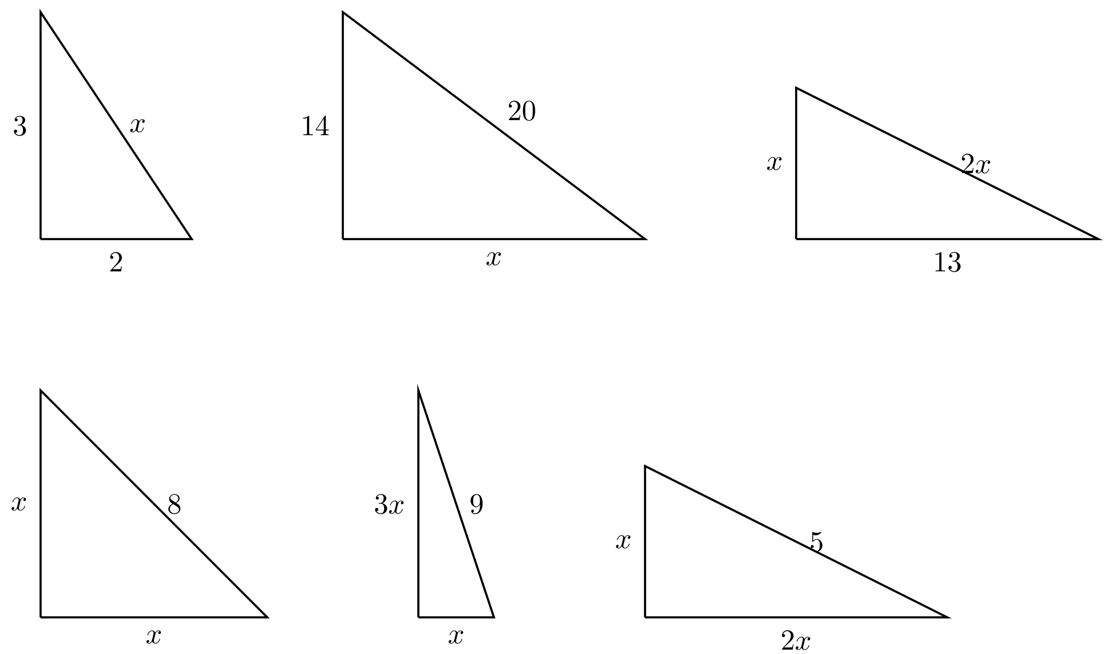
(q) $\sqrt[5]{\dfrac{32x^5}{y^5}}$

(r) $\sqrt[5]{\dfrac{100000x^7}{y^9}}$

(s) $\sqrt{\dfrac{72x^5y^{11}z^{13}}{75a^3}}$

(t) $\sqrt[4]{\dfrac{48a^6}{b^8}}$

2. Find x in the following right angled triangles (simplify the radicals):



3. Find the distance between the following pairs of points on the coordinate plane.
   (a) $(3, 8)$ and $(-9, -7)$
   (b) $(2, 0)$ and $(0, 2)$
   (c) $(3, -5)$ and $(9, -7)$
   (d) $(2, 4)$ and $(1, 2)$

1.4 Operations on Radical expressions

In this lesson we continue working with radical expressions following the properties presented in lesson 28. Recall

- For a natural number n, and real numbers a, b, we say $a = \sqrt[n]{b}$ if and only if $a^n = b$. That is, $\left(\sqrt[n]{b}\right)^n = b$.
- On the other hand, $\sqrt[n]{x^n} = |x|$ if n is an **even** natural number, and x is any real number. If x is allowed only positive values, then $\sqrt[n]{x^n} = x$.
- If n is an **odd** natural number, then $\sqrt[n]{x^n} = x$ for any real number x.

- For n an **even** natural number, and a negative, $\sqrt[n]{a}$ is not a real number. But if n is an **odd** natural number, then $\sqrt[n]{a}$ is a real number for any real number a.

- For n any natural number, and a, b any real numbers such that $\sqrt[n]{a}$, $\sqrt[n]{b}$ are real numbers, we have $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.

- For n any natural number, and a, b any real numbers such that $b \neq 0$, and $\sqrt[n]{a}$, $\sqrt[n]{b}$ are real numbers, we have $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

- **Caution :** $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$ and $\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$.

Now we can proceed with some examples of operations on radical expressions. A strategy for simplifying radicals is to look for perfect n-th power factors of $\sqrt[n]{\quad}$.

1. $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$. (Here, we treat terms involving $\sqrt{2}$ as like terms. For instance, $3x + 5x = 8x$.)

2. $9\sqrt[3]{5} - 12\sqrt[3]{5} = -3\sqrt[3]{5}$. (We treat terms involving $\sqrt[3]{5}$ as like terms. For instance, $9x - 12x = -3x$.)

3. $4\sqrt[6]{10} - 8\sqrt[6]{10} + 22\sqrt[6]{10} = 18\sqrt[6]{10}$.

4. $-5\sqrt{12} + 7\sqrt{3} - \sqrt{75}$. Here we cannot proceed unless we simplify the individual radical terms.

$$\begin{aligned} -5\sqrt{12} + 7\sqrt{3} - \sqrt{75} &= -5\sqrt{4 \times 3} + 7\sqrt{3} - \sqrt{25 \times 3} \\ &= -5\sqrt{4} \times \sqrt{3} + 7\sqrt{3} - \sqrt{25} \times \sqrt{3} \\ &= -5 \times 2 \times \sqrt{3} + 7\sqrt{3} - 5 \times \sqrt{3} \\ &= -10\sqrt{3} + 7\sqrt{3} - 5\sqrt{3} = -8\sqrt{3}. \end{aligned}$$

5. $3\sqrt[4]{5} + 7\sqrt[4]{9}$. This cannot be simplified further as $\sqrt[4]{5}$ and $\sqrt[4]{9}$ are unlike radical expressions.

6. $3\sqrt[4]{5} - 8\sqrt[3]{5}$. This cannot be simplified further as $\sqrt[4]{5}$ and $\sqrt[3]{5}$ are unlike radical expressions.

7. $7\sqrt[3]{5} - 8\sqrt[3]{\frac{40}{27}} + \frac{\sqrt[3]{320}}{2}$. This needs careful analysis.

$$\begin{aligned}
7\sqrt[3]{5} - 8\sqrt[3]{\frac{40}{27}} + \frac{\sqrt[3]{320}}{2} &= 7\sqrt[3]{5} - 8\frac{\sqrt[3]{40}}{\sqrt[3]{27}} + \frac{\sqrt[3]{320}}{2} \\
&= 7\sqrt[3]{5} - 8\frac{\sqrt[3]{8\times 5}}{\sqrt[3]{27}} + \frac{\sqrt[3]{64\times 5}}{2} \\
&= 7\sqrt[3]{5} - 8\frac{\sqrt[3]{8}\times\sqrt[3]{5}}{\sqrt[3]{27}} + \frac{\sqrt[3]{64}\times\sqrt[3]{5}}{2} \\
&= 7\sqrt[3]{5} - \frac{8}{1}\times\frac{2\sqrt[3]{5}}{3} + \frac{\overset{2}{\cancel{4}}\sqrt[3]{5}}{\underset{1}{\cancel{2}}} \\
&= 7\sqrt[3]{5} - \frac{16\sqrt[3]{5}}{3} + 2\sqrt[3]{5} \\
&= \frac{3}{3}\times 7\sqrt[3]{5} - \frac{16\sqrt[3]{5}}{3} + \frac{3}{3}\times 2\sqrt[3]{5} \qquad \textit{The least common denominator is } 3 \\
&= \frac{21\sqrt[3]{5}}{3} - \frac{16\sqrt[3]{5}}{3} + \frac{6\sqrt[3]{5}}{3} \\
&= \frac{21\sqrt[3]{5} - 16\sqrt[3]{5} + 6\sqrt[3]{5}}{3} \\
&= \frac{11\sqrt[3]{5}}{3}
\end{aligned}$$

8. $3\sqrt[4]{32} - \frac{9}{\sqrt[4]{8}}$. We first simplify the radical expressions, and rationalize the denominator.

$$\begin{aligned}
3\sqrt[4]{32} - \frac{9}{\sqrt[4]{8}} &= 3\sqrt[4]{16\times 2} - \frac{9\times\sqrt[4]{2}}{\sqrt[4]{8}\times\sqrt[4]{2}} \\
&= 3\sqrt[4]{16}\times\sqrt[4]{2} - \frac{9\times\sqrt[4]{2}}{\sqrt[4]{16}} \\
&= 6\sqrt[4]{2} - \frac{9\sqrt[4]{2}}{2} \\
&= \frac{2}{2}\times 6\sqrt[4]{2} - \frac{9\sqrt[4]{2}}{2} \qquad \textit{The least common denominator is } 2 \\
&= \frac{12\sqrt[4]{2}}{2} - \frac{9\sqrt[4]{2}}{2} \\
&= \frac{12\sqrt[4]{2} - 9\sqrt[4]{2}}{2} = \frac{3\sqrt[4]{2}}{2}
\end{aligned}$$

9. $\sqrt[3]{4x} - \dfrac{7x}{\sqrt[3]{16x^2}}$. Again, first rationalize the denominator.

$$\begin{aligned}
\sqrt[3]{4x} - \frac{7x}{\sqrt[3]{16x^2}} &= \sqrt[3]{4x} - \frac{7x \times \sqrt[3]{4x}}{\sqrt[3]{16x^2} \times \sqrt[3]{4x}} \\
&= \sqrt[3]{4x} - \frac{7x\sqrt[3]{4x}}{\sqrt[3]{64x^3}} \\
&= \sqrt[3]{4x} - \frac{7\cancel{x}^{1}\ \sqrt[3]{4x}}{4\cancel{x}^{1}} \\
&= \frac{4}{4} \times \frac{\sqrt[3]{4x}}{1} - \frac{7\sqrt[3]{4x}}{4} \\
&= \frac{4\sqrt[3]{4x}}{4} - \frac{7\sqrt[3]{4x}}{4} \\
&= \frac{4\sqrt[3]{4x} - 7\sqrt[3]{4x}}{4} = -\frac{3\sqrt[3]{4x}}{4}
\end{aligned}$$

10. $-5\sqrt[3]{7} + 8\sqrt[4]{9} + 9\sqrt[3]{7} - 11\sqrt[4]{9}$. We have two distinct kinds of radical expressions. So, combining like radicals,

$$\begin{aligned}
-5\sqrt[3]{7} + 8\sqrt[4]{9} + 9\sqrt[3]{7} - 11\sqrt[4]{9} &= -5\sqrt[3]{7} + 9\sqrt[3]{7} + 8\sqrt[4]{9} - 11\sqrt[4]{9} \\
&= 4\sqrt[3]{7} - 3\sqrt[4]{9}.
\end{aligned}$$

11. $\sqrt[3]{7} \cdot \sqrt[3]{11} = \sqrt[3]{7 \cdot 11} = \sqrt[3]{77}$ which cannot be simplified further.

12. $\sqrt[3]{5} \cdot \sqrt[4]{11}$. This cannot be simplified any further. Note that the cube-root and the fourth root are distinct kinds of roots.

13. $\sqrt[3]{5} \cdot \sqrt[4]{5}$. This cannot be simplified any further. Note that the cube-root and the fourth root are distinct kinds of roots.

14. $\sqrt[3]{4}\left(\sqrt[3]{5} - 2\sqrt[3]{2}\right)$. Multiplication distributes over subtraction. So,

$$\begin{aligned}
\sqrt[3]{4}\left(\sqrt[3]{5} - 2\sqrt[3]{2}\right) &= \sqrt[3]{4} \times \sqrt[3]{5} - \sqrt[3]{4} \times 2\sqrt[3]{2} \\
&= \sqrt[3]{4 \times 5} - 2\sqrt[3]{4 \times 2} \\
&= \sqrt[3]{20} - 2\sqrt[3]{8} \\
&= \sqrt[3]{20} - 2 \times 2 = \sqrt[3]{20} - 4.
\end{aligned}$$

15. $\left(\sqrt{x}-\sqrt{y}\right)^2$. Assume that the variables are non-negative.

$$\begin{aligned}\left(\sqrt{x}-\sqrt{y}\right)^2 &= \left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)\\ &= \sqrt{x}\sqrt{x}-\sqrt{x}\sqrt{y}-\sqrt{y}\sqrt{x}+\sqrt{y}\sqrt{y}\\ &= x-\sqrt{xy}-\sqrt{yx}+y\\ &= x-\sqrt{xy}-\sqrt{xy}+y = x-2\sqrt{xy}+y\end{aligned}$$

16. Note that $(\sqrt{x})^2-\left(\sqrt{y}\right)^2 = x-y$. Thus, $(\sqrt{x})^2-\left(\sqrt{y}\right)^2 \neq \left(\sqrt{x}-\sqrt{y}\right)^2$.

17. $\left(\sqrt{x}+\sqrt{y}\right)^2$. Assume that the variables are non-negative.

$$\begin{aligned}\left(\sqrt{x}+\sqrt{y}\right)^2 &= \left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)\\ &= \sqrt{x}\sqrt{x}+\sqrt{x}\sqrt{y}+\sqrt{y}\sqrt{x}+\sqrt{y}\sqrt{y}\\ &= x+\sqrt{xy}+\sqrt{yx}+y\\ &= x+\sqrt{xy}+\sqrt{xy}+y = x+2\sqrt{xy}+y\end{aligned}$$

18. Note that $(\sqrt{x})^2+\left(\sqrt{y}\right)^2 = x+y$. Thus, $(\sqrt{x})^2+\left(\sqrt{y}\right)^2 \neq \left(\sqrt{x}+\sqrt{y}\right)^2$.

19. $\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)$. Assume that the variables are non-negative. We follow the rules of polynomial multiplication.

$$\begin{aligned}\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right) &= \sqrt{x}\sqrt{x}+\sqrt{x}\sqrt{y}-\sqrt{y}\sqrt{x}-\sqrt{y}\sqrt{y}\\ &= x+\sqrt{xy}-\sqrt{yx}-y\\ &= x+\sqrt{xy}-\sqrt{xy}-y = x-y\end{aligned}$$

20. $\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}-\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)$ Assume that the variables are non-negative.

$$\begin{aligned}&\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}-\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)\\ &= \sqrt[3]{x}\sqrt[3]{x^2}-\sqrt[3]{x}\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{x}\sqrt[3]{y^2}+\sqrt[3]{y}\sqrt[3]{x^2}-\sqrt[3]{y}\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y}\sqrt[3]{y^2}\\ &= \sqrt[3]{x^3}-\sqrt[3]{x^2y}+\sqrt[3]{xy^2}+\sqrt[3]{yx^2}-\sqrt[3]{yxy}+\sqrt[3]{y^3}\\ &= x-\sqrt{x^2y}+\sqrt[3]{xy^2}+\sqrt[3]{x^2y}-\sqrt[3]{xy^2}+y = x+y.\end{aligned}$$

21. $\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)$ Assume that the variables are non-negative.

$$\begin{aligned}&\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)\\ &= \sqrt[3]{x}\sqrt[3]{x^2}+\sqrt[3]{x}\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{x}\sqrt[3]{y^2}-\sqrt[3]{y}\sqrt[3]{x^2}-\sqrt[3]{y}\sqrt[3]{x}\sqrt[3]{y}-\sqrt[3]{y}\sqrt[3]{y^2}\\ &= \sqrt[3]{x^3}+\sqrt[3]{x^2y}+\sqrt[3]{xy^2}-\sqrt[3]{yx^2}-\sqrt[3]{yxy}-\sqrt[3]{y^3}\\ &= x+\sqrt{x^2y}+\sqrt[3]{xy^2}-\sqrt[3]{x^2y}-\sqrt[3]{xy^2}-y = x-y.\end{aligned}$$

The examples 16, 17, and 18 are the radical versions of certain algebraic formulae. Recall and compare the formulae below.

Name	Recall	Compare
Difference of squares	$(a-b)(a+b)=a^2-b^2$	$\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)=x-y$
Sum of cubes	$(a+b)(a^2-ab+b^2)$ $=a^3+b^3$	$\left(\sqrt[3]{x}+\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}-\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)$ $=x+y$
Difference of cubes	$(a-b)(a^2+ab+b^2)$ $=a^3-b^3$	$\left(\sqrt[3]{x}-\sqrt[3]{y}\right)\left(\sqrt[3]{x^2}+\sqrt[3]{x}\sqrt[3]{y}+\sqrt[3]{y^2}\right)$ $=x-y$

These formulae help us to rationalize the denominators of certain complicated radical expressions. Here are some examples:

1.

$$\begin{aligned}\frac{2}{\sqrt{3}-\sqrt{5}} &= \frac{2\times\left(\sqrt{3}+\sqrt{5}\right)}{\left(\sqrt{3}-\sqrt{5}\right)\times\left(\sqrt{3}+\sqrt{5}\right)}\\ &= \frac{2\left(\sqrt{3}+\sqrt{5}\right)}{3-5}\\ &= \frac{\cancelto{1}{2}\left(\sqrt{3}+\sqrt{5}\right)}{-\cancelto{1}{2}} = -\left(\sqrt{3}+\sqrt{5}\right)\end{aligned}$$

2.

$$\begin{aligned}\frac{4}{\sqrt[3]{7}-\sqrt[3]{9}} &= \frac{4\times\left(\sqrt[3]{49}+\sqrt[3]{63}+\sqrt[3]{81}\right)}{\left(\sqrt[3]{7}-\sqrt[3]{9}\right)\times\left(\sqrt[3]{49}+\sqrt[3]{63}+\sqrt[3]{81}\right)}\\ &= \frac{4\left(\sqrt[3]{49}+\sqrt[3]{63}+\sqrt[3]{81}\right)}{7-9}\\ &= \frac{\cancelto{2}{4}\left(\sqrt[3]{49}+\sqrt[3]{63}+\sqrt[3]{81}\right)}{-\cancelto{1}{2}}\\ &= -2\left(\sqrt[3]{49}+\sqrt[3]{63}+\sqrt[3]{81}\right)\end{aligned}$$

3. Here is a general case, which will be relevant when we work with complex numbers:

$$\begin{aligned}\frac{1}{a+b\sqrt{c}} &= \frac{1\times(a-b\sqrt{c})}{(a+b\sqrt{c})\times(a-b\sqrt{c})}\\ &= \frac{a-b\sqrt{c}}{a^2-(b\sqrt{c})^2}\\ &= \frac{a-b\sqrt{c}}{a^2-b^2c}.\end{aligned}$$

Classroom Exercises : Simplify

(a) $\sqrt{5}+11\sqrt{5}$

(b) $-4\sqrt[4]{2}+7\sqrt[4]{2}$

(c) $12\sqrt[5]{2}-7\sqrt[5]{64}$

(d) $\sqrt{32}-\sqrt{60}+\sqrt{162}$

(e) $2\sqrt{45}-5\sqrt{20}+\sqrt{3}$

(f) $3\sqrt{8}+2\sqrt{18}-\sqrt{50}$

(g) $-3\sqrt{12}+2\sqrt{27}-4\sqrt{48}$

(h) $\sqrt{5}(\sqrt{5}+\sqrt{3})$

(i) $\sqrt{2}(\sqrt{8}-\sqrt{2})$

(j) $\sqrt{3}(\sqrt{15}+2\sqrt{3})$

(k) $\sqrt[5]{3}\cdot\sqrt[5]{7}$

(l) $\sqrt[5]{3}\cdot\sqrt[7]{5}$

(m) $\sqrt[4]{3}\left(\sqrt[4]{4}-8\sqrt[4]{5}\right)$

(n) $(\sqrt{3}+\sqrt{5})^2$

(o) $(\sqrt{7}-\sqrt{3})^2$

(p) $\left(\sqrt[3]{5}+\sqrt[4]{7}\right)^2$

(q) $(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})$

(r) $(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})$

(s) $\left(\sqrt{5}+\sqrt{7}\right)\left(\sqrt{5}-\sqrt{7}\right)$

(t) $\dfrac{1}{\sqrt[5]{16}}$

(u) $2\sqrt[5]{2}-3\sqrt[5]{64}+\dfrac{1}{\sqrt[5]{16}}$

(v) $\sqrt[3]{4m}-\dfrac{2m}{\sqrt[3]{2m^2}}$

Rationalize the denominators :

(a) • $\dfrac{3}{\sqrt{5}}$ • $\dfrac{4}{\sqrt{2}}$ • $-\dfrac{3}{\sqrt{11}}$ • $\dfrac{2}{\sqrt[3]{5}}$ • $\dfrac{4}{\sqrt[3]{2}}$

(b) • $\dfrac{1}{3+\sqrt{2}}$ • $\dfrac{4}{3-\sqrt{2}}$ • $-\dfrac{3}{1+\sqrt{11}}$ • $\dfrac{1}{\sqrt{7}+\sqrt{5}}$ • $\dfrac{1}{\sqrt{7}-\sqrt{5}}$

(c) • $\dfrac{1}{1+\sqrt[3]{2}}$ • $\dfrac{4}{1-\sqrt[3]{2}}$ • $-\dfrac{3}{1+\sqrt[3]{11}}$ • $\dfrac{1}{\sqrt[3]{7}-\sqrt[3]{5}}$ • $\dfrac{1}{\sqrt[3]{7}+\sqrt[3]{5}}$

1.4.1 Homework Exercises

Perform the following operations.

(1) $3\sqrt{5}+7\sqrt{5}$ • $4\sqrt{2}-2\sqrt{2}$ • $-6\sqrt{3}+4\sqrt{3}$

(2) $-4\sqrt{7}-8\sqrt{7}$ • $-\sqrt{5}+7\sqrt{5}$ • $-4\sqrt[3]{7}+12\sqrt[3]{7}$

(3) $8\sqrt[10]{5}-9\sqrt[10]{5}+12\sqrt[10]{5}$ • $\sqrt[3]{7}-\sqrt[10]{7}$ • $\sqrt[3]{7}-\sqrt[10]{7}+21\sqrt[3]{7}+16\sqrt[10]{7}$

(4) $3\sqrt{2}-5\sqrt{32}+2\sqrt{162}$ • $-7\sqrt{5}+11\sqrt{20}-12\sqrt{45}$ • $\dfrac{10}{\sqrt{5}}+\sqrt{5}$

(5) $-\dfrac{4}{\sqrt{2}}+9\sqrt{2}$ • $\dfrac{6}{\sqrt{3}}+11\sqrt{3}$ • $\sqrt{2}\cdot\sqrt{18}$

(6) $\sqrt{20}\cdot\sqrt{5}$ • $\sqrt{5}\left(3+\sqrt{7}-8\sqrt{5}\right)$ • $\sqrt{2}\left(\sqrt{8}+2\sqrt{2}-7\sqrt{50}\right)$

(7) $5\sqrt{3}\left(6\sqrt{3}-11\sqrt{12}+9\sqrt{75}\right)$ • $\left(3+\sqrt{3}\right)^2$ $\left(\sqrt{5}-\sqrt{3}\right)^2$

(8) $\left(3+\sqrt{3}\right)\left(9-\sqrt{5}\right)$ • $\left(-2+\sqrt{5}\right)\left(1+\sqrt{7}-\sqrt{5}\right)$ • $\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)$

(9) $\left(2+\sqrt{5}\right)\left(1-2\sqrt{5}+\sqrt{3}\right)$ • $\left(2-\sqrt{5}\right)\left(1+2\sqrt{5}+\sqrt{2}\right)$

Rationalize the denominators

1. $\dfrac{1}{\sqrt{3}}$; $\dfrac{3}{\sqrt{6}}$; $-\dfrac{4}{\sqrt{2}}$; $\dfrac{1}{\sqrt[3]{3}}$; $\dfrac{7}{\sqrt[5]{4}}$.

2. $-\dfrac{4}{\sqrt{12}}$; $\dfrac{3}{\sqrt[3]{4}}$; $\dfrac{3}{2-\sqrt{5}}$; $\dfrac{3}{3+\sqrt{7}}$.

3. $\dfrac{4}{2-\sqrt[3]{7}}$; $\dfrac{4}{2+\sqrt[3]{7}}$.

1.5 Solving Radical Equations

Radical equations are equations involving radical expressions. In this lesson we will learn to solve certain simple radical equations. **Remember to check that your solutions are correct.**

Example 1: Solve $\sqrt{x} = 3$ for x.

$\sqrt{x} = 3$

$(\sqrt{x})^2 = 3^2$ *Square both sides*

$x = 9.$ *This is a possible solution.*

Checking whether the solution is correct:

$\sqrt{9} = 3.$ *We see that $x = 9$ is a solution.*

Example 2: Solve $2\sqrt{3x} = 5$ for x.

$2\sqrt{3x} = 5$

$\sqrt{3x} = \frac{5}{2}$ *Divide both sides by 2;*

$\left(\sqrt{3x}\right)^2 = \left(\frac{5}{2}\right)^2$ *Square both sides;*

$3x = \frac{25}{4}$

$x = \frac{25}{4} \div 3$ *Divide both sides by 3*

$x = \frac{25}{12}.$ *This is a possible solution.*

Checking whether the solution is correct:

$2\sqrt{3 \times \frac{25}{12}} = 2\sqrt{\frac{25}{4}} = 2 \times \frac{5}{2} = 5.$ *We see that $x = \frac{25}{12}$ is a solution.*

Example 3: Solve $2\sqrt{3x - 4} = 5$ for x.

$2\sqrt{3x - 4} = 5$

$\sqrt{3x - 4} = \frac{5}{2}$ *Divide both sides by 2;*

$\left(\sqrt{3x - 4}\right)^2 = \left(\frac{5}{2}\right)^2$ *Square both sides;*

$3x - 4 = \frac{25}{4}$

$3x = \frac{25}{4} + 4$ *Add 4 to both sides;*

$3x = \frac{25}{4} + \frac{16}{4} = \frac{41}{4}$ *Simplifying the right hand side*;

$x = \frac{41}{4} \div 3$ *Divide both sides by* 3;

$x = \frac{41}{12}.$ *This is a possible solution.*

Checking whether the solution is correct:

$2\sqrt{3 \times \frac{41}{12} - 4} = 2\sqrt{\frac{41}{4} - 4} = 2 \times \sqrt{\frac{25}{4}} = 2 \times \frac{5}{2} = 5.$ *We see that* $x = \frac{41}{12}$ *is a solution.*

Example 4: Solve $4\sqrt{3x-2} = -1$ for x.

$4\sqrt{3x-2} = -1$

$\sqrt{3x-2} = -\frac{1}{4}$ *Divide both sides by* 4;

$\left(\sqrt{3x-2}\right)^2 = \left(-\frac{1}{4}\right)^2$ *Square both sides*;

$3x - 2 = \frac{1}{16}$

$3x = \frac{1}{16} + 2$ *Add* 2 *to both sides*;

$3x = \frac{1}{16} + \frac{32}{16} = \frac{33}{16}$ *Simplifying the right hand side*;

$x = \frac{33}{16} \div 3$ *Divide both sides by* 3;

$x = \frac{11}{16}.$ *This is a possible solution.*

Checking whether the solution is correct:

$4\sqrt{3 \times \frac{11}{16} - 2} = 4\sqrt{\frac{33}{16} - 2} = 4 \times \sqrt{\frac{1}{16}} = 4 \times \frac{1}{4} = 1 \neq -1.$ *We see that* $x = \frac{11}{16}$ *is* ***not*** *a solution.*

There is no solution to the given equation.

Example 5: Solve $3\sqrt{2x-5} + 2 = 7$ for x.

$3\sqrt{2x-5} + 5 = 7$

$3\sqrt{2x-5} = 2$ *Subtract* 5 *from both sides*;

$\sqrt{2x-5} = \frac{2}{3}$ *Divide both sides by* 3;

$\left(\sqrt{2x-5}\right)^2 = \left(\frac{2}{3}\right)^2$ *Square both sides;*

$2x - 5 = \frac{4}{9}$

$2x = \frac{4}{9} + 5$ *Add 5 to both sides;*

$2x = \frac{4}{9} + \frac{45}{9} = \frac{49}{9}$ *Simplifying the right hand side;*

$x = \frac{49}{9} \div 2$ *Divide both sides by 2;*

$x = \frac{49}{18}.$ *This is a possible solution.*

Checking whether the solution is correct:

$3\sqrt{2 \times \frac{49}{18} - 5} + 2 = 3\sqrt{\frac{49}{9} - 5} + 2 = 3 \times \sqrt{\frac{4}{9}} + 2$

$= 3 \times \frac{2}{3} + 5 = 2 + 5 = 7.$ *We see that $x = \frac{11}{16}$ is a solution.*

Example 6: Solve $\sqrt{2x+3} = x$ for x.

$\sqrt{2x+3} = x$

$\left(\sqrt{2x+3}\right)^2 = x^2$ *Square both sides;*

$2x + 3 = x^2$

$0 = x^2 - 2x - 3$ *Subtract $2x$ and subtract 3 from both sides;*

$0 = (x-3)(x+1)$ *Factoring the right hand side;*

$(x-3) = 0$ or $(x+1) = 0$ *Solving for x*

$x = 3$ or $x = -1$ *These are possible solutions.*

Checking whether the solutions are correct:

$\sqrt{2 \times 3 + 3} = \sqrt{6+3} = \sqrt{9} = 3$ *So, $x = 3$ is a solution.*

$\sqrt{2 \times (-1) + 3} = \sqrt{-2+3} = 1 \neq -1$ *So, $x = -1$ is not a solution.*

Example 7: Solve $\sqrt{10(x+3)} = x + 3$ for x.

$\sqrt{10(x+3)} = (x+3)$ *Place parentheses around the right hand side;*

$\left(\sqrt{10(x+3)}\right)^2 = (x+3)^2$ *Square both sides;*

$10(x+3) = x^2 + 6x + 9$ *Multiply the right hand side;*

$10x + 30 = x^2 + 6x + 9$

$0 = x^2 + 6x + 9 - 10x - 30$ *Subtract* $10x$ *and subtract* 30 *from both sides*;

$0 = x^2 - 4x - 21$ *Simplify the right hand side;*

$0 = (x-7)(x+3)$ *Factoring the right hand side*;

$(x-7) = 0$ or $(x+3) = 0$ *Solving for* x

$x = 7$ or $x = -3$ *These are possible solutions.*;

Checking whether the solutions are correct:

$\sqrt{10(7+3)} = \sqrt{10 \times 10} = \sqrt{100} = 10 = 7 + 3$ *So,* $x = 7$ *is a solution.*

$\sqrt{10 \times (-3+3)} = \sqrt{10 \times 0} = \sqrt{0} = 0 = -3 + 3$ *So,* $x = -3$ *is a solution.*

Example 8: Solve $\sqrt{x-3} + \sqrt{2x+8} = 0$

$\sqrt{x-3} + \sqrt{2x+8} = 0$

$\sqrt{x-3} = -\sqrt{2x+8}$ *Place the two radical expressions on opposite sides of =;*

$\left(\sqrt{x-3}\right)^2 = \left(-\sqrt{2x+8}\right)^2$ *Square both sides;*

$x + 3 = 2x + 8$

$0 = 2x + 8 - x - 3$ *Subtract* x *and* 3 *from both sides;*

$0 = x + 5$

$-5 = x$ *Subtract* 5 *from both sides.*

Checking whether the solutions are correct:

$\sqrt{-5-3} + \sqrt{2(-5)+8} = \sqrt{-8} + \sqrt{-2}$ *These are not real numbers.*

There is no solution to this problem.

Example 9: Sometimes you will need to square twice. Solve the equation $\sqrt{3x+7} + \sqrt{x+3} = 2$ for x.

$\sqrt{3x+7} + \sqrt{x+3} = 2$ *First isolate one of the radicals;*

$\sqrt{3x+7} = 2 - \sqrt{x+3}$	*Subtract* $\sqrt{x+3}$ *from both sides;*
$\left(\sqrt{3x+7}\right)^2 = \left(2 - \sqrt{x+3}\right)^2$	*Square both sides;*
$3x+7 = 4 - 4\sqrt{x+3} + x + 3$	*Multiplying and simplifying the right hand side;*
$3x+7 = x+7-4\sqrt{x+3}$	*Further simplifying the right hand side;*
$3x+7-x-7 = -4\sqrt{x+3}$	*Subtracting* x *and* 7 *from both sides;*
$2x = -4\sqrt{x+3}$	*Simplifying the left hand side;*
$(2x)^2 = \left(-4\sqrt{x+3}\right)^2$	*Square both sides;*
$4x^2 = 16(x+3)$	
$4x^2 = 16x+48$	
$4x^2-16x-48 = 0$	*Subtracting* $16x$ *and* 48 *from both sides;*
$4(x-6)(x+2) = 0$	*Factoring the left hand side;*
$x = 6$ or $x = -2$	*These are the possible solutions.*

Checking whether the solutions are correct:

$\sqrt{3(6)+7} + \sqrt{6+3} = \sqrt{18+7} + \sqrt{9} = \sqrt{25} + \sqrt{9} = 5+3 = 8 \neq 2.$ $\quad x = 6$ *is not a solution*

$\sqrt{3(-2)+7} + \sqrt{-2+3} = \sqrt{-6+7} + \sqrt{1} = \sqrt{1} + \sqrt{1} = 1+1 = 2.$ $\quad x = -2$ *is a solution*

Example 10: Solve the literal equation $\sqrt{2x^2+7} = y$ for x.

$\sqrt{2x^2+7)} = y$	
$\left(\sqrt{2x^2+7}\right)^2 = y^2$	*Square both sides*;
$2x^2+7 = y^2$	
$2x^2 = y^2-7$	*Subtract* 7 *from both sides*;
$x^2 = \dfrac{y^2-7}{2}$	*Divide both sides by* 2.
$\sqrt{x^2} = \pm\sqrt{\dfrac{y^2-7}{2}}$	*Take square-roots of both sides;*
$x = \pm\sqrt{\dfrac{y^2-7}{2}}$	*Note the two options* $\pm$. *We have solved for the variable* x.

Classroom Exercises: Solve the following equations for x.

1. • $\sqrt{x} = 4$ • $\sqrt{x} = 7$ • $\sqrt{x} = -5$
2. • $\sqrt{x+3} = 3$ • $\sqrt{x-4} = 2$ • $\sqrt{x-2} = -1$
3. • $3\sqrt{x+5} = 2$ • $2\sqrt{x-7} = 1$ • $4\sqrt{x+3} = -7$
4. • $\sqrt{3x+5} - 5 = 3$ • $\sqrt{2x-7} + 3 = 8$ • $\sqrt{5x+1} + 6 = 3$
5. • $\sqrt{3x+16} - 2 = x$ • $\sqrt{3-x} + 3 = x$ • $1 + \sqrt{15-3x} = 2 + x$
6. • $\sqrt{3-2x} - \sqrt{x+7} = 0$ • $\sqrt{2x+6} - \sqrt{x+2} = 0$ • $\sqrt{2x+5} - \sqrt{x+2} = 0$
7. • $\sqrt{3-2x} - \sqrt{x+7} = 1$ • $\sqrt{2x+6} - \sqrt{x+2} = 1$ • $\sqrt{2x+5} - \sqrt{x+2} = 1$

Solve the following literal equations for the indicated variables.

• $c = \sqrt{(a-3)^2 + (b-4)^2}$ for b. • $y = \sqrt{3x^2+4}$ for x.

• $A = \sqrt{\dfrac{B}{C}}$ for C.

1.5.1 Homework Exercises

Solve the following equations for x.

1. • $\sqrt{x} = 8$ • $\sqrt{x} = 5$ • $\sqrt{x} = -3$
2. • $\sqrt{x+2} = 4$ • $\sqrt{x-4} = 6$ • $\sqrt{x+3} = -7$
3. • $5\sqrt{x+2} = 3$ • $4\sqrt{x-3} = 5$ • $3\sqrt{x-5} = -1$
4. • $\sqrt{2x-5} - 3 = 4$ • $\sqrt{5x+2} + 1 = 3$ • $\sqrt{3x-7} + 7 = 2$
5. • $\sqrt{x+7} - 1 = x$ • $3 + \sqrt{37-3x} = x$ • $\sqrt{\dfrac{5x+11}{6}} = x + 1$
6. • $\sqrt{3x-1} + \sqrt{x-4} = 0$ • $\sqrt{2x+3} + \sqrt{x} = 0$ • $\sqrt{3x+4} - \sqrt{x} = 0$
7. • $\sqrt{x-1} + \sqrt{x-4} = 3$ • $\sqrt{x+3} + \sqrt{x} = 3$ • $\sqrt{3x+4} - \sqrt{x} = 2$

Solve the following literal equations for the indicated variables.

• $c = \sqrt{(a-3)^2 + (b-4)^2}$ for a. • $y = \sqrt{3x^2-4}$ for x.

• $A = \sqrt{\dfrac{B}{C}}$ for B.

1.6 Rational Exponents

Recall from subsection 1.2.1 and section 1.3 the notion of n-th root. For $n > 1$ an integer, we say that $\sqrt[n]{b} = a$ if and only if $a^n = b$.

We introduce rational exponents for the ease of computations.

For a real number b and any integer $n > 1$, we set

$$b^{\frac{1}{n}} = \sqrt[n]{b}.$$

Notice that if n is even, then the n-th root of a negative number is a complex number. We therefore assume that b is positive whenever n is even.

Examples:

- $4^{\frac{1}{2}} = \sqrt{4} = 2$
- $64^{\frac{1}{3}} = \sqrt[3]{64} = 4$
- $(-32)^{\frac{1}{5}} = \sqrt[5]{-32} = -2$
- $81^{\frac{1}{4}} = \sqrt[4]{81} = 3.$
- $-4^{\frac{1}{2}} = -\sqrt{4} = -2$
- $-64^{\frac{1}{3}} = -\sqrt[3]{64} = -4$
- $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$
- $-81^{\frac{1}{4}} = -\sqrt[4]{81} = -3.$

Classroom Exercises: Evaluate

(a) $25^{\frac{1}{2}}$ (b) $625^{\frac{1}{4}}$ (c) $343^{\frac{1}{3}}$ (d) $128^{\frac{1}{7}}$

(e) $-25^{\frac{1}{2}}$ (f) $-625^{\frac{1}{4}}$ (g) $(-343)^{\frac{1}{3}}$ (h) $-128^{\frac{1}{7}}$

One immediate consequence of this notation is that for any real number a and integers m, n with $n > 1$, we have

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

Examples:

- $4^{\frac{3}{2}} = \left(\sqrt{4}\right)^3 = 2^3 = 8$
- $64^{\frac{2}{3}} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16$
- $(-32)^{\frac{3}{5}} = \left(\sqrt[5]{-32}\right)^3 = (-2)^3 = -$
- $-4^{\frac{3}{2}} = -\left(\sqrt{4}\right)^3 = -2^3 = -8$
- $-64^{\frac{2}{3}} = -\left(\sqrt[3]{64}\right)^2 = -4^2 = -16$
- $125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2 = 25.$
- $-32^{\frac{2}{5}} = -\left(\sqrt[5]{32}\right)^2 = -2^2 = -4$
- $81^{\frac{3}{4}} = \left(\sqrt[4]{81}\right)^3 = 3^3 = 27$
- $-81^{\frac{2}{4}} = -\left(\sqrt[4]{81}\right)^2 = -3^2 = -9.$

Classroom Exercises: Evaluate

(a) $25^{\frac{3}{2}}$ (b) $625^{\frac{3}{4}}$ (c) $343^{\frac{2}{3}}$ (d) $128^{\frac{2}{7}}$

(e) $-25^{\frac{3}{2}}$ (f) $-625^{\frac{3}{4}}$ (g) $(-343)^{\frac{2}{3}}$ (h) $-128^{\frac{4}{7}}$

We can now refer to the properties described in subsection 1.2.1 and section 1.3 and state them as follows:

For any real numbers a, b and **rational numbers** m, n

$a^m \cdot a^n = a^{m+n}$, $\qquad \dfrac{a^m}{a^n} = a^{m-n}$ *for* $a \neq 0$;

$(ab)^m = a^m b^m$ $\qquad \left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ *for* $b \neq 0$;

$(a^m)^n = a^{mn}$, $a^{-m} = \dfrac{1}{a^m}$ *for* $a \neq 0$.

Examples:

- $3^{\frac{4}{5}} \cdot 3^{\frac{6}{5}} = 3^{\frac{4}{5}+\frac{6}{5}} = 3^{\frac{10}{5}} = 3^2 = 9$
- $4^{\frac{1}{8}} \cdot 4^{\frac{3}{8}} = 4^{\frac{1}{8}+\frac{3}{8}} = 4^{\frac{4}{8}} = 4^{\frac{1}{2}} = \sqrt{4} = 2$
- $x^{\frac{3}{7}} \cdot x^{\frac{5}{7}} = x^{\frac{3}{7}+\frac{5}{7}} = x^{\frac{8}{7}}$
- $y^{\frac{2}{5}} \cdot y^{\frac{3}{4}} = y^{\frac{2}{5}+\frac{3}{4}} = y^{\frac{8}{20}+\frac{15}{20}} = y^{\frac{23}{20}}$
- $\dfrac{32^{\frac{6}{5}}}{32^{\frac{4}{5}}} = 32^{\frac{6}{5}-\frac{4}{5}} = 32^{\frac{2}{5}} = \left(\sqrt[5]{32}\right)^2 = 2^2 = 4$
- $\dfrac{25^{\frac{2}{3}}}{25^{\frac{1}{6}}} = 125^{\frac{2}{3}-\frac{1}{6}} = 25^{\frac{4}{6}-\frac{1}{6}} = 25^{\frac{3}{6}} = 25^{\frac{1}{2}} = \sqrt{25} = 5$
- $\dfrac{x^{\frac{2}{3}}}{x^{\frac{2}{5}}} = x^{\frac{2}{3}-\frac{2}{5}} = x^{\frac{10}{15}-\frac{6}{15}} = x^{\frac{4}{15}}$
- $\dfrac{b^{\frac{5}{6}}}{b^{\frac{1}{12}}} = b^{\frac{5}{6}-\frac{1}{12}} = b^{\frac{10}{12}-\frac{1}{12}} = b^{\frac{9}{12}} = b^{\frac{3}{4}}$
- $(125 \cdot 8)^{\frac{2}{3}} = 125^{\frac{2}{3}} \cdot 8^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 \left(\sqrt[3]{8}\right)^2$

  $= 5^2 \cdot 2^2 = 25 \cdot 4 = 100$
- $(81 \cdot 16)^{\frac{1}{4}} = 81^{\frac{1}{4}} \cdot 16^{\frac{1}{4}} = \sqrt[4]{81} \cdot \sqrt[4]{16} = 3 \cdot 2 = 6$
- $(x \cdot y)^{\frac{3}{7}} = x^{\frac{3}{7}} \cdot y^{\frac{3}{7}}$
- $(p \cdot q)^{\frac{4}{5}} = p^{\frac{4}{5}} \cdot q^{\frac{4}{5}}$
- $\left(\dfrac{125}{8}\right)^{\frac{2}{3}} = \dfrac{125^{\frac{2}{3}}}{8^{\frac{2}{3}}} = \dfrac{\left(\sqrt[3]{125}\right)^2}{\left(\sqrt[3]{8}\right)^2} = \dfrac{5^2}{2^2} = \dfrac{25}{4}$
- $\left(\dfrac{81}{16}\right)^{\frac{1}{4}} = \dfrac{81^{\frac{1}{4}}}{16^{\frac{1}{4}}} = \dfrac{\sqrt[4]{81}}{\sqrt[4]{16}} = \dfrac{3}{2}$
- $\left(\dfrac{x}{y}\right)^{\frac{3}{7}} = \dfrac{x^{\frac{3}{7}}}{y^{\frac{3}{7}}}$
- $\left(\dfrac{p}{q}\right)^{\frac{4}{5}} = \dfrac{p^{\frac{4}{5}}}{q^{\frac{4}{5}}}$
- $\left(125^{\frac{1}{5}}\right)^{\frac{10}{3}} = 125^{\frac{1}{5}\times\frac{10}{3}} = 125^{\frac{2}{3}}$

  $= \left(\sqrt[3]{125}\right)^2 = 5^2 = 25$
- $\left(64^{\frac{3}{4}}\right)^{\frac{2}{3}} = 64^{\frac{3}{4}\times\frac{2}{3}} = 64^{\frac{1}{2}} = \sqrt{64} = 8$
- $\sqrt{16}^{-\frac{1}{4}} = \dfrac{1}{16^{\frac{1}{4}}} = \dfrac{1}{\sqrt[4]{16}} = \dfrac{1}{2}$
- $-27^{-\frac{1}{3}} = \dfrac{-1}{27^{\frac{1}{3}}} = \dfrac{-1}{\sqrt[3]{27}} = \dfrac{-1}{3} = -\dfrac{1}{3}$
- $\left(x^{\frac{2}{3}}\right)^{\frac{4}{5}} = x^{\frac{2}{3}\times\frac{4}{5}} = x^{\frac{8}{15}}$
- $x^{-\frac{3}{7}} = \dfrac{1}{x^{\frac{3}{7}}}$

Classroom Exercises: Evaluate

(a) • $5^{\frac{3}{4}} \cdot 5^{\frac{5}{4}}$ • $-36^{\frac{1}{8}} \cdot 36^{\frac{3}{8}}$ • $25^{\frac{1}{3}} \cdot 25^{\frac{1}{6}}$

(b) • $\dfrac{25^{\frac{7}{8}}}{25^{\frac{3}{8}}}$ • $\dfrac{-49^{\frac{21}{10}}}{49^{\frac{3}{5}}}$ • $\dfrac{1^{\frac{12}{23}}}{1^{\frac{34}{45}}}$

(c) • $(49 \cdot 25)^{\frac{1}{2}}$ • $(243 \cdot 32)^{\frac{2}{5}}$ • $(125 \cdot 1000)^{\frac{2}{3}}$

(d) • $\left(\dfrac{27}{8}\right)^{\frac{1}{3}}$ • $-\left(\dfrac{81}{16}\right)^{\frac{3}{4}}$ • $\left(-\dfrac{27}{125}\right)^{\frac{2}{3}}$

(e) • $\left(125^{\frac{3}{4}}\right)^{\frac{8}{9}}$ • $\left(16^{\frac{4}{3}}\right)^{\frac{9}{16}}$ • $\left(1000^{\frac{1}{2}}\right)^{\frac{4}{3}}$

(f) • $27^{-\frac{1}{3}}$ • $-64^{-\frac{1}{3}}$ • $(-125)^{-\frac{2}{3}}$

Classroom Exercises: Simplify and write your results with positive exponents:

(g) • $(x^{\frac{1}{2}})^{\frac{2}{3}} \cdot x$ • $y^{\frac{3}{5}} \cdot (y^{\frac{2}{3}})^{\frac{3}{4}}$ • $\dfrac{z^{\frac{3}{5}}}{z^{\frac{1}{3}} \cdot z^{\frac{1}{2}}}$

(h) • $(4x^2y^{\frac{2}{3}})^{\frac{2}{3}}$ • $(3x^{-\frac{2}{5}}y^{\frac{1}{4}})^{\frac{3}{4}}$ • $\left(\dfrac{3x^4y^{-\frac{1}{2}}}{z^{-3}}\right)^{\frac{2}{3}}$

1.6.1 Homework Exercises

Evaluate

(1) • $81^{\frac{1}{2}}$ • $81^{\frac{1}{4}}$ • $125^{\frac{1}{3}}$ • $32^{\frac{1}{5}}$

(2) • $-81^{\frac{1}{2}}$ • $-81^{\frac{1}{4}}$ • $(-125)^{\frac{1}{3}}$ • $-32^{\frac{1}{5}}$

(3) • $81^{\frac{3}{2}}$ • $81^{\frac{3}{4}}$ • $125^{\frac{2}{3}}$ • $32^{\frac{2}{5}}$

(4) • $-81^{\frac{3}{2}}$ • $-81^{\frac{3}{4}}$ • $(-125)^{\frac{2}{3}}$ • $-32^{\frac{4}{5}}$

Evaluate

(5) • $10^{\frac{1}{3}} \cdot 10^{\frac{8}{3}}$ • $-16^{\frac{2}{10}} \cdot 16^{\frac{3}{10}}$ • $27^{\frac{1}{6}} \cdot 27^{\frac{7}{6}}$

(6) • $\dfrac{16^{\frac{7}{4}}}{16^{\frac{3}{4}}}$ • $\dfrac{-49^{\frac{4}{10}}}{49^{\frac{6}{5}}}$ • $\dfrac{1^{\frac{12}{232}}}{1^{\frac{34}{453}}}$

(7) • $(81 \cdot 25)^{\frac{1}{2}}$ • $(243 \cdot 32)^{\frac{3}{5}}$ • $(125 \cdot 1000)^{\frac{1}{3}}$

(8) • $\left(\dfrac{27}{125}\right)^{\frac{1}{3}}$ • $-\left(\dfrac{81}{16}\right)^{\frac{1}{4}}$ • $\left(-\dfrac{64}{125}\right)^{\frac{2}{3}}$

(9) • $\left(27^{\frac{3}{4}}\right)^{\frac{8}{9}}$ • $\left(1000^{\frac{4}{3}}\right)^{\frac{9}{16}}$ • $\left(1000^{\frac{3}{2}}\right)^{\frac{2}{9}}$

(10) • $8^{-\frac{1}{3}}$ • $-16^{-\frac{1}{4}}$ • $(-27)^{-\frac{2}{3}}$

Simplify and write your results with positive exponents:

(11) • $(x^{\frac{1}{2}})^{\frac{4}{3}} \cdot x$ • $y^{\frac{3}{5}} \cdot (y^{\frac{2}{3}})^{\frac{3}{5}}$ • $\dfrac{z^{\frac{1}{5}}}{z^{\frac{2}{3}} \cdot z^{\frac{1}{2}}}$

(12) • $(4x^3y^{\frac{1}{3}})^{\frac{2}{3}}$ • $(3x^{-\frac{3}{5}}y^{\frac{3}{4}})^{\frac{1}{4}}$ • $\left(\dfrac{3x^2y^{-\frac{1}{3}}}{z^{-5}}\right)^{-\frac{2}{3}}$

1.7 Complex numbers

Recall the argument why $\sqrt{9} = 3$. Note, $3 \times 3 = 9$. Therefore $\sqrt{9} = 3$. Now recall the argument why $\sqrt{-9}$ cannot be a real number. The square of a real number cannot be negative. Therefore, there is no real number which is $\sqrt{-9}$.

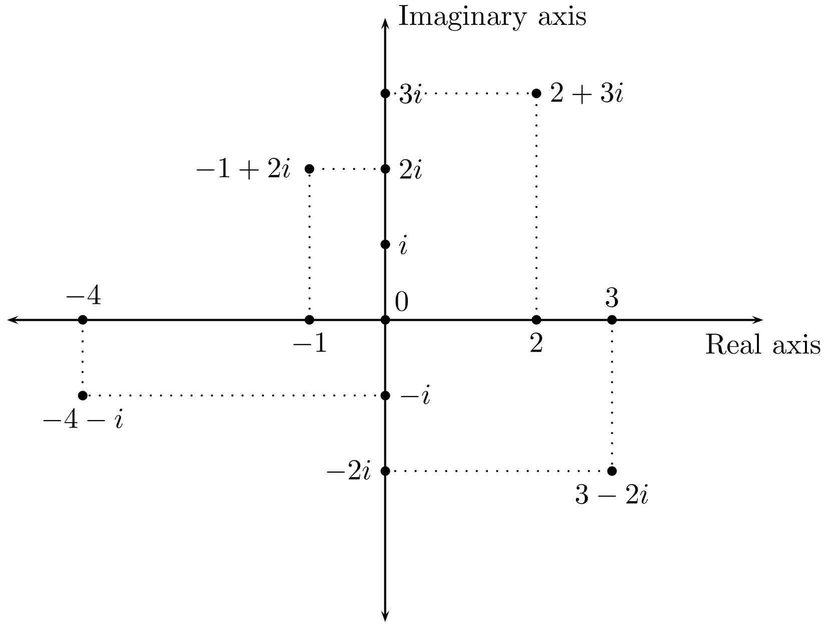
Complex numbers allow us to take square-roots of any real number. We first define

$$i = \sqrt{-1}. \textit{ Therefore, } i^2 = \left(\sqrt{-1}\right)^2 = -1.$$

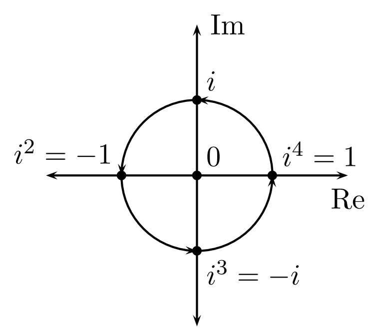
The letter "i" stands for "imaginary."

Every complex number z can be written in the form of $z = a + bi$ where a, b are real numbers. The real number a is called the **real part** of z, and bi is called the **imaginary part** of z. For instance, the real part of the compex number $3 + 4i$ is 3, and its imaginary part is $4i$. Similarly, the real part of the complex number $\left(\frac{1}{2} - \frac{7}{9}i\right)$ is $\frac{1}{2}$ and its imaginary part is $-\frac{7}{9}i$.

Complex numbers are arranged on a plane. The horizontal axis is the real number line and the vertical axis is the imaginary line. These two lines intersect at the complex number $0 = 0 + 0i$. A complex number is then plotted on this plane. The complex number $a + bi$ is positioned at the point (a, b) as in coordinate geometry. Here are some examples :



Power of i	Value
i^1	i
i^2	-1
i^3	$i^2 \times i = -1 \times i = -i$
i^4	$i^2 \times i^2 = -1 \times -1 = 1$



This table allows us to find other powers of i.

- $i^{11} = i^{8+3} = i^8 \times i^3 = 1 \times -i = -i$. Notice that we write $11 = 8 + 3$ because 8 is a multiple of 4. In other words, we separate out as many 4's as possible from the exponent first. Mathematically speaking, we divide 11 by 4, and get a remainder of 3. That is, $11 = 4 \times 2 + 3$. Geometrically, this can be seen by going around 0 **counter-clockwise** starting from 1 in the above diagram, and reading, "$i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1, i^7 = -i$, $i^8 = 1, i^9 = i, i^{10} = -1, i^{11} = -i$."

- $i^{245} = i^{244+1} = i^{244} \times i^1 = 1 \times i = i$. Here, dividing 245 by 4 gives us a remainder of 1. Check: $245 = 4 \times 61 + 1 = 244 + 1$.

- $i^{98} = i^{96+2} = i^{96} \times i^2 = 1 \times -1 = -1$. Again, dividing 98 by 4 gives us a remainder of 2. Check: $98 = 4 \times 24 + 2$.

- $i^{-1} = \dfrac{1}{i} = \dfrac{i^4}{i} = i^3 = -i$. Geometrically, this is obtained by going **clockwise** around 0 starting from 1, and reading, $i^{-1} = -i$.

- $i^{-2} = \dfrac{1}{i^2} = \dfrac{i^4}{i^2} = i^2 = -1$.

- $i^{-3} = \dfrac{1}{i^3} = \dfrac{i^4}{i^3} = i$.

- $i^{-7} = \dfrac{1}{i^7} = \dfrac{1}{i^{4+3}} = \dfrac{1}{i^4 \times i^3} = \dfrac{1}{i^3} = \dfrac{i^4}{i^3} = i$

- $i^{-54} = \dfrac{1}{i^{54}} = \dfrac{1}{i^{52+2}} = \dfrac{1}{i^{52} \times i^2} = \dfrac{1}{1 \times -1} = \dfrac{1}{-1} = -1$. Check: When 54 is divided by 4, the remainder is 2. We get $54 = 4 \times 13 + 2 = 52 + 2$.

Addition and subtraction of complex numbers follow the same rules as radical expressions. This is expected because i is the radical expression $\sqrt{-1}$. Here are some examples:

1. $(2 + 3i) + (5 + 4i) = (2 + 5) + (3i + 4i) = 7 + 7i$.

2. $(2 + 3i) - (5 + 4i) = 2 + 3i - 5 - 4i = (2 - 5) + (3i - 4i) = -3 - i$.

3. $(\sqrt{3} + \sqrt[3]{5}i) + (3 + 5i) = (\sqrt{3} + 3) + (\sqrt[3]{5} + 5)i$

4. $(\sqrt{3} + \sqrt[3]{5}i) - (3 + 5i) = \sqrt{3} + \sqrt[3]{5}i - 3 - 5i = (\sqrt{3} - 3) + (\sqrt[3]{5} - 5)i$

5.

$$\left(\frac{2}{5} - 7i\right) + \left(2 + \frac{1}{3}i\right)$$
$$= \frac{2}{5} - 7i + 2 + \frac{1}{3}i$$
$$= \left(\frac{2}{5} + 2\right) + \left(-7i + \frac{1}{3}i\right) \qquad \textit{Combine the real parts and the imaginary parts}$$
$$= \left(\frac{2}{5} + \frac{2 \times 5}{5}\right) + \left(\frac{-7 \times 3}{3}i + \frac{1}{3}i\right) \qquad \textit{Get common denominators}$$
$$= \left(\frac{2}{5} + \frac{10}{5}\right) + \left(\frac{-21}{3}i + \frac{1}{3}i\right)$$
$$= \left(\frac{2 + 10}{5}\right) + \left(\frac{-21 + 1}{3}i\right) = \frac{12}{5} - \frac{20}{3}i$$

6.

$$\left(\frac{2}{5} - 7i\right) - \left(2 + \frac{1}{3}i\right)$$
$$= \frac{2}{5} - 7i - 2 - \frac{1}{3}i \qquad \textit{Distribute the sign}$$
$$= \left(\frac{2}{5} - 2\right) + \left(-7i - \frac{1}{3}i\right) \qquad \textit{Combine the real parts and the imaginary parts}$$
$$= \left(\frac{2}{5} - \frac{2 \times 5}{5}\right) + \left(\frac{-7 \times 3}{3}i - \frac{1}{3}i\right) \qquad \textit{Get common denominators}$$
$$= \left(\frac{2}{5} - \frac{10}{5}\right) + \left(\frac{-21}{3}i - \frac{1}{3}i\right)$$
$$= \left(\frac{2 - 10}{5}\right) + \left(\frac{-21 - 1}{3}i\right) = -\frac{8}{5} - \frac{22}{3}i$$

Multiplication of complex numbers also follow the same rules as for multiplication of radical expressions. Keep in mind that $i^2 = -1$. Because of this, the product of two complex numbers can be written in the form of $a + bi$, with no higher power of i appearing. Here are some examples:

- $3(4 + 5i) = 12 + 15i$
- $3i(4 - 5i) = 12i - 15\cancel{i^2}^{\times(-1)} = 12i + 15 = 15 + 12i$.
- $\sqrt{-4}\sqrt{-9} = 2i \cdot 3i = 6i^2 = -6$.

- $\sqrt{-5}\sqrt{-11} = \sqrt{5}i \cdot \sqrt{11}i = \sqrt{55}i^2 = -\sqrt{55}.$
- $(2+3i)(5+4i).$

$$\begin{aligned}
(2+3i)(5+4i) &= 2(5+4i) + 3i(5+4i) && \textit{Distribution} \\
&= 10 + 8i + 15i \underbrace{+12i^2}_{-12} \quad (i^2 \to \times(-1)) && i^2 \textit{ is equal to } (-1) \\
&= 10 + 8i + 15i - 12 \\
&= (10-12) + (8i+15i) && \textit{Combining real and imaginary parts} \\
&= -2 + 23i.
\end{aligned}$$

- $(\sqrt{3} + \sqrt[3]{5}i)(3+5i).$

$$\begin{aligned}
(\sqrt{3} + \sqrt[3]{5}i)(3+5i) &= \sqrt{3}(3+5i) + \sqrt[3]{5}i(3+5i) && \textit{Distribute} \\
&= 3\sqrt{3} + 5\sqrt{3}i + 3\sqrt[3]{5}i + 5\sqrt[3]{5}i^2 \quad (i^2 \to \times(-1)) && i^2 \textit{ is equal to } (-1) \\
&= 3\sqrt{3} + 5\sqrt{3}i + 3\sqrt[3]{5}i - 5\sqrt[3]{5} \\
&= (3\sqrt{3} - 5\sqrt[3]{5}) + (5\sqrt{3} + 3\sqrt[3]{5})i && \textit{This cannot be simplified further.}
\end{aligned}$$

- $\left(\frac{2}{5} - 7i\right)\left(2 + \frac{1}{3}i\right)$

$$\begin{aligned}
\left(\frac{2}{5} - 7i\right)\left(2 + \frac{1}{3}i\right) &= \frac{2}{5}\left(2 + \frac{1}{3}i\right) - 7i\left(2 + \frac{1}{3}i\right) && \textit{Distribute} \\
&= \frac{2}{5} \times \frac{2}{1} + \frac{2}{5} \times \frac{1}{3}i - \frac{7}{1} \times \frac{2}{1}i - \frac{7}{1} \times \frac{1}{3}i^2 \quad (i^2 \to \times(-1)) && i^2 \textit{ is equal to } (-1) \\
&= \frac{4}{5} + \frac{2}{15}i - \frac{14}{1}i + \frac{7}{3} \\
&= \left(\frac{4}{5} + \frac{7}{3}\right) + \left(\frac{2}{15} - \frac{14}{1}\right)i && \textit{Collect the real and imaginary parts} \\
&= \left(\frac{4 \times 3}{5 \times 3} + \frac{7 \times 5}{3 \times 5}\right) + \left(\frac{2}{15} - \frac{14 \times 15}{1 \times 15}\right)i && \textit{Get common denominators} \\
&= \left(\frac{12}{15} + \frac{35}{15}\right) + \left(\frac{2}{15} - \frac{210}{1 \times 15}\right)i \\
&= \frac{47}{15} + \frac{-208}{15}i = \frac{47}{15} - \frac{208}{15}i
\end{aligned}$$

Before we can divide complex numbers we need to know the concepts of the **complex conjugate** and the **norm** of a complex number. The **complex conjugate** of the complex number $a + bi$ is $a - bi$. For example

- The complex conjugate of $3 + 4i$ is $3 - 4i$
- The complex conjugate of $5 - 7i$ is $5 - (-7i) = 5 + 7i$.
- The complex conjugate of $\sqrt{11} - \sqrt[3]{7}i$ is $\sqrt{11} + \sqrt[3]{7}i$.
- The complex conjugate of $\frac{3}{4} + \frac{2}{11}i$ is $\frac{3}{4} - \frac{2}{11}i$.
- The complex conjugate of 10 is 10. Note, $10 = 10 + 0i$ and therefore, its complex conjugate is $10 - 0i$ which is 10.
- The complex conjugate of $10i$ is $-10i$. Note, $10i = 0 + 10i$ and therefore, its complex conjugate is $0 - 10i$ which is $-10i$.

The **norm** of the complex number $a + bi$ is $\sqrt{(a + bi)(a - bi)} = \sqrt{a^2 + b^2}$. The norm of $a + bi$ is denoted by $||a + bi||$. The norm is the the distance of the complex number from 0 in the complex plane.

Examples:

- $||3 + 4i|| = \sqrt{(3 + 4i)(3 - 4i)} = \sqrt{9 - 12i + 12i - 16\cancelto{(-1)}{i^2}} = \sqrt{9 + 16} = \sqrt{25} = 5.$
- $||5 - 7i|| = \sqrt{(5 - 7i)(5 + 7i)} = \sqrt{25 + 35i - 35i - 49\cancelto{(-1)}{i^2}} = \sqrt{25 + 49} = \sqrt{74}.$
- $$\begin{aligned} ||\sqrt{11} - \sqrt[3]{7}i|| &= \sqrt{(\sqrt{11} - \sqrt[3]{7}i)(\sqrt{11} + \sqrt[3]{7}i)} \\ &= \sqrt{(\sqrt{11})^2 + \sqrt{11}\sqrt[3]{7}i - \sqrt{11}\sqrt[3]{7}i - (\sqrt[3]{7})^2\cancelto{(-1)}{i^2}} \\ &= \sqrt{11 + \sqrt[3]{49}} \end{aligned}$$
- $$\begin{aligned} ||\frac{3}{4} + \frac{2}{11}i|| &= \sqrt{\left(\frac{3}{4} + \frac{2}{11}i\right)\left(\frac{3}{4} - \frac{2}{11}i\right)} \\ &= \sqrt{\frac{3}{4} \times \frac{3}{4} - \frac{3}{4} \times \frac{2}{11}i + \frac{3}{4} \times \frac{2}{11}i - \frac{2}{11} \times \frac{2}{11}\cancelto{(-1)}{i^2}} \end{aligned}$$

$$= \sqrt{\frac{9}{16} + \frac{4}{121}} = \sqrt{\frac{1089 + 64}{1936}} = \sqrt{\frac{1153}{1936}} = \frac{\sqrt{1153}}{44}$$

- $||10|| = \sqrt{10 \times 10} = \sqrt{100} = 10$
- $||10i|| = \sqrt{10i \times -10i} = \sqrt{-100\overset{(-1)}{i^2}} = \sqrt{100} = 10.$

Before we learn to divide complex numbers, recall how we rationalized the denominator of $\frac{1}{a + b\sqrt{c}}$. Now we are ready to divide complex numbers:

- $3 \div (4 + 5i)$

$$\begin{aligned} 3 \div (4 + 5i) &= \frac{3}{(4 + 5i)} && \textit{Multiply the numerator and denominator} \\ &= \frac{3 \times (4 - 5i)}{(4 + 5i) \times (4 - 5i)} && \textit{by the complex conjugate of the denominator} \\ &= \frac{12 - 15i}{16 - 20i + 20i - 25\overset{(-1)}{i^2}} && \textit{Multiply the complex numbers} \\ &= \frac{12 - 15i}{16 + 25} \\ &= \frac{12 - 15i}{41} = \frac{12}{41} - \frac{15}{41}i \end{aligned}$$

- $3i \div (4 - 5i)$

$$\begin{aligned} 3i \div (4 - 5i) &= \frac{3i}{(4 - 5i)} && \textit{Multiply the numerator and denominator} \\ &= \frac{3i \times (4 + 5i)}{(4 - 5i) \times (4 + 5i)} && \textit{by the complex conjugate of the denominator} \\ &= \frac{12i + 15\overset{(-1)}{i^2}}{16 + 20i - 20i - 25\overset{(-1)}{i^2}} && \textit{Multiply the complex numbers} \\ &= \frac{-15 + 12i}{16 + 25} \\ &= \frac{-15 + 12i}{41} = -\frac{15}{41} + \frac{12}{41}i \end{aligned}$$

- $$\begin{aligned}(2+3i)\div(5+4i) &= \frac{2+3i}{(5+4i)}\\ &= \frac{(2+3i)\times(5-4i)}{(5+4i)\times(5-4i)}\\ &= \frac{10-8i+15i-12i^2}{25-20i+20i-16i^2} \quad (i^2 \to (-1))\\ &= \frac{10+12-8i+15i}{25+16} = \frac{22+7i}{41} = \frac{22}{41}+\frac{7}{41}i\end{aligned}$$

- $(\sqrt{3}+\sqrt[3]{5}i)\div(\sqrt{7}+5i)$

$$= \frac{(\sqrt{3}+\sqrt[3]{5}i)}{(\sqrt{7}+5i)}$$ *Multiply the numerator and denominator*

$$= \frac{(\sqrt{3}+\sqrt[3]{5}i)\times(\sqrt{7}-5i)}{(\sqrt{7}+5i)\times(\sqrt{7}-5i)}$$ *by the complex conjugate of the denominator*

$$= \frac{\sqrt{3}\sqrt{7}-5\sqrt{3}i+\sqrt[3]{5}\sqrt{7}i-5\sqrt[3]{5}i^2}{\sqrt{7}\sqrt{7}-5\sqrt{7}i+5\sqrt{7}i-25i^2} \quad (i^2 \to (-1))$$ *Multiply the complex numbers*

$$= \frac{\sqrt{21}+5\sqrt[3]{5}+(-5\sqrt{3}+\sqrt[3]{5}\sqrt{7})i}{7+25}$$

$$= \frac{\sqrt{21}+5\sqrt[3]{5}+(-5\sqrt{3}+\sqrt[3]{5}\sqrt{7})i}{32}$$

$$= \frac{\sqrt{21}+5\sqrt[3]{5}}{32}+\frac{(-5\sqrt{3}+\sqrt[3]{5}\sqrt{7})}{32}i$$

- $\left(\frac{2}{5}-7i\right)\div\left(2+\frac{1}{3}i\right)$

$$= \frac{\left(\frac{2}{5}-7i\right)}{\left(2+\frac{1}{3}i\right)}$$ *Multiply the numerator and denominator*

$$= \frac{\left(\frac{2}{5}-7i\right)\times\left(2-\frac{1}{3}i\right)}{\left(2+\frac{1}{3}i\right)\times\left(2-\frac{1}{3}i\right)}$$ *by the complex conjugate of the denominator*

$$= \frac{\frac{2}{5}\times\frac{2}{1}-\frac{2}{5}\times\frac{1}{3}i-\frac{7}{1}\times\frac{2}{1}i+\frac{7}{1}\times\frac{1}{3}\cancelto{(-1)}{i^2}}{4-2\times\frac{1}{3}i+2\times\frac{1}{3}i-\frac{1}{3}\times\frac{1}{3}\cancelto{(-1)}{i^2}}$$ *Multiply the complex numbers*

$$= \frac{\frac{4}{5}-\frac{7}{3}+\left(-\frac{2}{15}-\frac{14}{1}\right)i}{4+\frac{1}{9}}$$

$$= \frac{\frac{4\times 3}{5\times 3}-\frac{7\times 5}{3\times 5}+\left(-\frac{2}{15}-\frac{14\times 15}{15}\right)i}{\frac{4\times 9}{9}+\frac{1}{9}}$$

$$= \frac{\frac{12-35}{15}+\left(\frac{-2-210}{15}\right)i}{\frac{36+1}{9}}$$

$$= \frac{-\frac{\cancelto{9}{27}}{\cancelto{5}{15}}-\frac{212}{15}i}{\frac{37}{9}} = \frac{-\frac{9}{5}-\frac{212}{15}i}{\frac{37}{9}}$$

$$= \left(-\frac{9}{5}-\frac{212}{15}i\right)\frac{9}{37} = -\frac{9\times 9}{5\times 37}-\frac{212\times 9}{15\times 37}i$$

$$= -\frac{81}{187}-\frac{1908}{525}i$$

Classroom Exercises:

(a) Write the real and imaginary parts of

- $\frac{1}{3}+\sqrt{3}i;\quad -\sqrt[3]{4}+\frac{2}{7}i;\quad -3-7i;\quad 4-8i;\quad 0;\quad 1;\quad i.$

(b) What is the complex conjugate of each of the following?

- $7;\quad 7i;\quad 12-3i;\quad \frac{2}{7}+\frac{3}{4}i;\quad \sqrt{11}+\sqrt[4]{5}i.$

(c) Write the following in the form $a+bi$.

(i) $i^2;\quad i^3;\quad i^4;\quad i^7;\quad i^{98};\quad i^{-6};\quad i^{-43}.$

(ii) $(3-2i)+(-7+12i);\quad (\sqrt{11}+5i)+(8i-\sqrt[3]{7});\quad \left(\frac{2}{7}+\frac{3}{4}i\right)+\left(-\frac{5}{2}+\frac{7}{12}i\right).$

(iii) $(3-2i)-(-7+12i)$; $(\sqrt{11}+5i)-(8-\sqrt[3]{7}i)$; $\left(\frac{2}{7}+\frac{3}{4}i\right)-\left(-\frac{5}{2}+\frac{7}{12}i\right)$.

(iv) $-5(3-4i)$; $\sqrt{-9}\sqrt{-16}$; $\sqrt{-7}\sqrt{-3}$; $-5i(3-4i)$; $(3-2i)(-7+12i)$; $(\sqrt{11}+5i)(8-\sqrt[3]{7}i)$; $\left(\frac{2}{7}+\frac{3}{4}i\right)\left(-\frac{5}{2}+\frac{7}{12}i\right)$.

(v) $\frac{-5}{(3-4i)}$; $\frac{-5i}{(3-4i)}$; $\frac{(3-2i)}{(-7+12i)}$; $\frac{(\sqrt{11}+5i)}{(8-\sqrt{7}i)}$; $\frac{\left(\frac{2}{7}+\frac{3}{4}i\right)}{\left(-\frac{5}{2}+\frac{7}{12}i\right)}$.

1.7.1 Homework Exercises

Write the real and imaginary parts of

1. $\frac{2}{9}+3i$; $-\sqrt{5}+4i$; $-8-9i$; $9-11i$.
2. 7; $3i$; $-4i$.

What is the complex conjugate of each of the following?

3. -3; $-3i$; $-9+12i$; $\sqrt[4]{3}-8i$; $\frac{2}{5}+\frac{7}{9}i$.

Write the following in the form $a+bi$

4. i^3; i^{33}; i^{333}; i^{-7}; i^{-13}; i^{-220}.
5. $(-9+12i)+(-2-6i)$; $(\sqrt[4]{3}-8i)+(\sqrt[4]{5}+7i)$; $\left(\frac{2}{5}+\frac{7}{9}i\right)+\left(\frac{1}{3}+\frac{5}{12}i\right)$.
6. $(-9+12i)-(-2-6i)$; $(\sqrt[4]{3}-8i)-(\sqrt[4]{5}+7i)$; $\left(\frac{2}{5}+\frac{7}{9}i\right)-\left(\frac{1}{3}+\frac{5}{12}i\right)$.
7. $4(3+12i)$; $\sqrt{-4}\sqrt{-25}$; $\sqrt{-3}\sqrt{-2}$; $4i(-3+12i)$; $(-9+12i)(-2-6i)$; $(\sqrt{3}-8i)(\sqrt{5}+7i)$; $\left(\frac{2}{5}+\frac{7}{9}i\right)\left(\frac{1}{3}+\frac{5}{12}i\right)$.
8. $\frac{4}{(3+12i)}$; $\frac{4i}{(-3+12i)}$; $\frac{(-9+12i)}{(-2-6i)}$; $\frac{(\sqrt{3}-8i)}{(\sqrt{5}+7i)}$; $\frac{\left(\frac{2}{5}+\frac{7}{9}i\right)}{\left(\frac{1}{3}+\frac{5}{12}i\right)}$.

Chapter 2

Quadratic Functions

2.1 Solving Quadratic Equations By Factoring

An equation is a mathematical statement involving an equality (=). A quadratic equation in one variable, x, can be written in the form

$$ax^2 + bx + c = 0 \qquad \textit{for real numbers } a, b, c, a \neq 0.$$

To solve a quadratic equation is to find those numbers which, when substituted for x, satisfy the equation.

In this section, we will solve quadratic equations in which the quadratic polynomial can be factored. Preliminaries on multiplication and factorization of polynomials are given in the appendix of this book.

First, the **Zero product law:**

$$\textit{When } a \cdot b = 0 \quad \textit{then } a = 0 \textit{ or } b = 0.$$

That is, if the product of two numbers is zero, then one or the other has to be zero. Notice that, this is true only for zero. For instance, if $a \cdot b = 6$, then it does not mean that $a = 6$ or $b = 6$. Indeed, $2 \cdot 3 = 6$. Thus, zero alone has this property.

Using the Zero product law, we proceed.

Examples :

- $x(x-1) = 0$

  $x = 0$ or $x - 1 = 0$ *(Zero product law.)*

  $x = 0$ or $x = 1$ *(Solve the two linear equations.)*

- $$\begin{aligned} x(x-1) &= 6 && \textit{(We need a zero on one side of the equation.)} \\ x^2 - x &= 6 && \textit{(Subtract 6 to get a zero.)} \\ x^2 - x - 6 &= 0 && \textit{(Factor the left.)} \\ (x-3)(x+2) &= 0 && \\ x - 3 = 0 \quad &\text{or} \quad x + 2 = 0 && \textit{(Zero product law)} \\ x = 3 \quad &\text{or} \quad x = -2 && \textit{(Solve the two linear equations.)} \end{aligned}$$

- $$\begin{aligned} 3x(2x-1) &= -2(x-1) && \textit{(We need a zero.)} \\ 3x(2x-1) + 2(x-1) &= 0 && \textit{(Adding } 2(x-1) \textit{ to get a zero.)} \\ 6x^2 - 3x + 2x - 2 &= 0 && \textit{(Simplify the left.)} \\ 6x^2 - x - 2 &= 0 && \textit{(Now factor the left .)} \\ (2x+1)(3x-2) &= 0 && \\ 2x + 1 = 0 \quad &\text{or} \quad 3x - 2 = 0 && \textit{(Zero product law)} \\ 2x = -1 \quad &\text{or} \quad 3x = 2 && \\ x = -\frac{1}{2} \quad &\text{or} \quad x = \frac{2}{3} && \end{aligned}$$

- $$\begin{aligned} 4x^3 &= \frac{15x^2(1-x)}{(3x-1)} && \textit{(First cross-multiply.)} \\ 4x^3(3x-1) &= 15x^2(1-x) && \textit{(Get a zero on the right.)} \\ 4x^3(3x-1) - 15x^2(1-x) &= 0 && \textit{(Simplify the left.)} \\ 12x^4 - 4x^3 - 15x^2 + 15x^3 &= 0 && \\ 12x^4 + 11x^3 - 15x^2 &= 0 && \textit{(GCF = } x^2\textit{.)} \\ x^2(12x^2 + 11x - 15) &= 0 && \textit{(Factor the left.)} \\ x^2(4x-3)(3x+5) &= 0 && \\ x^2 = 0, \quad 4x - 3 = 0, \quad &\text{or} \quad 3x + 5 = 0 && \end{aligned}$$

$x = 0,$ or $4x = 3,$ or $3x = -5$

$x = 0,$ or $x = \frac{3}{4},$ or $x = -\frac{5}{3}$

$x = 0,$ or $x = \frac{3}{4},$ or $x = -1\frac{2}{3}$

- $8x^4 - 22x^2 + 15 = 0$

  $8x^4 - 12x^2 - 10x^2 + 15 = 0$ *Using ac-method, with* $8 \times 15 = 120$;

  $4x^2(2x^2 - 3) - 5(2x^2 - 3) = 0$

  $(2x^2 - 3)(4x^2 - 5) = 0$ *The left hand side is factored*;

  $2x^2 - 3 = 0$ or $4x^2 - 5 = 0$ *The Zero property*;

  $2x^2 = 3$ or $4x^2 = 5$

  $x^2 = \frac{3}{2}$ or $x^2 = \frac{5}{2}$

  $x = \pm\sqrt{\frac{3}{2}}$ or $x = \pm\sqrt{\frac{5}{2}}$ *Notice the two options* $\pm$;

  $x = \pm\frac{\sqrt{6}}{2}$ or $x = \pm\frac{\sqrt{10}}{2}$ *Rationalize the denominator.*

- $x - 2\sqrt{x} - 15 = 0$

  $(\sqrt{x})^2 - 2\sqrt{x} - 15 = 0$

$(\sqrt{x} - 5)(\sqrt{x} + 2) = 0$ *Factoring the left hand side*;

$\sqrt{x} - 5 = 0$ or $\sqrt{x} + 2 = 0$ *The Zero property*;

$\sqrt{x} = 5$ or $\sqrt{x} = -2$

$x = 25$ or $x = 4$ *Squaring both sides. Check which of these are solutions*;

$x = 25$ *is a solution to this equation, but* $x = 4$ *is not a solution.*

Classroom Exercises : Solve the following equations.

(a) $x^2 + 3x = 0$ (b) $x^2 + 3x = 10$ (c) $6x(x + 2) = x - 4$

(d) $12x = \frac{x - 3}{x - 1}$ (e) $x^4 = \frac{15x^3(x - 1)}{(4x - 1)}$ (f) $x^4 - 13x^2 = -36$

(g) $12x^4 + 5x^2 = 2$ (h) $2x - 9\sqrt{x} = 5$

2.1.1 Homework Exercises

Solve the following equations:

(1) • $x^2 + 7x = 0$ • $x^2 + 7x = -12$ • $x(2x + 25) = 14(x - 1)$

(2) • $2x(x + 2) = 15(x - 1)$ • $6x = \dfrac{-(x + 12)}{(x - 3)}$ • $x^6 = \dfrac{2x^5(2x - 1)}{3(x - 1)}$

(3) • $x^4 - 11x^2 = -28$ • $6x^4 + 13x^2 = 5$ • $x - 7\sqrt{x} = 18$

2.2 Completing the square and the quadratic formula

A polynomial of degree 2 is called, a **quadratic polynomial**. A **quadratic equation** is an equation which can be written in the form of $P = 0$ for a quadratic polynomial P. In our course we will be concerned with **quadratic equations with real coefficients in one variable**. Any quadratic equation with real coefficients in one variable, x, can be written in the form

$$ax^2 + bx + c = 0 \quad \text{for } a, b, c \in \mathbb{R}, a \neq 0.$$

Examples of quadratic equations with real coefficients are

$$2x^2 + 3x + 4 = 0, \quad -3x^2 + 4x - 5 = 0, \quad \frac{1}{4}x^2 - \frac{2}{3}x + 7 = 0,$$
$$\sqrt{3}x^2 - \frac{1}{5}x + \sqrt{2} = 0, \quad 3x^2 + 4x = 0, \quad 3x^2 + 2 = 0, \cdots .$$

We derive the complete squares formulae:

$$\begin{aligned}(x + h)^2 &= (x + h)(x + h)\\ &= x^2 + hx + hx + h^2\\ &= x^2 + 2hx + h^2.\end{aligned}$$

Likewise,

$$\begin{aligned}(x - h)^2 &= (x - h)(x - h)\\ &= x^2 - hx - hx + h^2\\ &= x^2 - 2hx + h^2.\end{aligned}$$

Every quadratic equation can be written in terms of a **complete square** as follows.

$$a(x - h)^2 = t \quad \text{for } a, h, t \in \mathbb{R}, a \neq 0.$$

The process of converting any quadratic polynomial in one variable to a polynomial in the form of $a(x - h)^2 - t$ complete square is called **completing the square**.

Example 1:

$$x^2 + 4x + 4 = (x+2)^2. \quad \textit{(this is just the complete square formula).}$$

Example 2:

$$\begin{aligned} x^2 + 4x &= \underbrace{x^2 + 4x + 4} - 4 \\ &= (x+2)^2 - 4. \end{aligned}$$

Example 3:

$$\begin{aligned} x^2 + 4x + 7 &= \underbrace{x^2 + 4x + 4} - 4 + 7 \\ &= (x+2)^2 + 3. \end{aligned}$$

Example 4:

$$\begin{aligned} x^2 - 8x &= \underbrace{x^2 - 8x + (-4)^2} - (4)^2 \\ &= \quad (x-4)^2 - 16. \end{aligned}$$

By now the reader must have recognized the pattern.
Example 5:

$$\begin{aligned} x^2 + bx &= \underbrace{x^2 + bx + \left(\frac{b}{2}\right)^2} - \left(\frac{b}{2}\right)^2 \\ &= \quad \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 \end{aligned}$$

Now consider the general (that is of the form $ax^2 + \cdots$) quadratic polynomial:
Example 6:

$$\begin{aligned} 2x^2 + 5x &= 2\left(x^2 + \frac{5}{2}x\right) \\ &= 2\left(\underbrace{x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2} - \left(\frac{5}{4}\right)^2\right) \quad \left(\textit{since } \frac{5}{2} \div 2 = \frac{5}{2} \times \frac{1}{2} = \frac{5}{4}\right); \\ &= \quad 2\left(\left(x + \frac{5}{4}\right)^2 - \frac{25}{16}\right) \end{aligned}$$

$$= 2\left(x + \frac{5}{4}\right)^2 - 2 \times \frac{25}{16} \quad (\textit{distribute multiplication by } 2)$$
$$= 2\left(x + \frac{5}{4}\right)^2 - \frac{25}{8}.$$

In general, we see the following:
Example 7:

$$\begin{aligned}
ax^2 + bx &= a\left(x^2 + \frac{b}{a}x\right) \\
&= a\left(\underbrace{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2} - \left(\frac{b}{2a}\right)^2\right) \quad \left(\textit{since } \frac{b}{a} \div 2 = \frac{b}{a} \times \frac{1}{2} = \frac{b}{2a}\right); \\
&= \quad a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) \\
&= a\left(x + \frac{b}{2a}\right)^2 - a \times \frac{b^2}{4a^2} \quad (\textit{distribute the multiplication by } a) \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a}.
\end{aligned}$$

Example 8:

$$\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(\underbrace{x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2} - \left(\frac{b}{2a}\right)^2\right) + c \\
&= \quad a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right) + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - a \times \frac{b^2}{4a^2} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \left(\underbrace{\frac{b^2}{4a} - c}\right) \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}
\end{aligned}$$

Classroom Exercises : Complete the following as squares:

(a) $x^2 - 6x$ (b) $x^2 + 8x$

(c) $x^2 + 7x$ (d) $x^2 + 9x$

(e) $3x^2 + 6x + 4$ (f) $2x^2 - 4x$

(g) $2x^2 - 7x + 3$ (h) $3x^2 + 7x - 4$

Once we know to complete a quadratic polynomial to a square, we can solve a quadratic equation. To solve a quadartic equation is to find the value of x which satisfies the equation.

Let us consider some of the examples considered above.

Example 1:

$$
\begin{aligned}
x^2 + 4x + 4 &= 0 \\
(x+2)^2 &= 0 \quad (\textit{completing of the square}) \\
(x+2) &= \sqrt{0} = 0 \quad (\textit{taking square-roots of both sides}) \\
x &= -2 \quad (\textit{subtract } 2 \textit{ from both sides}).
\end{aligned}
$$

Example 3: Here we get **complex solutions**.

$$
\begin{aligned}
x^2 + 4x + 7 &= 0 \\
(x+2)^2 + 3 &= 0 \quad (\textit{completing of the square}) \\
(x+2)^2 &= -3 \quad (\textit{subtract } 3 \textit{ from both sides}) \\
(x+2) &= \pm\sqrt{-3} = \pm\sqrt{3}i \quad (\textit{taking square-roots of both sides}) \\
x &= -2 \pm \sqrt{3}i \quad (\textit{subtract } 2 \textit{ from both sides}).
\end{aligned}
$$

Example 6:

$$
\begin{aligned}
2x^2 + 5x &= 0 \\
2\left(x + \frac{5}{4}\right)^2 - \frac{25}{8} &= 0 \quad (\textit{completing the square}) \\
2\left(x + \frac{5}{4}\right)^2 &= \frac{25}{8} \quad (\textit{add } \frac{25}{8} \textit{ to both sides}) \\
\left(x + \frac{5}{4}\right)^2 &= \frac{25}{8} \div 2 = \frac{25}{8} \times \frac{1}{2} \quad (\textit{divide both sides by } 2) \\
\left(x + \frac{5}{4}\right)^2 &= \frac{25}{16} \\
\left(x + \frac{5}{4}\right) &= \pm\sqrt{\frac{25}{16}}
\end{aligned}
$$

$$
\begin{aligned}
x+\frac{5}{4} &= \pm\frac{5}{4} \\
x &= -\frac{5}{4}\pm\frac{5}{4} \\
x &= -\frac{5}{4}+\frac{5}{4} \text{ or } x=-\frac{5}{4}-\frac{5}{4} \\
x &= 0 \text{ or } x=-\frac{10}{4} \\
x &= 0 \text{ or } x=-\frac{5}{2}.
\end{aligned}
$$

A more general example:

$$
\begin{aligned}
3x^2-7x+4 &= 0 \\
3\left(x^2-\frac{7}{3}\right)+4 &= 0 \\
3\left(\left(x-\frac{7}{6}\right)^2-\frac{49}{36}\right)+4 &= 0 \quad (\textit{check for yourself, the completing of square}) \\
3\left(x-\frac{7}{6}\right)^2-3\times\frac{49}{36}+4 &= 0 \quad (\textit{distribute multiplication by } 3) \\
3\left(x-\frac{7}{6}\right)^2-\underbrace{\frac{49}{12}+4} &= 0 \\
3\left(x-\frac{7}{6}\right)^2-\frac{49}{12}+\frac{48}{12} &= 0 \\
3\left(x-\frac{7}{6}\right)^2-\frac{1}{12} &= 0 \\
3\left(x-\frac{7}{6}\right)^2 &= \frac{1}{12} \quad (\textit{add } \frac{1}{12} \textit{ to both sides}) \\
\left(x-\frac{7}{6}\right)^2 &= \frac{1}{12}\div 3=\frac{1}{12}\times\frac{1}{3} \\
\left(x-\frac{7}{6}\right)^2 &= \frac{1}{36} \\
\left(x-\frac{7}{6}\right) &= \pm\sqrt{\frac{1}{36}} \\
\left(x-\frac{7}{6}\right) &= \pm\frac{1}{6}
\end{aligned}
$$

$$x = \frac{7}{6} \pm \frac{1}{6}$$
$$x = \frac{7}{6} + \frac{1}{6} \ \textit{or}\ x = \frac{7}{6} - \frac{1}{6}$$
$$x = \frac{8}{6} \ \textit{or}\ x = \frac{6}{6}$$
$$x = \frac{4}{3} \ \textit{or}\ x = 1.$$

Classroom Exercises : Solve by completing the squares:

(a) $x^2 - 6x = 0$ (b) $x^2 + 8x = 0$

(c) $x^2 + 7x = 0$ (d) $x^2 + 9x = 0$

(e) $3x^2 + 6x - 4 = 0$ (f) $2x^2 - 4x = 0$

(g) $2x^2 - 7x + 3 = 0$ (h) $3x^2 + 7x = 0$

(i) $x^2 + 9x + 4 = 0$ (j) $2x^2 - 5x + 8 = 0$

We now come to the **quadratic formula**. Consider the general quadratic equation

$$\begin{aligned}
ax^2 + bx + c &= 0 \quad (\textit{ for } a,b,c \in \mathbb{R}, a \neq 0)\\
a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} &= 0 \quad (\textit{completing the square})\\
a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \quad \left(\textit{add } \frac{b^2 - 4ac}{4a} \textit{ to both sides}\right)\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \div a = \frac{b^2 - 4ac}{4a} \times \frac{1}{a} \quad (\textit{divide both sides by } a)\\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2}\\
\left(x + \frac{b}{2a}\right) &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}\\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}\\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Classroom Exercses :
Solve by using the quadratic formula:

(a) $x^2 - 7x = 0$ (here, $a = 1, b = -7, c = 0$ and now substitute in the quadratic formula).

(b) $x^2 + 9x - 3 = 0$.

(c) $2x^2 - 5x + 8 = 0$.

(d) $x^2 - 8 = 0$.

(e) $x^2 + 5x = 0$.

(f) $x^2 - 3x + 4 = 0$.

(g) $3x^2 + 5x - 3 = 0$.

(h) $4x^2 - 7x - 8 = 0$.

2.2.1 Homework Exercises

1. Solve by completing the square:

   (a) $x^2 - 8x + 5 = 0$

   (b) $x^2 + 9x - 4 = 0$

   (c) $3x^2 - x + 5 = 0$

   (d) $2x^2 + 3x = 0$

2. Solve by using the quadratic formula:

   (a) $x^2 - 8x + 5 = 0$

   (b) $x^2 + 9x - 4 = 0$

   (c) $3x^2 - x + 5 = 0$

   (d) $2x^2 + 3x = 0$

2.3 Introduction to Parabolas

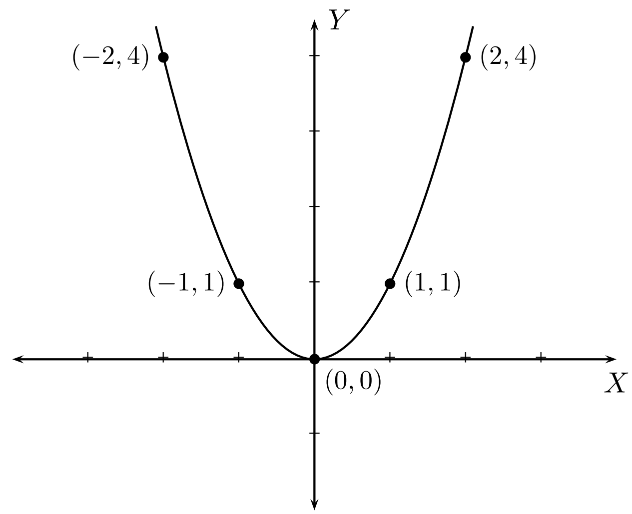
In this section we learn to graph quadratic equations. In other words, we want to plot points (x, y) on the coordinate plane which satisfy the equation

$$y = ax^2 + bx + c \quad \text{for } a, b, c \in \mathbb{R}, a \neq 0.$$

The real number a is called the **leading coefficient** of the quadratic polynomial $ax^2 + bx + c$. We start with the simplest of quadratic equations.

This is the graph of $y = x^2$.

x	$y = x^2$
0	0
-1	1
-2	4
1	1
2	4



The graph of $y = x^2$ is a **parabola**. The point $(0,0)$ is its **vertex** and this parabola **opens up**. Notice that the parabola is initially **decreasing** and then changes its direction and becomes **increasing**. The number 0 is the **critical number** when the parabola changes its direction. The critical number is the x-coordinate of the vertex for a parabola. The precise definition of a critical number requires knowledge of calculus which is beyond the scope of this course. But we will develop an intuitive idea of the critical number for a parabola by the following examples.

y	Critical number	y	Critical number
$y = (x-3)^2$	$x = 3$	$y = (x+5)^2$	$x = -5$
$y = \left(x - \frac{1}{2}\right)^2$	$x = \frac{1}{2}$	$y = \left(x + \frac{2}{3}\right)^2$	$x = -\frac{2}{3}$
$y = (x - \sqrt{3})^2 + 5$	$x = \sqrt{3}$	$y = (x+4)^2 - 6$	$x = -4$
$y = 2\left(x - \frac{1}{5}\right)^2 - 7$	$x = \frac{1}{5}$	$y = -3\left(x + \frac{2}{5}\right)^2 + 125$	$x = -\frac{2}{5}$

The reader may have noticed the significance of completing of squares for finding the critical number.

Example : Consider $y = 2x^2 + 3x - 4$.

$$\begin{aligned}
y &= 2x^2 + 3x - 4 \\
&= 2\left(x^2 + \frac{3}{2}x\right) - 4 \\
&= 2\left(x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right) - 4 \quad (\textit{Completing the square}) \\
&= 2\left(x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{9}{16}\right) - 4 \\
&= 2\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) - 2 \times \frac{9}{16} - 4 \quad (\textit{distribute multiplication by } 2) \\
&= 2\left(x + \frac{3}{4}\right)^2 - \frac{9}{8} - 4 \\
&= 2\left(x + \frac{3}{4}\right)^2 - \frac{41}{8}.
\end{aligned}$$

Hence, the critical number is $x = -\frac{3}{4}$.

y	Critical number	y	Critical number
$y = 3x^2 + 4x - 1$	$x = -\frac{4}{2 \times 3} = -\frac{2}{3}$	$y = -2x^2 + 5x + 12$	$x = -\frac{5}{2 \times -2} = +\frac{5}{4}$
$y = 4x^2 + \frac{2}{3}x - 1$	$x = -\frac{2}{3} \div (2 \times 4) = -\frac{1}{12}$	$y = -2x^2 + 7$	$x = -\frac{0}{2 \times -2} = 0$

In general, the critical number of

$$y = ax^2 + bx + c \text{ is given by } x = -\frac{b}{2a}.$$

Classroom Exercises : Find the critical number of

(a) $y = x^2 + 5x$

(b) $y = 2x^2 - 7$

(c) $y = -3x^2$

(d) $y = -2(x + 4)^2$

(e) $y = (x - \sqrt{5})^2$

(f) $y = 2(x+1)^2 - 17$

(g) $y = 2x^2 - 3x + 5$

(h) $y = 3x^2 + \frac{1}{2}x + 7$

(i) $y = \frac{2}{3}x^2 - \frac{3}{5}x + 5$

Since the critical number is the x-coordinate of the **vertex**, we can then find the vertex by calculating the y-coordinate. The y-coordinate of the vertex is obtained by substituting the critical number for x in the expression for y. The sign of the leading coefficient tells us whether the parabola will **open up** or **open down**. Recall that the leading coefficient of $ax^2 + bx + c$ or of $a(x-h)^2 + k$ is a. If $a > 0$, then the parabola opens up. If $a < 0$, then the parabola opens down. The following table gives examples.

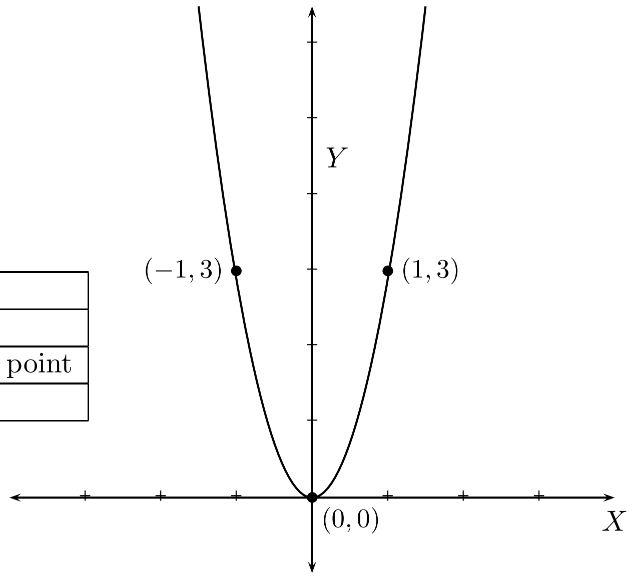
$y = a(x-h)^2 + k$ $y = ax^2 + bx + c$	Critical number h	y-coordinate k	Vertex (h, k)	Parabola opens
$y = 3x^2$	0	$3(0)^2 = 0$	$(0, 0)$	up $(3 > 0)$
$y = -4x^2$	0	$-4(0)^2 = 0$	$(0, 0)$	down $(-4 < 0)$
$y = 2(x-3)^2$	3	$2(3-3)^2 = 0$	$(3, 0)$	up $(2 > 0)$
$y = -5(x+1)^2$	-1	$-5(-1+1)^2 = 0$	$(-1, 0)$	down $(-5 < 0)$
$y = 2(x-\sqrt{3})^2 + 1$	$\sqrt{3}$	$2(\sqrt{3}-\sqrt{3})^2 + 1 = 1$	$(\sqrt{3}, 1)$	up $(2 > 0)$
$y = \left(x - \frac{1}{2}\right)^2 - 4$	$\frac{1}{2}$	$y = \left(\frac{1}{2} - \frac{1}{2}\right)^2 - 4 = -4$	$\left(\frac{1}{2}, -4\right)$	up $(1 > 0)$
$y = 3x^2 + 4x + 5$	$-\frac{4}{2 \times 3} = -\frac{2}{3}$	$3\left(-\frac{2}{3}\right)^2 + 4\left(-\frac{2}{3}\right) + 5$ $= 3\frac{2}{3}$	$\left(-\frac{2}{3}, 3\frac{2}{3}\right)$	up $(3 > 0)$
$y = -2x^2 + 5x - 3$	$-\frac{5}{2 \times -2} = \frac{5}{4}$	$-2\left(\frac{5}{4}\right)^2 + 5\left(\frac{5}{4}\right) - 3$ $= \frac{1}{8}$	$\left(\frac{5}{4}, \frac{1}{8}\right)$	down $(-2 < 0)$

To graph the parabola, it is convenient to find the vertex, and two points which are **equidistant** from the vertex on the parabola. These two points will then be symmetric about the **axis of symmetry**. To find these two points, we take two numbers on either side of the critical number which are both equidistant from the critical number. Then substitute these values to find the required points. Lastly, we can see the **range** as those y-values attained by the quadratic function. We graph the quadratic equations listed above.

This is the graph of $y = 3x^2$.

x	$y = 3x^2$	Description of point
-1	$3(-1)^2 = 3$	Left of vertex
0	0	Vertex and x,y-intercept point
1	$3(1)^2 = 3$	Right of vertex

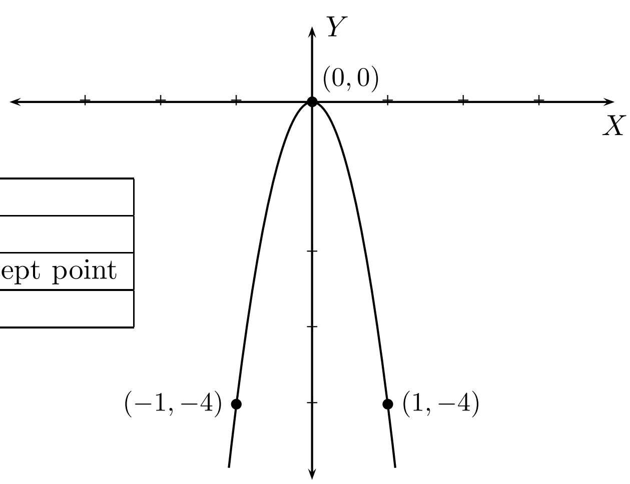
Axis of symmetry is the line
$x = 0$ (the y-axis).
The x-intercept is 0, and the y-intercept is 0.
The range is $[0, \infty)$.



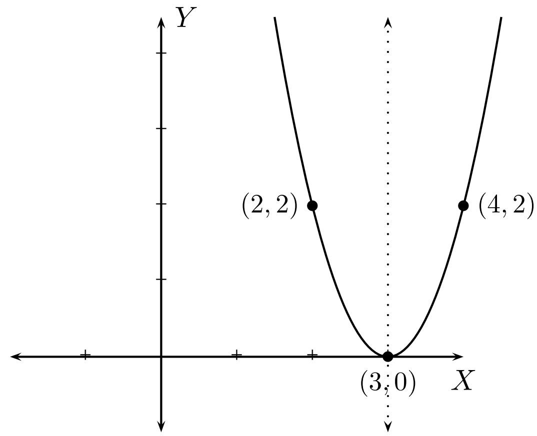
This is the graph of $y = -4x^2$.

x	$y = -4x^2$	Description of point
-1	$-4(-1)^2 = -4$	Left of vertex
0	0	Vertex and x,y-intercept point
1	$-4(1)^2 = -4$	Right of vertex

Axis of symmetry is the line
$x = 0$ (the y-axis).
The x-intercept is 0, and the y-intercept is 0.
The range is $(-\infty, 0]$.



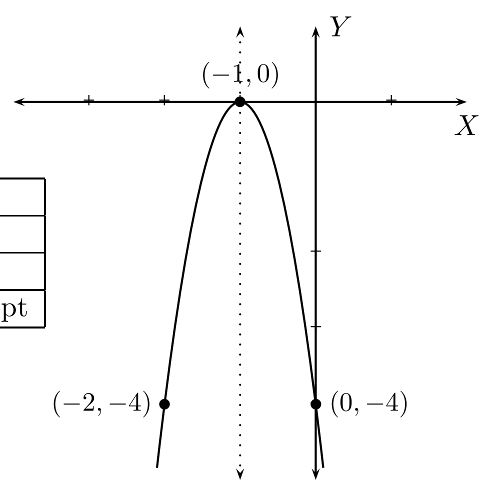
This is the graph of $y = 2(x-3)^2$.



x	$y = 2(x-3)^2$	Description of point
2	$2(2-3)^2 = 2$	Left of vertex
3	0	Vertex and x-intercept point
4	$2(4-3)^2 = 2$	Right of vertex
0	$2(0-3)^2 = 18$	y-intercept point

Axis of symmetry is the line $x = 3$.
The x-intercept is 3, and the y-intercept is 18.
The range is $[0, \infty)$.

This is the graph of $y = -4(x+1)^2$.

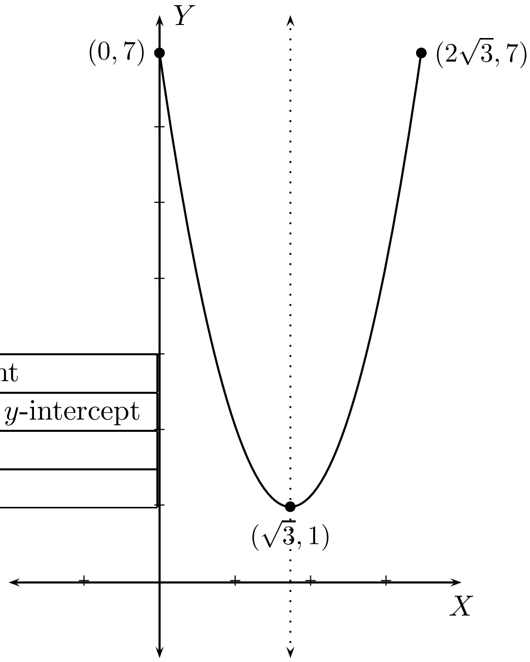


x	$y = -4(x+1)^2$	Description of point
-2	$-4(-2+1)^2 = -4$	Left of vertex
-1	0	Vertex and x-intercept point
0	$-4(0+1)^2 = -4$	Right of vertex and y-intercept

Axis of symmetry is the line $x = -1$.
The x-intercept is -1, and the y-intercept is -4.
The range is $(-\infty, 0]$.

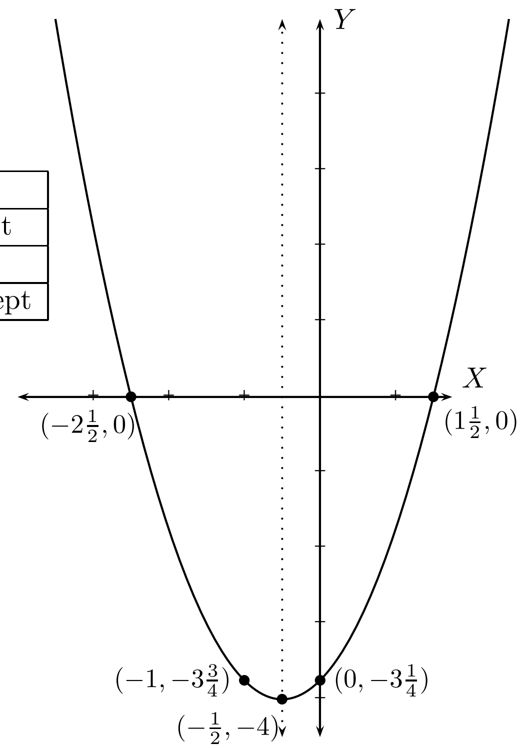
This is the graph of $y = 2(x - \sqrt{3})^2 + 1$.

x	$y = 2(x - \sqrt{3})^2 + 1$	Description of point
0	$2(0 - \sqrt{3})^2 + 1 = 7$	Left of vertex and y-intercept
$\sqrt{3}$	1	Vertex
$2\sqrt{3}$	$2(2\sqrt{3} - \sqrt{3})^2 + 1 = 7$	Right of vertex



Axis of symmetry is the line $x = \sqrt{3}$.
There is no x-intercept, and the y-intercept is 7.
The range is $[1, \infty)$.

This is the graph of $y = \left(x + \frac{1}{2}\right)^2 - 4$.



x	$y = (x + \frac{1}{2})^2 - 4$	Description of point
-1	$(-1 + \frac{1}{2})^2 - 4 = -3\frac{3}{4}$	Left of vertex and y-intercept
$-\frac{1}{2}$	-4	Vertex
0	$3(0 + \frac{1}{2})^2 - 4 = -3\frac{3}{4}$	Right of vertex and y-intercept

Axis of symmetry is the line $x = -\dfrac{1}{2}$.

The y-intercept is $-3\dfrac{3}{4}$.

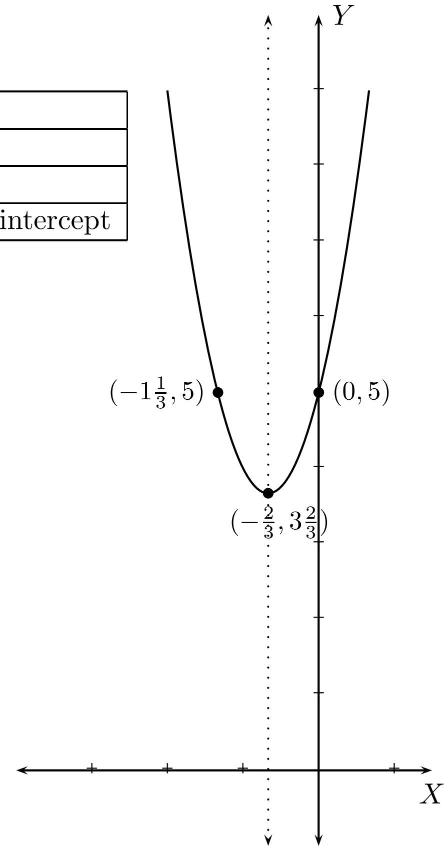
To find the x-intercept, set $y = 0$ and solve for x.

The x-intercepts are $1\dfrac{1}{2}$ and $-2\dfrac{1}{2}$.

The range is $[-4, \infty)$.

This is the graph of $y = 3x^2 + 4x + 5$.

x	$y = 3x^2 + 4x + 5$	Description of point
$-1\frac{1}{3}$	$3(-1\frac{1}{3})^2 + 4(-1\frac{1}{3}) + 5 = 5$	Left of vertex
$-\frac{2}{3}$	$3\frac{2}{3}$	Vertex
0	$3(0)^2 + 4(0) + 5 = 5$	Right of vertex and y-intercept



Axis of symmetry is the line $x = -\dfrac{2}{3}$.

The y-intercept is 5.

There is no x-intercept.

The range is $[3\frac{2}{3}, \infty)$.

This is the graph of $y = -2x^2 + 5x - 3$.
Find the x-intercepts by setting $y = 0$ and solving for x.

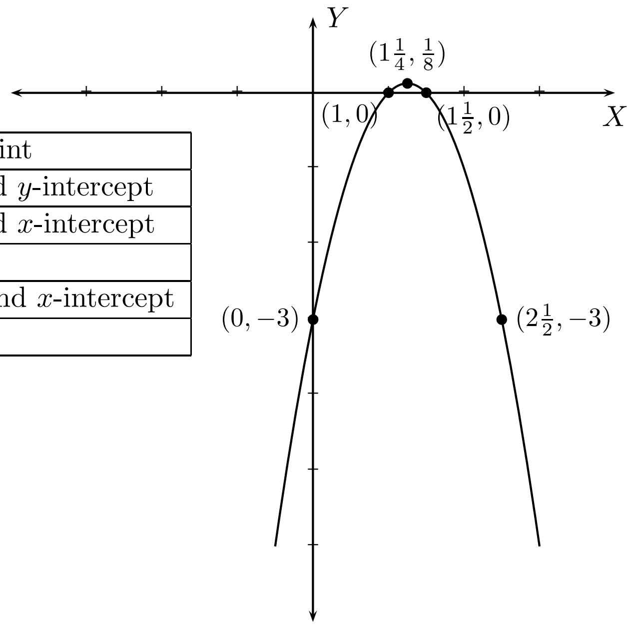
x	$y = -2x^2 + 5x - 3$	Description of point
0	-3	Left of vertex and y-intercept
1	0	Left of vertex and x-intercept
$1\frac{1}{4}$	$\frac{1}{8}$	Vertex
$1\frac{1}{2}$	0	Right of vertex and x-intercept
$2\frac{1}{2}$	-3	Right of vertex

Axis of symmetry is the line $x = 1\dfrac{1}{4}$.

The x-intercepts are 1 and $1\dfrac{1}{2}$

The y-intercept is -3.

The range is $(-\infty, \frac{1}{8}]$.



Classroom Exercises: For each of the following quadratic equations find the critical number, vertex, x, y-intercepts, if any, axis of symmetry, and two points on the parabola symmetric about the axis of symmetry. Tell whether the parabola opens up or down, and explain why. Graph the parabola and find the range.

(a) $y = -x^2$

(b) $y = 3(x - 1)^2$

(c) $y = -2(x + 1)^2$

(d) $y = (x - 3)^2 + 2$

(e) $y = -3\left(x + \dfrac{1}{2}\right)^2 + 4$

(f) $y = x^2 + 6x$

(g) $y = x^2 + 6$

(h) $y = x^2 - 4x$

(i) $y = x^2 - 4$

(j) $y = x^2 + 5x + 4$

(k) $y = x^2 + 5x$

(l) $y = x^2 - 7x$

(m) $y = 2x^2 - 4x + 5$

(n) $y = 3x^2 - 6x + 5$

(o) $y = 3x^2 + 2x + 2$

2.3.1 Homework Exercises

For each of the following quadratic equations find the critical number, vertex, x, y-intercepts, if any, axis of symmetry, and two points on the parabola symmetric about the axis of symmetry. Tell whether the parabola opens up or down, and explain why. Graph the parabola and find the range.

1. $y = -3x^2$
2. $y = 2(x - 1)^2$
3. $y = -3(x + 2)^2$
4. $y = (x - 2)^2 + 1$
5. $y = -2\left(x + \frac{1}{3}\right)^2 + 8$
6. $y = x^2 + 8x$
7. $y = x^2 - 6x$
8. $y = x^2 + 8$
9. $y = x^2 - 6$
10. $y = 3x^2 + 6x - 2$
11. $y = 2x^2 - 12x + 4$
12. $y = x^2 + 3x - 4$
13. $y = 3x^2 + 3x + 1$

2.4 Solving Word Problems Using Quadratic Equations

In this section we use techniques of quadratic equations to address certain word problems.
Examples :

- The difference of two numbers is 11, while their product is -30. What are the two numbers?

  Solution. *Let the smaller number be x.*
  Then, the bigger number is $(x+11)$
  Their product is -30.
  Thus,

  $$\begin{aligned} x(x+11) &= -30; \\ x^2 + 11x &= -30; \\ x^2 + 11x + 30 &= 0; \\ (x+5)(x+6) &= 0. \end{aligned}$$

  By the Zero Product law, we have

  $$\begin{aligned} x+5 = 0 \ \textit{or} \ x+6 &= 0; \\ x = -5 \ \textit{or} \ x &= -6. \end{aligned}$$

  Since the bigger number is 11 more than the smaller number, when the smaller number is -5, the bigger number is 6; and, when the smaller number is -6, the bigger number is 5.

- The sum of the squares of two consecutive even integers is 340. Find the two integers.

  Solution. *Let the smaller even integer be x.*
  Then, the bigger even integer is $(x+2)$. (The next even integer will be two more than the previous one).
  The sum of their squares is 340.
  That is,

  $$\begin{aligned} x^2 + (x+2)^2 &= 340; \\ x^2 + x^2 + 2x + 4 &= 340; \\ 2x^2 + 4x + 4 &= 340; \\ 2x^2 + 4x + 4 - 340 &= 0; \\ 2x^2 + 4x - 336 &= 0; \\ 2(x^2 + 2x - 168) &= 0; \quad \textit{(Divide both sides by 2)} \\ x^2 + 2x - 168 &= 0; \\ (x+14)(x-12) &= 0 \end{aligned}$$

By the Zero Product Law, we have

$$x + 14 = 0 \text{ or } x - 12 = 0;$$
$$x = -14 \text{ or } x = 12.$$

Since the bigger even integer is two more than the smaller integer, when the smaller integer is -14*, the bigger one is* -12*; and, when the smaller integer is* 12*, the bigger one is* 14*.*

- The sum of the squares of two consecutive odd integers is 394. Find the two integers.

  Solution. *Let the smaller odd integer be* x*.*
  Then, the bigger odd integer is $(x + 2)$*. (The next odd integer will be two more than the previous one).*
  The sum of their squares is 394*.*
  That is,

$$\begin{aligned} x^2 + (x + 2)^2 &= 394; \\ x^2 + x^2 + 2x + 4 &= 394; \\ 2x^2 + 4x + 4 &= 394; \\ 2x^2 + 4x + 4 - 394 &= 0; \\ 2x^2 + 4x - 390 &= 0; \end{aligned}$$

$$\begin{aligned} 2(x^2 + 2x - 195) &= 0; \quad \textit{(Divide both sides by 2)} \\ x^2 + 2x - 195 &= 0; \\ (x + 15)(x - 13) &= 0 \end{aligned}$$

  By the Zero Product Law, we have

$$x + 15 = 0 \text{ or } x - 13 = 0;$$
$$x = -15 \text{ or } x = 13.$$

  Since the bigger odd integer is two more than the smaller integer, when the smaller integer is -15*, the bigger one is* -13*; and, when the smaller integer is* 13*, the bigger one is* 15*.*

- The height of a triangle is 3 inches more than the base. Find the height and the base, if the area of the triangle is 44 square inches.

  Solution. *Let the base of the triangle be* x *inches.*
  Then, the height of the triangle is $(x + 3)$ *inches.*

The area of the triangle is 44 *square inches.*
Therefore,

$$\begin{aligned}\frac{x(x+3)}{2} &= 44; \quad \textit{Multiply both sides by } 2\\ x(x+3) &= 2\times 44;\\ x^2+3x &= 88;\\ x^2+3x-88 &= 0;\\ (x+11)(x-8) &= 0.\end{aligned}$$

Hence, by the Zero Product Law, we have

$$\begin{aligned} x+11=0 \ \textit{or}\ x-8=0;\\ x=-11 \ \textit{or}\ x=8.\end{aligned}$$

Note that x denotes the base, and hence cannot be negative. Thus, the base is 8 *inches, and the height is* 11 *inches (since the height is* 3 *inches more than the base).*

- The length of a rectangle is twice its width. Find the length and the width if the area of the rectangle is 128 square centimeters.

  Solution. *Let the width of a rectangle be x cm.*
  Then the length is $2x$ cm.
  The area of the rectangle is 128 *sq. cm.*
  Therefore,

$$\begin{aligned} 2x\times x &= 128;\\ 2x^2 &= 128; \quad \textit{Divide both sides by } 2;\\ x^2 &= 64; \quad \textit{Take square roots}\\ x &= \pm 8.\end{aligned}$$

  Note that x represents the width of the rectangle, and therefore cannot be a negative number. Hence, the width is 8 *cm while the length is* 16 *cm (since the length is twice the width).*

Classroom Exercises :

(a) The difference of two numbers is 8, while their product is −15. What are the two numbers?

(b) The sum of the squares of two consecutive even integers is 244. Find the two integers.

(c) The sum of the squares of two consecutive odd integers is 290. Find the two integers.

(d) The height of a triangle is 3 inches more than the base. Find the height and the base, if the area of the triangle is 35 square inches.

(e) The length of a rectangle is twice its width. Find the length and the width if the area of the rectangle is 98 square centimeters.

2.4.1 Homework Exercises

State clearly what your variables stand for. Provide detailed work.

1. The difference of two numbers is 13, while their product is -42. What are the two numbers?

2. The sum of the squares of two consecutive even integers is 100. Find the two integers.

3. The sum of the squares of two consecutive odd integers is 202. Find the two integers.

4. The height of a triangle is 3 inches more than the base. Find the height and the base, if the area of the triangle is 20 square inches.

5. The length of a rectangle is twice its width. Find the length and the width if the area of the rectangle is 72 square centimeters.

Chapter 3

Rational expressions

3.1 Introduction

Recall that a rational number is a fraction of integers where the denominator is nonzero. A **rational expression** is a fraction of polynomials where the denominator is nonzero. An expression of the form $\dfrac{P(x)}{Q(x)}$ where $P(x), Q(x)$ are polynomials with $Q(x)$ a nonzero polynomial is a rational expression in one variable x. One may similarly define rational expressions in several variables.

We say that $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function. We present some examples of rational functions and examples of functions which are not rational.

Rational functions	Not rational functions
$f(x) = \dfrac{3x+4}{x^2+3x-5}$	$a(x) = \dfrac{\lvert 2x+3\rvert}{4x^2+5x}$. Note that $\lvert 2x+3\rvert$ is not a polynomial function.
$g(x) = \dfrac{10x^{20}-\sqrt{7}x^8+11}{3x^7+5x^6-\sqrt{3}}$	$b(x) = \dfrac{10x^{20}-7\sqrt{x}+11}{3x^7+5x^6-\sqrt{3}}$. Note that $\sqrt{x}$ cannot be a term in a polynomial.
$h(x) = \dfrac{4x^2-5x+6}{3}$	$c(x) = \dfrac{\sqrt{4x^2-5x}}{3}$. Note that $\sqrt{4x^2-5x}$ is not a polynomial.
$k(x) = \dfrac{4}{7x^2+11x-8}$	$d(x) = \dfrac{4}{7x^2+11x^{\frac{3}{2}}-8}$. Note, $x^{\frac{3}{2}}$ cannot be a term in a polynomial.

Classroom Exercises: State which of the following are rational functions. Explain why the others are not rational functions.

$$f(x) = \frac{3x^2-5x+6}{7x^{100}+8x^{45}-\sqrt{3}x+1}; \quad g(x) = \frac{3x^2-5\sqrt{x}+6}{7x^{100}+8x^{45}-3x+1}; \quad h(x) = \frac{3x^2-5x+6}{7x^{100}+\lvert 8x^{45}-3x+1\rvert}; \quad k(x) = \frac{3}{2}.$$

Given a rational function $f(x)$ and a real number a, $f(a)$ denotes the result when $f(x)$ is evaluated at $x = a$.

Example: Let $f(x) = \dfrac{3x^2-5}{4x+3}$. Find $f(0), f(1), f(-2)$, and (-10).

$$f(0) = \frac{3(0)^2-5}{4(0)+3} = \frac{-5}{3} = -\frac{5}{3}.$$
$$f(1) = \frac{3(1)^2-5}{4(1)+3} = \frac{-2}{7} = -\frac{2}{7}.$$
$$f(-2) = \frac{3(-2)^2-5}{4(-2)+3} = \frac{7}{-5} = -\frac{7}{5}.$$
$$f(-10) = \frac{3(-10)^2-5}{4(-10)+3} = \frac{295}{-37} = -7\frac{36}{37}.$$

Classroom Exercises: Let $f(x) = \dfrac{4x^2+7x-5}{3x-2}$. Find $f(0), f(2), f(-1)$, and $f(-3)$.

The reader may have noticed that evaluating a rational expression can result in an undefined outcome.

Examples:

• Let $f(x) = \dfrac{4x^2-5x+3}{x-2}$. Note that

$$f(2) = \frac{4(2)^2-5(2)+3}{2-2} = \frac{9}{0} \textit{ is undefined.}$$

• Let $g(x) = \dfrac{3x+5}{2x-3}$. Note

$$g\left(\frac{3}{2}\right) = \frac{3\left(\frac{3}{2}\right)+5}{2\left(\frac{3}{2}\right)-3} = \frac{\frac{9}{2}+5}{3-3} = \frac{\left(\frac{19}{2}\right)}{0} \textit{ is undefined.}$$

• Let $h(x) = \dfrac{4x-3}{x^2-8x-20}$. To find the values for which $h(x)$ is undefined, we set the denominator to 0. That is,

$x^2 - 8x - 20 = 0$ *Set the denominator to* 0;

$(x-10)(x+2) = 0$ *Factor the left hand side*;

$x - 10 = 0$ or $x + 2 = 0$ *Use the Zero property*;

$x = 10$ or $x = -2$ *Solve for x. We have found the values for which $f(x)$ undefined.*

• Let $k(x) = \dfrac{3x-5}{4x^2+8x-5}$. To find the values for which $k(x)$ is undefined, we set the denominator to 0. That is,

$4x^2+8x-5=0$	*Set the denominator to* 0;
$(2x+5)(2x-1)=0$	*Factor the left hand side*;
$2x+5=0$ or $2x-1=0$	*The Zero property*;
$x=-\dfrac{5}{2}$ or $x=\dfrac{1}{2}$	*Solve for x. These are the values for which $k(x)$ is undefined.*

Classroom Exercises:

Find values of x for which the given rational functions are undefined.

(a) $f(x) = \dfrac{4x^2-7x+3}{x-5}$ (b) $g(x) = \dfrac{4x^2-7x+3}{x+7}$ (c) $h(x) = \dfrac{4x^2-7x+3}{x^2}$

(d) $k(x) = \dfrac{4x^2-7x+3}{x^2+8x+7}$ (e) $a(x) = \dfrac{4x^2-7x+3}{6x^2-11x-10}$ (f) $b(x) = \dfrac{4x^2-7x+3}{x^3-9x}$

(g) $c(x) = \dfrac{4x^2-7x+3}{4x^2-9}$ (h) $d(x) = \dfrac{4x^2-7x+3}{2x^2-5}$

The values for which a rational expression is undefined are not to be confused with the values for which the expression is equal to 0. When $f(a)=0$, we say that $x=a$ is a **zero** of $f(x)$.

Example: • Let $f(x) = \dfrac{x-3}{x+2}$.

$$f(-2) = \frac{-2-3}{-2+2} = \frac{-5}{0} \text{ is undefined.}$$

$$f(3) = \frac{3-3}{3+2} = \frac{0}{5} = 0. \text{ Hence } x=3 \text{ is a } \textbf{zero} \text{ of } f(x).$$

• Let $f(x) = \dfrac{x-3}{x-3}$. Note that $f(3) = \dfrac{3-3}{3-3} = \dfrac{0}{0}$ is undefined. In this case, $x=3$ is not a zero of $f(x)$.

Classroom Exercises:

Find the zeroes of each of the following rational functions. Further find those values for which the functions are undefined.

(a) $f(x) = \dfrac{x+5}{x-4}$ (b) $f(x) = \dfrac{2x-3}{3x-4}$ (c) $f(x) = \dfrac{x^2-2x-8}{x^2-4}$

(d) $f(x) = \dfrac{x^2-x-6}{x^2-3x-28}$ (e) $f(x) = \dfrac{6x^2-5x-6}{x^2+3x-10}$

3.1.1 Homework Exercises

(1) State which of the following are rational functions. Explain why the others are not rational functions.

$$f(x) = \frac{4x+5}{8x^3 - \sqrt{5}x^2 + 5x - 1}; \quad g(x) = \frac{4|x|+5}{8x^3 - \sqrt{5}x^2 + 5x - 1}; \quad h(x) = \frac{\sqrt{4x+5}}{8x^3 - \sqrt{5}x^2 + 5x - 1}; \quad k(x) = 0.$$

(2) Let $f(x) = \dfrac{5x^2 - 7x + 3}{2x+5}$. Find $f(0), f(3), f(-2)$, and $f(-1)$.

(3) Find values of x for which the given rational functions are undefined.

(a) $f(x) = \dfrac{5x^2+11x-7}{x-6}$ (b) $g(x) = \dfrac{5x^2+11x-7}{x+3}$ (c) $h(x) = \dfrac{5x^2+11x-7}{x^3}$

(d) $k(x) = \dfrac{5x^2+11x-7}{x^2+4x-96}$ (e) $a(x) = \dfrac{5x^2+11x-7}{6x^2+19x-7}$ (f) $b(x) = \dfrac{5x^2+11x-7}{x^3-4x}$

(g) $c(x) = \dfrac{5x^2+11x-7}{25x^2-16}$ (h) $d(x) = \dfrac{5x^2+11x-7}{3x^2-2}$

(4) Find the zeroes of each of the following rational functions. Further find those values for which the functions are undefined.

(a) $f(x) = \dfrac{x+7}{x+2}$ (b) $f(x) = \dfrac{5x-3}{4x-3}$ (c) $f(x) = \dfrac{x^2-x-12}{x^2-9}$

(d) $f(x) = \dfrac{x^2+3x-18}{x^2-2x-24}$ (e) $f(x) = \dfrac{12x^2+5x-2}{x^2+13x-30}$

3.2 Simplifying Rational Expressions

Rational expressions are simplified the same way as rational numbers are. The process is to reduce by dividing the numerator and denominator of the rational expression by a common factor until 1 is the only common factor.

Examples: Simplify the following rational functions:

- $f(x) = \dfrac{20}{45} = \dfrac{\cancel{5} \times 4}{\cancel{5} \times 9}$ *Factor the numerator and denominator*;
  $= \dfrac{4}{9}$ *Reduced form.*
- $f(x) = \dfrac{x^2 - 3x}{x^2+4x} = \dfrac{\cancel{x}(x-3)}{\cancel{x}(x+4)}$ *Factor the numerator and denominator*;
  $= \dfrac{(x-3)}{(x+4)}$ *Reduced form.*
- $f(x) = \dfrac{4x^2y^3}{12xy^5} = \dfrac{x}{3y^2}$ *Reduced form.*

- $f(x) = \dfrac{x^3+5x^2-84x}{x^3+13x^2+12x} = \dfrac{x(x^2+5x-84)}{x(x^2+13x+12)}$

  $= \dfrac{x(x+12)(x-7)}{x(x+12)(x+1)}$ *Factor the numerator and denominator*;

  $= \dfrac{(x-7)}{(x+1)}$ *Reduced form.*

- $f(x) = \dfrac{8x^2-10x-3}{12x^2+7x+1}$

  $= \dfrac{(2x-3)(4x+1)}{(3x+1)(4x+1)}$ *Factor the numerator and denominator*;

  $= \dfrac{(2x-3)}{(3x+1)}$ *Reduced form.*

- $f(x) = \dfrac{6x^3-2x^2+9x-3}{4x^3+2x^2+6x+3}$

  $= \dfrac{(2x^2+3)(3x-1)}{(2x^2+3)(2x+1)}$ *Factor the numerator and denominator*;

  $= \dfrac{(3x-1)}{(2x+1)}$ *Reduced form.*

- $f(x) = \dfrac{x^2-9}{x^3+27}$

  $= \dfrac{(x+3)(x-3)}{(x+3)(x^2-3x+9)}$ *Factor the numerator and denominator*;

  $= \dfrac{(x-3)}{(x^2-3x+9)}$ *Reduced form.*

- Here is a tricky example: $f(x) = \dfrac{5-x}{x-5} = \dfrac{-(x-5)}{x-5} = -1.$

Classroom Exercises: Simplify each rational expression:

(1) • $\dfrac{35}{75}$ • $\dfrac{6x^2y^3}{15x^5y}$ • $\dfrac{x^2-4}{x^2-2x}$ • $\dfrac{x^2-4}{x^2-4x}$ • $\dfrac{2x^2+3x}{x^3-6x^2}$

(2) • $\dfrac{-32a^2b^3c^4}{48ab^5c^4}$ • $\dfrac{25-x^2}{x^2-5x}$ • $\dfrac{x^3-5x^2+2x-10}{x^3-3x^2+2x-6}$ • $\dfrac{2x^2-x}{6x^2+x-2}$ • $\dfrac{3-x}{x-3}$

(3) • $\dfrac{x^2-2x-15}{x^2-x-12}$ • $\dfrac{x^2-4}{x^3-8}$ • $\dfrac{x^2-4}{x^2-2x}$ • $\dfrac{6x^2+13x-5}{6x^2+19x+10}$

3.2.1 Homework Exercises:

Simplify each rational expression:

(1) • $\dfrac{22}{32}$ • $\dfrac{-11x^3y^4}{-44xy^6}$ • $\dfrac{-21a^3b^4c^8}{14a^4b^8c^2}$ • $\dfrac{x^2-25}{x^2+5x}$ • $\dfrac{2x^3+11x^2+5x}{x^3+3x^2-10x}$

(2) • $\dfrac{x^2-x-56}{x^2+9x+14}$ • $\dfrac{x^3+3x^2+4x+12}{x^3-3x^2+4x-12}$ • $\dfrac{x^3-27}{x^2-9}$ • $\dfrac{7-x}{x-7}$ • $\dfrac{16-x^2}{x^3-4x^2}$

3.3 Multiplying and Dividing Rational Functions

Rational functions or expressions are multiplied or divided the same way rational numbers are, and then reduced. The reader is cautioned here that functions have not been fully defined. In particular, their domains have not been dealt with. We are using the word "functions" for the purposes of familiarity.

Examples: Find the products of the following rational functions:

- Let $f = \dfrac{3x^2y^3z^9}{13a^5bc^8}$ and $g = \dfrac{26a^3b^2c}{9x^5yz^{10}}$. Find $(f \cdot g)$.

$$\begin{aligned}(f \cdot g) &= \frac{3x^2y^3z^9}{13a^5bc^8} \times \frac{26a^3b^2c}{9x^5yz^{10}} \\ &= \frac{3x^2y^3z^9 \times 26a^3b^2c}{13a^5bc^8 \times 9x^5yz^{10}} && \textit{Multiplying rational expressions;} \\ &= \frac{2x^3y^2b}{3a^2c^7z} && \textit{Reduced form.}\end{aligned}$$

- Let $f(x) = \dfrac{3}{4}$ and $g(x) = \dfrac{12}{15}$. Find $(f \cdot g)(x)$.

$$(f \cdot g)(x) = f(x) \times g(x) = \frac{3}{4} \times \frac{12}{15} = \frac{3 \times 12}{4 \times 15} = \frac{\cancel{3}^{1} \times \cancel{12}^{3}}{\cancel{4}^{1} \times \cancel{15}^{5}} = \frac{3}{5}$$

- Let $f(x) = \dfrac{x^2 - 4}{x^2 - 7x + 12}$ and $g(x) = \dfrac{x^2 + 4x - 21}{x^2 + 7x + 10}$. Find $(f \cdot g)(x)$.

$$\begin{aligned}(f \cdot g)(x) &= f(x) \times g(x) \\ &= \frac{x^2 - 4}{x^2 - 7x + 12} \times \frac{x^2 + 4x - 21}{x^2 + 7x + 10} \\ &= \frac{(x^2 - 4) \times (x^2 + 4x - 21)}{(x^2 - 7x + 12) \times (x^2 + 7x + 10)} && \textit{Multiplying rational expressions;} \\ &= \frac{(x+2)(x-2) \times (x+7)(x-3)}{(x-4)(x-3) \times (x+2)(x+5)} && \textit{Factor the numerator and denominator;} \\ &= \frac{(x-2)(x+7)}{(x-4)(x+5)} && \textit{Reduce by common factors;} \\ &= \frac{x^2 + 5x - 14}{x^2 + x - 20} && \textit{The final answer.}\end{aligned}$$

- Let $f(x) = \dfrac{3x^2 + x - 4}{x^2 + x - 6}$ and $g(x) = \dfrac{x^2 - 4}{2x^2 + x - 3}$. Find $(f \cdot g)(x)$.

$$\begin{aligned}
(f \cdot g)(x) &= f(x) \times g(x) \\
&= \frac{3x^2+x-4}{x^2+x-6} \times \frac{x^2-4}{2x^2+x-3} \\
&= \frac{(3x^2+x-4)\times(x^2-4)}{(x^2+x-6)\times(2x^2+x-3)} && \textit{Multiplying rational expressions;} \\
&= \frac{(3x+4)(x-1)\times(x+2)(x-2)}{(x+3)(x-2)\times(x-1)(2x+3)} && \textit{Factor the numerator and denominator;} \\
&= \frac{(3x+4)(x-2)}{(x+3)(2x+3)} && \textit{Reduce by common factors;} \\
&= \frac{3x^2-2x-8}{2x^2+9x+0} && \textit{The final answer.}
\end{aligned}$$

- Let $f(a) = \dfrac{a^3-8}{a^3+3a^2+4a+12}$ and $g(a) = \dfrac{a^2-a-12}{a^2-4}$. Find $(f \cdot g)(a)$.

$$\begin{aligned}
(f \cdot g)(a) &= f(a) \times g(a) \\
&= \frac{a^3-8}{a^3+3a^2+4a+12} \times \frac{a^2-a-12}{a^2-4} \\
&= \frac{(a^3-8)\times(a^2-a-12)}{(a^3+3a^2+4a+12)\times(a^2-4)} && \textit{Multiplying rational expressions;} \\
&= \frac{(a-2)(a^2+2a+4)\times(a-4)(a+3)}{(a+3)(a^2+4)\times(a-2)(a+2)} && \textit{Factor the numerator and denominator;} \\
&= \frac{(a^2+2a+4)(a-4)}{(a^2+4)(a+2)} && \textit{Reduce by common factors;} \\
&= \frac{a^3-2a^2-4a-16}{a^3+2a^2+4a+8} && \textit{The final answer.}
\end{aligned}$$

Division of rational expressions or functions follows the same rules as division of rational numbers.

Examples:

- Let $f(x) = \dfrac{4}{5}$ and $g(x) = \dfrac{12}{25}$. Find $f(x) \div g(x)$.

$$f(x) \div g(x) = \frac{4}{5} \div \frac{12}{25} = \frac{4}{5} \times \frac{25}{12} = \frac{4 \times 25}{5 \times 12} = \frac{5}{3}.$$

- Let $f = -\dfrac{3x^2y^3}{4ab}$ and $g = \dfrac{2xy^5}{5a^2b^2}$. Find $f \div g$.

$$f \div g = -\frac{3x^2y^3}{4ab} \div \frac{2xy^5}{5a^2b^2} = -\frac{3x^2y^3}{4ab} \times \frac{5a^2b^2}{2xy^5} = -\frac{3x^2y^3 \times 5a^2b^2}{4ab \times 2xy^5} = -\frac{15xab}{8y^2}.$$

- Let $f(x) = \dfrac{x^2+2x-15}{x^2-12x+35}$ and $g(x) = \dfrac{9-x^2}{x^2-5x-14}$. Find $(f \div g)(x)$.

$$(f \div g)(x) = f(x) \div g(x) = \frac{x^2+2x-15}{x^2-12x+35} \div \frac{9-x^2}{x^2-5x-14}$$

$$
\begin{aligned}
&= \frac{x^2+2x-15}{x^2-12x+35} \times \frac{x^2-5x-14}{9-x^2} \\
&= \frac{(x^2+2x-15)\times(x^2-5x-14)}{(x^2-12x+35)\times(9-x^2)} \\
&= \frac{(x+5)(x-3)\times(x-7)(x+2)}{(x-7)(x-5)\times(3-x)(3+x)} \\
&= \frac{-(x+5)(3-x)\times(x-7)(x+2)}{(x-7)(x-5)\times(3-x)(3+x)} \qquad \textit{Note, } (x-3)=-(3-x); \\
&= \frac{-(x+5)(x+2)}{(x-5)(3+x)} \\
&= -\frac{x^2+7x+10}{x^2-2x-15}.
\end{aligned}
$$

- Let $f(x,y) = \dfrac{x^3-7x^2+5x-35}{x^2+xy-6y^2}$ and $g(x,y) = \dfrac{x^2-7x}{x^2+3xy-10y^2}$. Find $(f \div g)(x,y)$.

$$
\begin{aligned}
(f\div g)(x,y) = f(x,y)\div g(x,y) \quad &= \frac{x^3-7x^2+5x-35}{x^2+xy-6y^2} \div \frac{x^2-7x}{x^2+3xy-10y^2} \\
&= \frac{x^3-7x^2+5x-35}{x^2+xy-6y^2} \times \frac{x^2+3xy-10y^2}{x^2-7x} \\
&= \frac{(x^3-7x^2+5x-35)\times(x^2+3xy-10y^2)}{(x^2+xy-6y^2)\times(x^2-7x)} \\
&= \frac{(x^2+5)(x-7)\times(x+5y)(x-2y)}{(x+3y)(x-2y)\times x(x-7)} \\
&= \frac{(x^2+5)(x+5y)}{(x+3y)x} \\
&= \frac{x^3+5x^2y+5x+25y}{x^2+3xy}.
\end{aligned}
$$

- Let $f(x,y) = \dfrac{x^2-9}{2x^2+xy-y^2}$ and $g(x,y) = \dfrac{x^3+27}{3x^2+7xy+4y^2}$. Find $(f \div g)(x,y)$.

$$
\begin{aligned}
(f\div g)(x,y) = f(x,y)\div g(x,y) \quad &= \frac{x^2-9}{2x^2+xy-y^2} \div \frac{x^3+27}{3x^2+7xy+4y^2} \\
&= \frac{x^2-9}{2x^2+xy-y^2} \times \frac{3x^2+7xy+4y^2}{x^3+27} \\
&= \frac{(x^2-9)\times(3x^2+7xy+4y^2)}{(2x^2+xy+y^2)\times(x^3+27)} \\
&= \frac{(x+3)(x-3)\times(3x+4y)(x+y)}{(2x-y)(x+y)\times(x+3)(x^2-3x+9)} \\
&= \frac{(x-3)(3x+4y)}{(2x-y)(x^2-3x+9)} \\
&= \frac{3x^2+4xy-9x-12y}{2x^3-3x^2-x^2y+3xy+18x-9y}.
\end{aligned}
$$

- Find $(f \cdot g)(1)$ and $(f \div g)(1)$ when $f(x) = \dfrac{3x^2+3x-1}{4x+5}$ and $g(x) = \dfrac{4x-1}{x^2+x+5}$.

  $(f \cdot g)(1) = f(1) \cdot g(1) = \dfrac{3(1)^2+3(1)-1}{4(1)+5} \cdot \dfrac{4(1)-1}{(1)^2+(1)+5} = \dfrac{5}{9} \cdot \dfrac{3}{7} = \dfrac{5}{21}$.

  $(f \div g)(1) = f(1) \div g(1) = \dfrac{5}{9} \div \dfrac{3}{7} = \dfrac{5}{9} \times \dfrac{7}{3} = \dfrac{35}{27}$.

Classroom Exercises:

(1) Find

- $\dfrac{18}{25} \times \dfrac{35}{24} \div \dfrac{7}{8}$
- $-\dfrac{20x^2y^3z^4}{27ab^2c^3} \times \dfrac{18a^2bc^5}{25x^4yz^3}$
- $\dfrac{x^2-9}{x+2} \times \dfrac{x^2-4x-12}{x^3-27}$
- $\dfrac{x^3-2x^2+3x-6}{x^2-7x} \times \dfrac{x^2-49}{x^3+5x^2+3x+15}$
- $\dfrac{a-b}{a^2+3ab+2b^2} \times \dfrac{a^2-ab-6b^2}{ab-a^2}$
- $-\dfrac{16}{33} \div \left(-\dfrac{8}{9}\right)$
- $-\dfrac{12xy}{7pq^2r^3} \div \left(-\dfrac{9x^2y^3}{7p^2q^3r}\right)$
- $\dfrac{x^2-4x-21}{x^2-10x+24} \div \dfrac{x^2+11x+24}{x^2+x-20}$
- $\dfrac{x^3-3x^2y}{x-5} \div \dfrac{x^3-x^2y-6xy^2}{x^2-x-20}$
- $\dfrac{4x-4y}{y} \div \dfrac{3y-3x}{x}$

(2) Find $(f \cdot g)(2)$ and $(f \div g)(-1)$ when $f(x) = \dfrac{x^2-4x-21}{x^2-10x+24}$ and $g(x) = \dfrac{x^2+11x+24}{x^2+x-20}$.

(3) Find $(f \cdot g)(0)$ and $(f \div g)(-2)$ when $f(x) = \dfrac{x^3-3x^2}{x-5}$ and $g(x) = \dfrac{x^3+x^2-6x}{x^2+x-20}$

3.3.1 Homework Exercises:

(1) Find

- $\dfrac{12}{35} \times \dfrac{5}{6} \div \dfrac{2}{3}$
- $-\dfrac{14a^2bc^3}{15xyz^5} \times \left(-\dfrac{5x^2y^3z^4}{21abc^5}\right)$
- $\dfrac{x^3+125}{x+3} \times \dfrac{x^2+7x+12}{x^2-25}$
- $\dfrac{x^3-3x^2+5x-15}{x^2-7x+10} \times \dfrac{x^2-25}{x^3+2x^2+5x+10}$
- $\dfrac{xy-y^2}{x^+2xy} \times \dfrac{x-y}{x^2-4xy+4y^2}$
- $-\dfrac{15}{16} \div \dfrac{9}{7}$
- $\dfrac{100qr}{81x^2y^2z^2} \div \left(-\dfrac{5p^2q}{27xyz}\right)$
- $\dfrac{x^2-8x+15}{x^2+6x-16} \div \dfrac{x^2+4x-21}{x^2-2x}$
- $\dfrac{x^3-5x^2y}{x+2} \div \dfrac{x^3-4x^2y-5xy^2}{x^2-x-6}$
- $\dfrac{5p-5q}{p} \div \dfrac{10q-10p}{q}$

(2) Find $(f \cdot g)(1)$ and $(f \div g)(-2)$ when $f(x) = \dfrac{x^2-4x-21}{x^2-10x+24}$ and $g(x) = \dfrac{x^2+11x+24}{x^2+x-20}$.

(3) Find $(f \cdot g)(0)$ and $(f \div g)(-1)$ when $f(x) = \dfrac{x^3-3x^2}{x-5}$ and $g(x) = \dfrac{x^3+x^2-6x}{x^2+x-20}$

3.4 Adding and Subtracting rational functions

Rational functions or expressions are added or subtracted the same way as rational numbers. Again, it is important to note that we have not yet fully defined a function, or its domain.

Examples: Find $(f+g)$ and $(f-g)$ in each case.

- $f(x) = \frac{3}{4}$ and $g(x) = \frac{7}{6}$.

  The Least Common Denominator (LCD) is 12.

$$(f+g)(x) = \frac{3}{4} + \frac{7}{6} = \frac{9}{12} + \frac{14}{12} = \frac{23}{12}.$$
$$(f-g)(x) = \frac{3}{4} - \frac{7}{6} = \frac{9}{12} - \frac{14}{12} = -\frac{5}{12}.$$

- $f(x,y) = \frac{3x}{4y^2}$ and $g(x) = \frac{7x}{6y}$.

  Note that the denominators are $4y^2$ and $6y$. We need a **common denominator** preferably (but not necessarily) the **least common denominator** (LCD). We look for that expression which is a multiple of both $4y^2$ and $6y$. We see that the LCD here is $12y^2$, since no smaller degree polynomial or a smaller coefficient would give us a common multiple. Now we add or subtract as always.

$$\begin{aligned}
(f+g)(x,y) &= \frac{3x}{4y^2} + \frac{7x}{6y} \\
&= \frac{3x \times 3}{4y^2 \times 3} + \frac{7x \times 2y}{6y \times 2y} \\
&= \frac{9x}{12y^2} + \frac{14xy}{12y^2} = \frac{9x + 14xy}{12y^2} = \frac{x(9+14y)}{12y^2}. \\
(f-g)(x,y) &= \frac{3x}{4y^2} - \frac{7x}{6y} \\
&= \frac{3x \times 3}{4y^2 \times 3} - \frac{7x \times 2y}{6y \times 2y} \\
&= \frac{9x}{12y^2} - \frac{14xy}{12y^2} = \frac{9x - 14xy}{12y^2} = \frac{x(9-14y)}{12y^2}.
\end{aligned}$$

- $f(x) = \frac{10x+3}{5x^4}$ and $g(x) = \frac{7x-4}{5x^4}$.

$$(f+g)(x) = f(x) + g(x) = \frac{10x+3}{5x^4} + \frac{7x-4}{5x^4} = \frac{10x+3+7x-4}{5x^4} = \frac{17x-1}{5x^4}.$$
$$(f-g)(x) = f(x) - g(x) = \frac{10x+3}{5x^4} - \frac{7x-4}{5x^4} = \frac{10x+3-(7x-4)}{5x^4} = \frac{10x+3-7x+4}{5x^4} = \frac{3x+7}{5x^4}.$$

- $f(x) = \dfrac{7x^3 - 4x^2 + 2x}{x-3}$ and $g(x) = \dfrac{6x^3 + 5x^2 - 7}{3-x}$.

  Note that $(3-x) = -(x-3)$. Hence, $\dfrac{6x^3+5x^2-7}{3-x} = -\dfrac{6x^3+5x^2-7}{x-3}$ and the LCD is $(x-3)$.

$$\begin{aligned}
(f+g)(x) = f(x)+g(x) &= \frac{7x^3-4x^2+2x}{x-3} + \frac{6x^3+5x^2-7}{3-x} \\
&= \frac{7x^3-4x^2+2x}{x-3} - \frac{6x^3+5x^2-7}{x-3} && \textit{Note the change in sign;} \\
&= \frac{7x^3-4x^2+2x-(6x^3+5x^2-7)}{x-3} && \textit{A fraction bar is a} \\
&= \frac{7x^3-4x^2+2x-6x^3-5x^2+7}{x-3} = \frac{x^3-9x^2+2x+7}{x-3} && \textit{grouping symbol.} \\
(f-g)(x) = f(x)-g(x) &= \frac{7x^3-4x^2+2x}{x-3} - \frac{6x^3+5x^2-7}{3-x} \\
&= \frac{7x^3-4x^2+2x}{x-3} - \left(-\frac{6x^3+5x^2-7}{x-3}\right) && \textit{Note the change in sign;} \\
&= \frac{7x^3-4x^2+2x}{x-3} + \frac{6x^3+5x^2-7}{3-x} \\
&= \frac{7x^3-4x^2+2x+6x^3+5x^2-7}{x-3} \\
&= \frac{13x^3+x^2+2x-7}{x-3}.
\end{aligned}$$

- $f(x) = \dfrac{2x+1}{x}$ and $g(x) = \dfrac{4x}{x+3}$. With denominators being x and $(x+3)$ the LCD will be $x(x+3)$.

$$\begin{aligned}
(f+g)(x) = f(x)+g(x) &= \frac{2x+1}{x} + \frac{4x}{x+3} \\
&= \frac{(2x+1)\times(x+3)}{x(x+3)} + \frac{4x \times x}{x(x+3)} \\
&= \frac{2x^2+7x+3}{x(x+3)} + \frac{4x^2}{x(x+3)} \\
&= \frac{2x^2+7x+3+4x^2}{x(x+3)} = \frac{6x^2+7x+3}{x(x+3)}. \\
(f-g)(x) = f(x)-g(x) &= \frac{2x+1}{x} - \frac{4x}{x+3}
\end{aligned}$$

$$= \frac{(2x+1)\times(x+3)}{x(x+3)} - \frac{4x \times x}{x(x+3)}$$
$$= \frac{2x^2+7x+3}{x(x+3)} - \frac{4x^2}{x(x+3)}$$
$$= \frac{2x^2+7x+3-4x^2}{x(x+3)} = \frac{-2x^2+7x+3}{x(x+3)}.$$

- $f(x) = \dfrac{2x-1}{x-7}$ and $g(x) = \dfrac{x}{x+4}$. Here, the LCD = $(x-7)(x+4)$.

$$\begin{aligned}(f+g)(x) = f(x)+g(x) &= \frac{2x-1}{x-7} + \frac{x}{x+4}\\ &= \frac{(2x-1)(x+4)}{(x-7)(x+4)} + \frac{x(x-7)}{(x-7)(x+4)}\\ &= \frac{2x^2+7x-4}{(x-7)(x+4)} + \frac{x^2-7x}{(x-7)(x+4)}\\ &= \frac{2x^2+7x-4+x^2-7x}{(x-7)(x+4)}\\ &= \frac{3x^2-4}{(x-7)(x+4)} = \frac{\left(\sqrt{3}x+2\right)\left(\sqrt{3}x-2\right)}{(x-7)(x+4)}.\\ (f-g)(x) = f(x)-g(x) &= \frac{2x-1}{x-7} - \frac{x}{x+4}\\ &= \frac{(2x-1)(x+4)}{(x-7)(x+4)} - \frac{x(x-7)}{(x-7)(x+4)}\\ &= \frac{2x^2+7x-4}{(x-7)(x+4)} - \frac{x^2-7x}{(x-7)(x+4)}\\ &= \frac{2x^2+7x-4-(x^2-7x)}{(x-7)(x+4)}\\ &= \frac{2x^2+7x-4-x^2+7x}{(x-7)(x+4)} = \frac{x^2+14x-4}{(x-7)(x+4)}.\end{aligned}$$

- Let $f(x) = \dfrac{3}{x^2-3x}$ and $g(x) = \dfrac{4}{x^2+x-12}$. **Note that to find the LCD we need to first factor the denominators.** Here, the denominators are $x^2-3x = x(x-3)$ and $x^2+x-12 = (x+4)(x-3)$. Hence, a common multiple of these two expressions is $x(x-3)(x+4)$. In fact, the readers are asked to consider whether any smaller degree polynomial could possibly be a common multiple. The LCD is indeed $x(x-3)(x+4)$.

$$\begin{aligned}(f+g)(x) = f(x)+g(x) &= \frac{3}{x^2-3x}+\frac{4}{x^2+x-12}\\ &= \frac{3}{x(x-3)}+\frac{4}{(x+4)(x-3)}\\ &= \frac{3(x+4)}{x(x-3)(x+4)}+\frac{4x}{x(x-3)(x+4)}\\ &= \frac{3x+12}{x(x-3)(x+4)}+\frac{4x}{x(x-3)(x+4)}\\ &= \frac{3x+12+4x}{x(x-3)(x+4)}=\frac{7x+12}{x(x-3)(x+4)}.\\ (f+g)(x) = f(x)+g(x) &= \frac{3}{x^2-3x}-\frac{4}{x^2+x-12}\\ &= \frac{3}{x(x-3)}-\frac{4}{(x+4)(x-3)}\\ &= \frac{3(x+4)}{x(x-3)(x+4)}-\frac{4x}{x(x-3)(x+4)}\\ &= \frac{3x+12}{x(x-3)(x+4)}-\frac{4x}{x(x-3)(x+4)}\\ &= \frac{3x+12-4x}{x(x-3)(x+4)}=\frac{-x+12}{x(x-3)(x+4)}.\end{aligned}$$

- Find $(f+g)(2)$ and $(f-g)(-1)$ when $f(x)=\dfrac{1}{x^3-2}$ and $g(x)=\dfrac{3}{x+5}$.

$$\begin{aligned}(f+g)(2) = f(2)+g(2) &= \frac{1}{2^3-2}+\frac{3}{2+5}\\ &= \frac{1}{8-2}+\frac{3}{7}\\ &= \frac{1}{6}+\frac{3}{7}=\frac{7}{42}+\frac{18}{42}=\frac{25}{42}.\\ (f-g)(-1) = f(-1)-g(-1) &= \frac{1}{(-1)^3-2}-\frac{3}{(-1)+5}\\ &= \frac{1}{-1-2}-\frac{3}{4}\\ &= \frac{1}{-3}-\frac{3}{4}=-\frac{4}{12}-\frac{9}{12}=-\frac{13}{12}.\end{aligned}$$

Steps to find the LCD:

- First factor the denominators into irreducible factors (that is, factors which cannot be factored further).
- LCD is the product of the highest power of each distinct factor which appears in the above step.

Examples of LCD:

First Denominator	Second Denominator	LCD
$(x-3)(x+5)$	$x(x-3)$	$x(x-3)(x+5)$
$(x-3)^2(x+5)^3$	$x(x-3)^4$	$x(x-3)^4(x+5)^3$
$x^2(x-2)^3(x+7)^4$	$x(x-2)^2(x+7)^5(x+8)^2$	$x^2(x-2)^3(x+7)^5(x+8)^2$

Classroom Exercises: (1) Find $(f+g)$ and $(f-g)$ in the following cases:

f	g
$\frac{4x^2+5x-1}{5x^3}$	$\frac{2x^2+7x+2}{5x^3}$
$\frac{b}{a-b}$	$\frac{a}{a-b}$
$\frac{x^2-3x}{x-5}$	$\frac{2x}{5-x}$
$\frac{x-2}{x-3}$	$\frac{x}{x+3}$
$\frac{2x}{x^2-3x-4}$	$\frac{5}{x^2-1}$
$\frac{3}{x^2-16}$	$\frac{2}{x^3+64}$
$\frac{2}{x^3-x^2+3x-3}$	$\frac{1}{x^2+6x-7}$
$\frac{1}{5v^2+9v+4}$	$\frac{1}{10v^2+19v+9}$

(2) Given $f(x)=\dfrac{2}{6x^2+11x-10}$, $g(x)=\dfrac{1}{2x+5}$ and $h(x)=\dfrac{2}{3x-2}$, find

- $(f+g-h)(2)$
- $(f-g+h)(-1)$.

3.4.1 Homework Exercises:

(1) Find $(f+g)$ and $(f-g)$ in the following cases:

f	g
$\frac{x^2+5x-1}{2x^3}$	$\frac{2x^2+7x+2}{2x^3}$
$\frac{y}{y-x}$	$\frac{x}{x-y}$
$\frac{x^2-3x}{x-5}$	$\frac{2x}{5-x}$
$\frac{x-2}{x+4}$	$\frac{x}{x-4}$
$\frac{3}{x^2-4}$	$\frac{2}{x^3+8}$
$\frac{2}{x^3+2x^2+4x+8}$	$\frac{1}{x^2-7x-18}$
$\frac{2v-1}{8v^2+10v-3}$	$\frac{1}{4v^2-1}$

(2) Given $f(x)=\dfrac{2}{x^2+10x-1}$, $g(x)=\dfrac{1}{3x+5}$ and $h(x)=\dfrac{2}{5x-2}$, find

- $(f+g-h)(2)$
- $(f-g+h)(-1)$.

3.5 Complex expressions

As the title suggests, here we will simplify complex expressions and reduce them to rational expressions in their simplest forms.

Examples: Simplify:

- $\dfrac{2+\dfrac{1}{x+3}}{3-\dfrac{1}{x+3}}$. To simplify such complex functions, we separate the numerator and denominator and then proceed.

$$\begin{aligned}\frac{2+\dfrac{1}{x+3}}{3-\dfrac{1}{x+3}} &= \left(2+\frac{1}{x+3}\right)\div\left(3-\frac{1}{x+3}\right)\\ &= \left(\frac{2}{1}+\frac{1}{x+3}\right)\div\left(\frac{3}{1}-\frac{1}{x+3}\right)\end{aligned}$$

$$= \left(\frac{2(x+3)}{(x+3)} + \frac{1}{(x+3)}\right) \div \left(\frac{3(x+3)}{(x+3)} - \frac{1}{(x+3)}\right)$$
$$= \left(\frac{2(x+3)+1}{x+3}\right) \div \left(\frac{3(x+3)-1}{x+3}\right)$$
$$= \left(\frac{2x+6+1}{x+3}\right) \div \left(\frac{3x+9-1}{x+3}\right)$$
$$= \left(\frac{2x+7}{x+3}\right) \div \left(\frac{3x+8}{x+3}\right)$$
$$= \frac{2x+7}{x+3} \times \frac{x+3}{3x+8} = \frac{(2x+7)(x+3)}{(x+3)(3x+8)} = \frac{2x+7}{3x+8}.$$

- $\dfrac{\frac{m+3}{m-2} + \frac{m+1}{m+2}}{\frac{m-3}{m-2} - \frac{m}{m+2}}$. Here again, separate the numerator and denominator.

$$\frac{\frac{m+3}{m-2} + \frac{m+1}{m+2}}{\frac{m-3}{m-2} - \frac{m}{m+2}} = \left(\frac{m+3}{m-2} + \frac{m+1}{m+2}\right) \div \left(\frac{m-3}{m-2} - \frac{m}{m+2}\right)$$
$$= \left(\frac{(m+3)(m+2)}{(m-2)(m+2)} + \frac{(m+1)(m-2)}{(m-2)(m+2)}\right) \div \left(\frac{(m-3)(m+2)}{(m-2)(m+2)} - \frac{m(m-2)}{(m+2)(m-2)}\right)$$
$$= \left(\frac{m^2+5m+6}{(m-2)(m+2)} + \frac{m^2-m-2}{(m-2)(m+2)}\right) \div \left(\frac{m^2-m-6}{(m-2)(m+2)} - \frac{m^2-2m}{(m+2)(m-2)}\right)$$
$$= \left(\frac{m^2+5m+6+m^2-m-2}{(m-2)(m+2)}\right) \div \left(\frac{m^2-m-6-(m^2-2m)}{(m-2)(m+2)}\right)$$
$$= \left(\frac{2m^2+4m+4}{(m-2)(m+2)}\right) \div \left(\frac{m^2-m-6-m^2+2m)}{(m-2)(m+2)}\right)$$
$$= \left(\frac{2(m^2+2m+2)}{(m-2)(m+2)}\right) \div \left(\frac{m-6}{(m-2)(m+2)}\right)$$
$$= \frac{2(m^2+2m+2)}{(m-2)(m+2)} \times \frac{(m-2)(m+2)}{m-6}$$
$$= \frac{2(m^2+2m+2) \times (m-2)(m+2)}{(m-2)(m+2) \times (m-6)} = \frac{2(m^2+2m+2)}{m-6}.$$

- $1 + \dfrac{1}{1+\frac{1}{1+x}}$. This is an example of terminated continued fraction. First, convert the denominator into a single rational expression.

$$\begin{aligned} 1 + \frac{1}{1+x} &= \frac{1}{1} + \frac{1}{1+x} \\ &= \frac{1+x}{1+x} + \frac{1}{1+x} \\ &= \frac{1+x+1}{1+x} = \frac{2+x}{1+x}. \end{aligned}$$

$$\text{Now, } \frac{1}{1+\frac{1}{1+x}} = 1 \div \left(1 + \frac{1}{1+x}\right) = 1 \div \left(\frac{2+x}{1+x}\right) = 1 \times \left(\frac{1+x}{2+x}\right) = \frac{1+x}{2+x}.$$

$$\text{Lastly, } 1 + \frac{1}{1+\frac{1}{1+x}} = 1 + \frac{1+x}{2+x} = \frac{2+x}{2+x} + \frac{1+x}{2+x} = \frac{3+2x}{2+x}.$$

Classroom Exercises: Simplify the following complex expressions.

- $\dfrac{\frac{3}{x}+1}{\frac{4}{x}-1}$
- $\dfrac{\frac{x}{x+1}+\frac{1}{x}}{\frac{x+2}{x+1}-\frac{1}{x}}$
- $\dfrac{\frac{2x}{x-2}+\frac{x}{x+3}}{\frac{x}{x-2}-\frac{3x}{x+3}}$
- $\dfrac{\frac{a+b}{a-b}-\frac{a-b}{a+b}}{\frac{a+b}{a-b}+\frac{a-b}{a+b}}$
- $1 - \dfrac{1}{1+\frac{1}{1-\frac{1}{y}}}$.

3.5.1 Homework Exercises:

Simplify the following complex expressions.

- $\dfrac{\frac{2}{y}+1}{\frac{3}{y}-1}$
- $\dfrac{\frac{y}{y+2}+\frac{1}{y}}{\frac{y+1}{y+2}-\frac{1}{y}}$
- $\dfrac{\frac{2y}{y-3}+\frac{y}{y+2}}{\frac{y}{y-3}-\frac{4y}{y+2}}$
- $\dfrac{\frac{x+y}{x-y}+\frac{x-y}{x+y}}{\frac{x+y}{x-y}-\frac{x-y}{x+y}}$
- $1 + \dfrac{1}{x+\frac{1}{1+\frac{1}{x}}}$.

3.6 Solving Rational Equations

In this section we learn to solve rational equations. These are equations in which one or both sides contains a rational expression. Recall that we may add or subtract two sides of an equation by a number or expression and obtain an equivalent equation and hence obtain the same solutions. Likewise, we may multiply or divide the two sides of an equation by a nonzero real number or expression and obtain an equivalent equation and hence obtain the same solutions.

Examples Solve the following rational equations for x.

- $$\begin{aligned}
\frac{4x-1}{5}+\frac{2-x}{15}&=\frac{2x+1}{3} && \textit{The LCD is } 15\\
15\times\frac{(4x-1)}{15}+15\times\frac{2-x}{15}&=15\times\frac{(2x+1)}{3} && \textit{Multiply both sides of the equation by } 15\\
3(4x-1)+2-x&=5(2x+1)\\
12x-3+2-x&=10x+5 && \textit{This is a linear equation;}\\
11x-1&=10x+5\\
x&=6.
\end{aligned}$$

Always check your solution. Most importantly, make sure that your solution does not make any of your original expressions undefined. Here LHS stands for Left Hand Side, and RHS stands for the Right Hand Side.

$$LHS=\frac{4(6)-1}{5}+\frac{2-6}{15}=\frac{23}{5}-\frac{4}{15}=\frac{69}{15}-\frac{4}{15}=\frac{65}{15}=\frac{13}{3}$$

$$RHS=\frac{2(6)+1}{3}=\frac{13}{3}.$$

Since $LHS=RHS$, we have solved the equation.

- $$\begin{aligned}
\frac{x-5}{x-3}+\frac{x-3}{x}&=\frac{2x+3}{x^2-3x} && \textit{Factor the denominators;}\\
\frac{x-5}{x-3}+\frac{x-3}{x}&=\frac{2x+3}{x(x-3)} && \textit{Multiply both sides by LCD } =x(x-3)\\
x(x-3)\cdot\frac{(x-5)}{(x-3)}+x(x-3)\cdot\frac{(x-3)}{x}&=x(x-3)\cdot\frac{2x+3}{x(x-3)}\\
x(x-5)+(x-3)(x-3)&=2x+3\\
x^2-5x+x^2-6x+9&=2x+3\\
2x^2-11x+9&=2x+3\\
2x^2-13x+6&=0 && \textit{This is a quadratic equation;}\\
(2x-1)(x-6)&=0 && \textit{Factor the LHS;}\\
2x-1=0 &\ \textit{or}\ x-6=0 && \textit{Solve for } x\\
x=\frac{1}{2} &\ \textit{or}\ x=6
\end{aligned}$$

Check that these numbers do not make any of the original expressions undefined.

•

$$\frac{x}{x+5} + \frac{1}{x+1} = \frac{20}{x^2+6x+5} \qquad \textit{Factor the denominators;}$$

$$\frac{x}{x+5} + \frac{1}{x+1} = \frac{20}{(x+1)(x+5)} \qquad LCD = (x+1)(x+5)$$

$$(x+1)(x+5)\cdot\frac{x}{x+5} + (x+1)(x+5)\cdot\frac{1}{x+1} = (x+1)(x+5)\cdot\frac{20}{(x+1)(x+5)} \qquad \textit{Multiply by LCD;}$$

$$(x+1)x + (x+5) = 20 \qquad \textit{Simplify;}$$

$$x^2 + x + x + 5 = 20 \qquad \textit{Simplify, get zero on the RHS;}$$

$$x^2 + 2x - 15 = 0 \qquad \textit{Factor the LHS;}$$

$$(x+5)(x-3) = 0 \qquad \textit{Use the zero property;}$$

$$x+5=0 \textit{ or } x-3=0;$$

$$x=-5 \textit{ or } x=3.$$

Check your solutions. Notice that $x = -5$ makes one of the expressions undefined (see $\frac{x}{x+5}$). Hence, $x = -5$ is not a solution. The reader is encouraged to check that $x = 3$ is indeed a solution.

Summary:

- First clear the denominators by multiplying both sides of the equation by the LCD.
- Simplify both the sides.
- If you are left with a linear equation, then solve it as you normally would. If you are left with a higher degree equation, then factor (or use the quadratic formula for a quadratic equation that doesn't factor easily). Now proceed with the zero property.
- Once you have possible solution(s), make sure that your answers do not make any of the original expressions undefined.

Classroom Exercises: Solve for x:

(a) $\frac{3x-2}{5} + \frac{2x-1}{10} = \frac{1-x}{2}$

(b) $\frac{x}{2x-1} + \frac{1}{x-3} = \frac{5}{2x^2-7x+3}$

(c) $\frac{2x}{x-7} - \frac{x}{x-4} = \frac{12}{x^2-11x+28}$

(d) $\dfrac{2x}{x-1} - \dfrac{x}{x+3} = \dfrac{18}{x^2+2x-3}$

(e) $\dfrac{3x}{2x-1} - \dfrac{x}{x-1} = \dfrac{24}{2x^2-3x+1}$

(f) $\dfrac{x-2}{x-8} + \dfrac{3}{x-1} = \dfrac{3}{x^2-9x+8}$

(g) $\dfrac{4x}{3x+4} - \dfrac{x}{x+2} = \dfrac{21}{3x^2+10x+8}$

Here are some word problems which can be solved using the above methods.
(1) The sum of the reciprocals of two consecutive integers is 11 times the reciprocal of the product of the numbers. Find the two numbers.

Let the two consecutive integers be x and $(x+1)$.

$$\begin{aligned}
\frac{1}{x} + \frac{1}{(x+1)} &= 11 \times \frac{1}{x(x+1)} && \textit{Translating the first sentence.} \\
x(x+1)\cdot\frac{1}{x} + x(x+1)\cdot\frac{1}{(x+1)} &= x(x+1)\cdot\frac{11}{x(x+1)} && LCD = x(x+1) \\
(x+1) + x &= 11 \\
2x + 1 + 1 &= 11 \\
2x &= 10 \\
x &= 5.
\end{aligned}$$

Hence, the two numbers are 5 and 6.
(2) The current of a stream is 5 mi/h. A motorboat travels 4 miles downstream in the same time as it travels 3 miles upstream. What is the speed of the motorboat in still water?

Let the speed of the motorboat in still water be x mi/h.

Then, the speed of the motorboat downstream (with the flow of the stream) = $(x+5)$ mi/h.

The speed of the motorboat upstream (against the flow of the stream) = $(x-5)$ mi/h.

(Note that the motorboat has to be faster than the current of the stream and hence we are assuming that $x > 5$).

$$\textit{Time} = \frac{\textit{Distance}}{\textit{Speed}}.$$

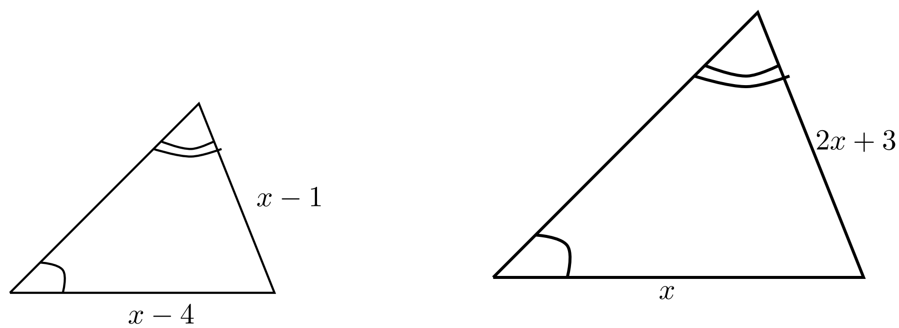
Since the time taken in downstream travel is equal to the time taken travelling upstream, we have

$$\frac{4}{x+5} = \frac{3}{x-5}.$$

We now can solve for x.

$$\begin{aligned} \frac{4}{(x+5)} &= \frac{3}{(x-5)} && \textit{Cross-multiply;} \\ 4(x-5) &= 3(x+5) \\ 4x-20 &= 3x+15 \\ x &= 35. \end{aligned}$$

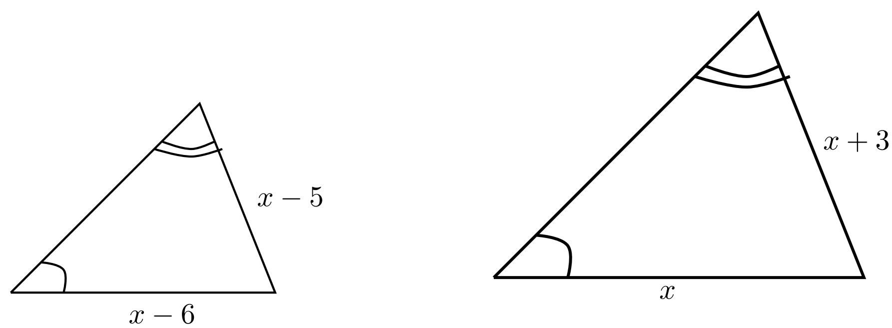
The speed of the motorboat in still water is 35 mi/h.
(3) Find x assuming that the following two triangles are similar.



Since corresponding sides of similar triangles are similar, we have

$$\begin{aligned} \frac{2x+3}{x-1} &= \frac{x}{x-4} && \textit{Cross-multiply to get} \\ (x-4)(2x+3) &= x(x-1) \\ 2x^2-5x-12 &= x^2-x \\ x^2-4x-12 &= 0 \\ (x-6)(x+2) &= 0 \\ x &= 6 \textit{ or } x=-2. \end{aligned}$$

Note that length cannot be -2. Hence, we have $x = 6$ units.
Classroom Exercises:
(1) The sum of the reciprocals of two consecutive odd integers is 16 times the reciprocal of the product of the numbers. Find the two numbers.
(2) The speed of a plane in still air is 550 mi/h. On a certain day the plane flies 9,000 miles with tailwind and flies back the same distance against wind of the same speed. The complete trip takes 33 hours. What is the rate of the tailwind?
(3) Find x assuming that the following two triangles are similar.

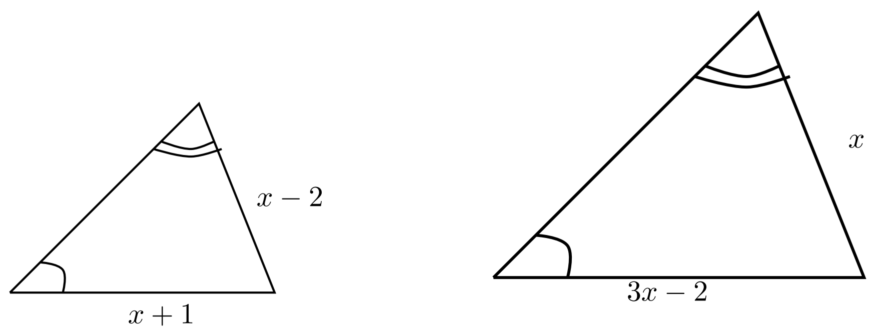


3.6.1 Homework Exercises:

(1) Solve for x:

1. $\dfrac{2y+3}{3} - \dfrac{2y+1}{4} = \dfrac{3-y}{12}$
2. $2x + \dfrac{2x-9}{x+2} = \dfrac{11}{x+2}$
3. $\dfrac{2x}{x+2} - \dfrac{x+2}{x+1} = \dfrac{5}{x^2+3x+2}$
4. $\dfrac{2(x-3)}{x-2} - \dfrac{1}{x-3} = \dfrac{x}{x^2-5x+6}$
5. $\dfrac{2x}{x+1} - \dfrac{x}{x-2} = -\dfrac{6}{x^2-x-2}$
6. $\dfrac{3x}{2x-1} + \dfrac{2x}{x-2} = \dfrac{60}{2x^2-5x+2}$
7. $\dfrac{3x}{x-4} - \dfrac{6}{x+1} = \dfrac{28}{x^2-3x-4}$

(2) The sum of the reciprocals of two consecutive even integers is 14 times the reciprocal of the product of the numbers. Find the two numbers.
(3) Rob can row a boat at 7 mi/h. On a certain day he rows 45 miles downstream and rows back the same distance upstream. The total time taken is 14 hours. What is the speed of the current?
(4) Find x assuming that the following two triangles are similar.


x − 2
x + 1
x
3x − 2

Chapter 4

Exponential and Logarithmic functions

4.1 Exponential functions

Definition 1. Let X and Y be nonempty sets. A function f from a subset of X to Y is a rule which assigns to every element x from X a **unique** element y in Y. The notation "$f : X \to Y$" represents the phrase "f is a function from X to Y" and $f(x) = y$ represents "f assigns to x value y."

Examples: • Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. Let f be an assignment given by $f(1) = a$, $f(2) = f(3) = b$. Then $f : X \to Y$ is a function.

But, the assignment g given by $g(1) = a$, $g(1) = b$, $g(2) = g(3) = c$ is not a function because g assigns two values to 1.

• Recall from Chapter 1 that $\mathbb{R}$ denotes the set of real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be given by the rule $f(x) = 3x^2 - 7x + 5$ for every x from $\mathbb{R}$. That is, for every real number x, f assigns the number $3x^2 - 7x + 5$. This is a function.

• $\mathbb{R} \setminus \{3\}$ denotes the set of all real numbers except 3, and define $f : \mathbb{R} \setminus \{3\} \to \mathbb{R}$ by the rule $f(x) = \dfrac{1}{x-3}$. Then f is indeed a function. Note that $\dfrac{1}{3-3}$ is not defined. That is why $f : \mathbb{R} \to \mathbb{R}$ is not a function, but $f : \mathbb{R} \setminus \{3\} \to \mathbb{R}$ is a function.

One of the most important functions is the **exponential function.**

Definition 2. Let b be a real number, $b > 0, b \neq 1$. The exponential function with base b is the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = b^x$.

It has to be pointed out that to truly understand this function, its values on irrational numbers, requires knowledge of the concept of limits. In this book we merely touch upon the basics of this function.

Examples:

• Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3^x$. Then,

$$f(1) = 3^1 = 3; \quad f(-1) = 3^{-1} = \frac{1}{3};$$
$$f(2) = 3^2 = 9; \quad f(-2) = 3^{-2} = \frac{1}{3^2} = \frac{1}{9};$$
$$f(3) = 3^3 = 27; \quad f(-3) = 3^{-3} = \frac{1}{3^3} = \frac{1}{27};$$
$$f(0) = 3^0 = 1; \quad f\left(\frac{1}{2}\right) = 3^{\frac{1}{2}} = \sqrt{3}.$$

• Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \left(\frac{2}{3}\right)^x$. Then,

$$f(1) = \left(\frac{2}{3}\right)^1 = \frac{2}{3}; \quad f(-1) = \left(\frac{2}{3}\right)^{-1} = \frac{2^{-1}}{3^{-1}} = \frac{3}{2};$$
$$f(2) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}; \quad f(-2) = \left(\frac{2}{3}\right)^{-2} = \frac{2^{-2}}{3^{-2}} = \frac{3^2}{2^2} = \frac{9}{4};$$
$$f(3) = \left(\frac{2}{3}\right)^3 = \frac{8}{27}; \quad f(-3) = \left(\frac{2}{3}\right)^{-3} = \frac{2^{-3}}{3^{-3}} = \frac{3^3}{2^3} = \frac{27}{8};$$
$$f(0) = \left(\frac{2}{3}\right)^0 = 1; \quad f\left(\frac{1}{2}\right) = \left(\frac{2}{3}\right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3} \text{ (by rationalizing the denominator).}$$

Caution: Note that $f(x) = 1^x$ is an uninteresting function, because $1^x = 1$ for every real number x. This is why we require that the base $b \neq 1$ for an exponential function $f(x) = b^x$.

The case $b < 0$ is not considered in our course for the reason that b^x need not be a real number for certain cases of x. For example, if $b = -3$ then $b^{\frac{1}{2}} = (-3)^{\frac{1}{2}} = \sqrt{-3}$ is a complex number.

Classroom Exercises:

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be the exponential function defined by $f(x) = 2^x$.

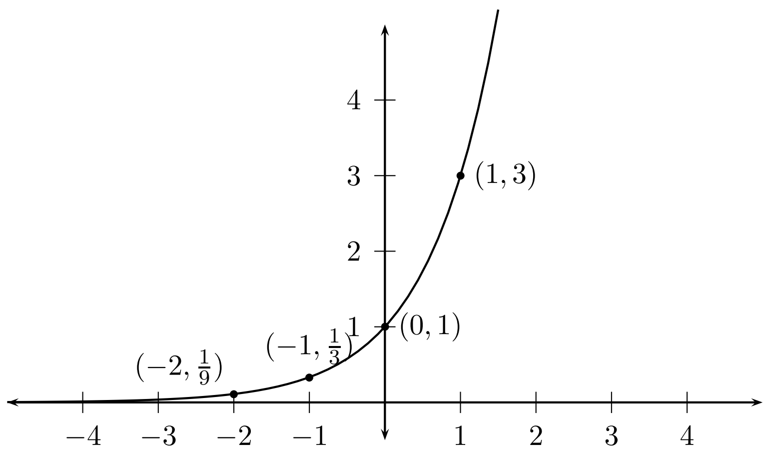
Find $f(1), f(2), f(3), f(-1), f(-2), f(-3), f(0), f\left(\frac{1}{2}\right), f\left(\frac{1}{3}\right), f\left(\frac{2}{3}\right)$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be the exponential function defined by $f(x) = \left(\frac{3}{5}\right)^x$.

Find $f(1), f(2), f(3), f(-1), f(-2), f(-3), f(0), f\left(\frac{1}{2}\right), f\left(\frac{1}{3}\right), f\left(\frac{2}{3}\right)$.

We now use this information to graph exponential functions. That is, we plot points (x, y) where $y = f(x)$ for an exponential function $f(x)$.

Example 1: Consider the exponential function $f(x) = 3^x$. We set $y = 3^x$ and plot some points, then smoothly join these points to get a curve.

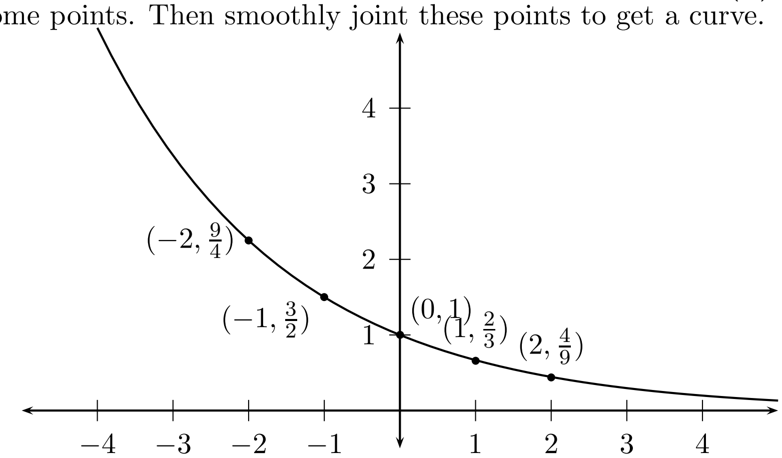


x	$y = 3^x$
1	3
2	9
0	1
−1	$\frac{1}{3}$
−2	$\frac{1}{9}$

Important features:

- First note that the y-values **increase** as the x-values increase. We say that f is an **increasing** function.
- Next, note that the curve is getting close to the x-axis on the left, but will never cross it (why not?). We describe this phenomenon by saying that the curve is **asymptotic** to the x-axis on the left.
- The curve passes through the point $(0, 1)$.

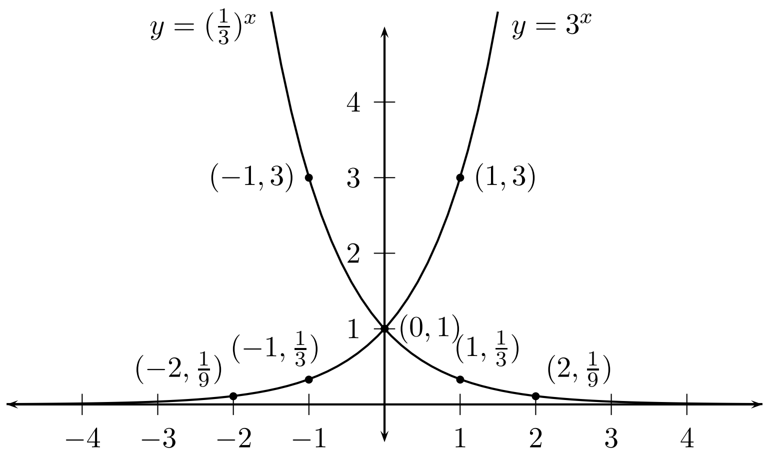
Example 2: Consider the exponential function $f(x) = \left(\frac{2}{3}\right)^x$. We set $y = \left(\frac{2}{3}\right)^x$ and plot some points. Then smoothly joint these points to get a curve.



x	$y = \left(\frac{2}{3}\right)^x$
1	$\frac{2}{3}$
2	$\frac{4}{9}$
0	1
−1	$\frac{3}{2}$
−2	$\frac{9}{4}$

Important features: The y-values **decrease** as the x-values increase. That is, the function f is **decreasing**. The curve is getting close to the x-axis on the right, but will never cross it. That is, the curve is **asymptotic** to the x-axis on the right. The curve passes through the point $(0, 1)$.

Example 3: Consider the exponential functions $f(x) = 3^x$ and the graph $g(x) = 3^{-x} = \left(\frac{1}{3}\right)^x$ simultaneously. That is, we graph the curves $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$.



x	$y = 3^x$
1	3
2	9
0	1
−1	$\frac{1}{3}$
−2	$\frac{1}{9}$

x	$y = 3^{-x} = \left(\frac{1}{3}\right)^x$
1	$\frac{1}{3}$
2	$\frac{1}{9}$
0	1
−1	3
−2	9

Important feature: The graph of $f(x) = 3^x$ and that of $g(x) = 3^{-x} = \left(\frac{1}{3}\right)^3$ are symmetric about the y-axis.

In general, we have the following important features of the graph of the exponential function $f(x) = b^x$ for $b > 0, b \neq 1$:

- **Case $b > 1$:** In this case, the function $y = b^x$ is increasing and asymptotic to the x axis on the left. This curve passes through the point $(0, 1)$.
- **Case $b < 1$:** In this case, the function $y = b^x$ is decreasing and asymptotic to the x axis on the right. This curve passes through the point $(0, 1)$.
- The curves $y = b^x$ and $y = b^{-x}$ are symmetric to each other about the y-axis.

Classroom Exercises:

(1) On separate coordinate plane, graph the following functions, plot at least 4 points on each, and state their important features: $f(x) = 2^x, f(x) = \left(\frac{3}{5}\right)^x$.

(2) On the same coordinate plane, graph the following functions, plot at least 4 points on each, state their important features, and how they relate to each other: $f(x) = 4^x$, $g(x) = 4^{-x} = \left(\frac{1}{4}\right)^x$.

One of the most important conclusions one draws from the graphs above is that these exponential functions are **one-to-one**. That is, the function $f(x) = b^x$ is such that $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.

In other words, $b^{x_1} = b^{x_2}$ implies that $x_1 = x_2$. This property can be used to solve some exponential equations.

Examples: Solve for the unknown variable:

- $2^x = 1$
  $2^x = 2^0$ *Note that* $1 = 2^0$;
  $x = 0$ *We have solved for* x *using the one-to-one property.*
- $3^{x-2} = 9$
  $3^{x-2} = 3^2$ *Note that* $9 = 3^2$;
  $x - 2 = 2$ *using the one-to-one property*;
  $x = 4$ *We have solved for* x.
- $\left(\frac{1}{2}\right)^{x+3} = 16$
  $\left(\frac{1}{2}\right)^{x+3} = 2^4$ *Note that* $16 = 2^4$;
  $(2^{-1})^{x+3} = 2^4$
  $2^{-(x+3)} = 2^4$
  $-(x + 3) = 4$ *using the one-to-one property*;

  $-x - 3 = 4$
  $-x = 7$ *Multiply both sides by* (-1);
  $x = -7$ *We have solved for* x.
- $5^{3y+1} = 125$
  $5^{3y+1} = 5^3$ *Note that* $125 = 5^3$;
  $3y + 1 = 3$ *using the one-to-one property*;

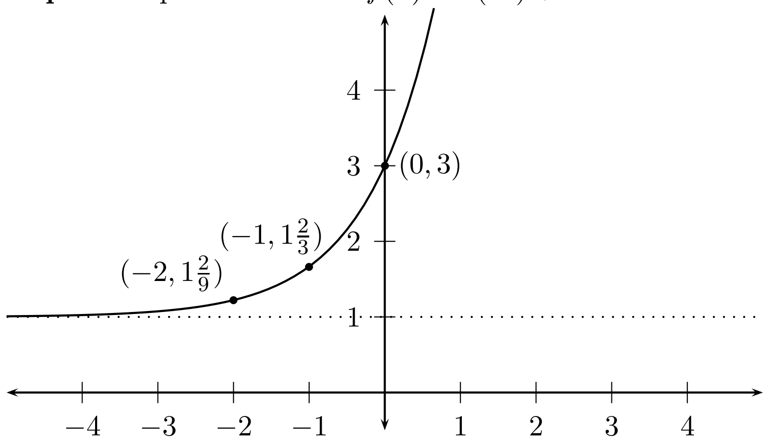
  $3y = 2$
  $y = \frac{2}{3}$ *We have solved for* y.

Classroom Exercises: Solve for the unknown variable:

• $3^x = 27$ • $2^{5x} = 32$ • $\left(\frac{1}{3}\right)^{x+2} = 9$ • $5^{100x} = 1$ • $7^{3x+1} = 49$

We now draw a more complicated graph of the kind $f(x) = a(b^x) + c$ for any real numbers a, b, c with $a, b \neq 0, b > 0, b \neq 1$, by plotting points.

Example: Graph the function $f(x) = 2(3^x) + 1$ and state its important features.



x	$y = 2(3^x) + 1$
1	$2(3) + 1 = 7$
2	$2(3^2) + 1 = 10$
0	$2(3^0) + 1 = 3$
-1	$2(3^{-1}) + 1 = 1\frac{2}{3}$
-2	$2(3^{-2}) + 1 = 1\frac{2}{9}$

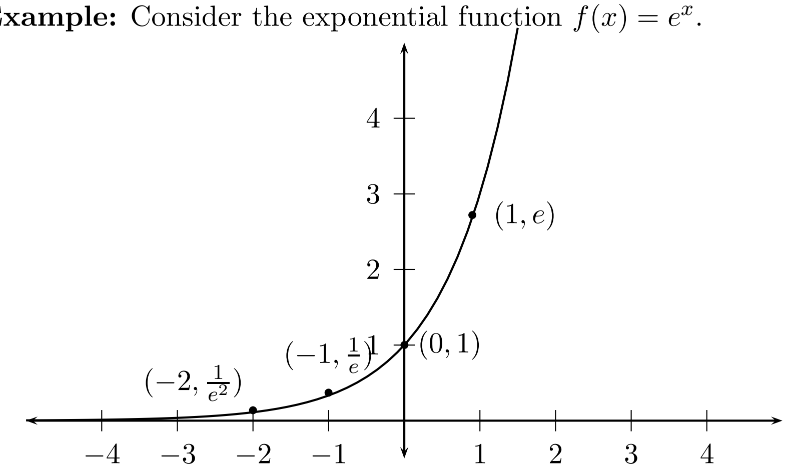
Important features: The graph $y = 2(3^x) + 1$ is increasing. It is asymptotic to the line $y = 1$ on the left. It passes through the point $(0, 3)$.

Classroom Exercises: Graph the following functions, and state their important features.

- $f(x) = 3(2^x) + 1$
- $f(x) = 3(2^x) - 1$
- $f(x) = -3(2^x) + 1$
- $f(x) = -3(2^x) - 1$
- $f(x) = 2^{x+1} = 2(2^x)$.

There is a special irrational number e sometimes called **Euler's number.** It is a number approximately equal to 2.71828. This number appears in banking, biology, physics, and even in humanities.

Example: Consider the exponential function $f(x) = e^x$.



x	$y = e^x$
1	e
2	e^2
0	1
-1	$\frac{1}{e}$
-2	$\frac{1}{e^2}$

Classroom Exercises: Graph $y = e^{2x}$, $y = e^{-x}$, and $y = 3e^x$.

4.1.1 Homework Exercises:

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the exponential function defined by $f(x) = 5^x$.

   Find $f(1), f(2), f(3), f(-1), f(-2), f(-3), f(0), f\left(\frac{1}{2}\right), f\left(\frac{1}{3}\right), f\left(\frac{2}{3}\right)$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be the exponential function defined by $f(x) = \left(\frac{2}{5}\right)^x$.

   Find $f(1), f(2), f(3), f(-1), f(-2), f(-3), f(0), f\left(\frac{1}{2}\right), f\left(\frac{1}{3}\right), f\left(\frac{2}{3}\right)$.

3. On separate coordinate plane, graph the following functions, plot at least 4 points on each, and state their important features: $f(x) = 5^x, f(x) = \left(\frac{2}{5}\right)^x$.

4. On the same coordinate plane, graph the following functions, plot at least 4 points on each, and state their important features, and how they relate to each other: $f(x) = 2^x$, $g(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$.

5. Solve for the unknown variable:
   - $2^x = 8$ • $3^{2x} = 27$ • $\left(\frac{1}{2}\right)^{x+2} = 8$ • $5^{x-1} = 25$ • $2^{3x+5} = 4$

6. Graph the following functions, and state their important features.
   - $f(x) = 2(3^x) + 1$ • $f(x) = 2(3^x) - 1$ • $f(x) = -2(3^x) + 1$
   - $f(x) = -2(3^x) - 1$ • $f(x) = 3^{x+1} = 3(3^x)$.

7. Graph $y = e^{3x}$, $y = 2e^x$, and $y = e^{-x}$.

4.2 Logarithmic functions

Let $b > 0, b \neq 1$. Recall the exponential function $f(x) = b^x$ from the previous section. From its graph note that for any $a > 0$, there is a **unique** real number x such that $b^x = a$. This is because $f(x) = b^x$ is a one-to-one function. Such an x is denoted by $\log_b a$, read "logarithm of a to the base b." That is,

$$b^x = a \quad \textit{if and only if} \quad x = \log_b a.$$

It follows that

$$\log_b(b^x) = x \textit{ for any real number } x \textit{ and } b^{\log_b a} = a \textit{ for any positive real number } a$$

$\log$ **is the mechanism to identify the exponent. Compare that with the n-th root function.**

Comparison table of roots versus logarithms:

$x^4 = 81$ $x = \pm\sqrt[4]{81}$ $x = \pm 3.$	$4^x = 81$ $x = \log_4 81.$
$x^5 = 32$ $x = \sqrt[5]{32}$ $x = 2.$	$5^x = 32$ $x = \log_5 32.$
$x^5 = -32$ $x = -\sqrt[5]{32}$ $x = -2.$	$5^x = -32$ This has no solution because no power of 5 can result in a negative number.

Equivalent Exponential equations and Logarithmic equations:

Exponential equations	Logarithmic equations
$3^4 = 81$	$4 = \log_3 81$
$5^0 = 1$	$0 = \log_5 1$
$2^4 = 16$	$4 = \log_2 16$
$3^{-2} = \left(\frac{1}{9}\right)$	$-2 = \log_3 \left(\frac{1}{9}\right)$
$2^{-3} = \left(\frac{1}{8}\right)$	$-3 = \log_2 \left(\frac{1}{8}\right)$
$5^{-3} = \left(\frac{1}{125}\right)$	$-3 = \log_5 \left(\frac{1}{125}\right)$

Classroom Exercises: Complete the following table:

Exponential equations	Logarithmic equations	Exponential equations	Logarithmic equations
$4^2 = 16$		$5^3 = 125$	
$10^3 = 1000$		$2^6 = 64$	
$7^0 = 1$		$4^{-1} = \frac{1}{4}$	
$5^{-2} = \frac{1}{25}$		$3^{-4} = \frac{1}{81}$	
$7^{-2} = \frac{1}{49}$		$4^3 = 64$	
	$4 = \log_5 625$		$4 = \log_{10} 10{,}000$
	$3 = \log_2 8$		$0 = \log_8 1$
	$-3 = \log_{10} \left(\frac{1}{1000}\right)$		$-2 = \log_8 \left(\frac{1}{64}\right)$

Now we are ready to solve logarithmic equations.

Examples: Solve for x:

(1)

$$
\begin{aligned}
x &= \log_3 81 \\
3^x &= 81 && \textit{Conversion to an exponential equation}; \\
3^x &= 3^4 \\
x &= 4. && \textit{We have solved for } x.
\end{aligned}
$$

(2)

$$
\begin{aligned}
2x-1 &= \log_2 16 \\
2^{2x-1} &= 16 && \textit{Conversion to an equivalent exponential equation}; \\
2^{2x-1} &= 2^4 \\
2x-1 &= 4 \\
2x &= 15 \\
x &= \frac{15}{2}. && \textit{We have solved for } x.
\end{aligned}
$$

(3)

$$
\begin{aligned}
3 &= \log_5 x \\
5^3 &= x && \textit{Conversion to an equivalent exponential equation}; \\
125 &= x. && \textit{We have solved for } x.
\end{aligned}
$$

(4)

$$
\begin{aligned}
4 &= \log_3(2x-5) \\
3^4 &= 2x-5 && \textit{Conversion to an equivalent exponential equation}; \\
81 &= 2x-5 \\
86 &= 2x \\
43 &= x. && \textit{We have solved for } x.
\end{aligned}
$$

(5)

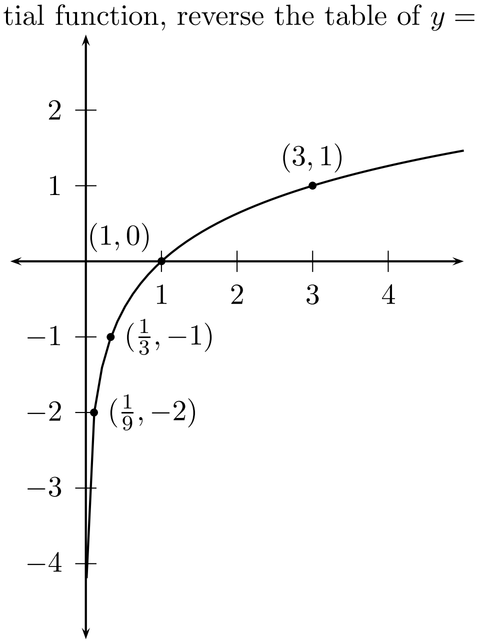
$$\begin{aligned}
-4 &= \log_3(2x-5) \\
3^{-4} &= 2x-5 && \textit{Conversion to an equivalent exponential equation;} \\
\frac{1}{81} &= 2x-5 \\
5+\frac{1}{81} &= 2x \\
\frac{406}{81} &= 2x \\
\frac{406}{81} \div 2 &= x \\
\frac{203}{81} &= x. && \textit{We have solved for } x.
\end{aligned}$$

Classroom Exercises: Solve for x:

- $x = \log_5 1$
- $3x + 5 = \log_3 81$
- $3 = \log_7 x$
- $2 = \log_4(3x + 5)$
- $-3 = \log_7 x$
- $-2 = \log_4(3x + 5)$

We are now ready to graph logarithmic equations.

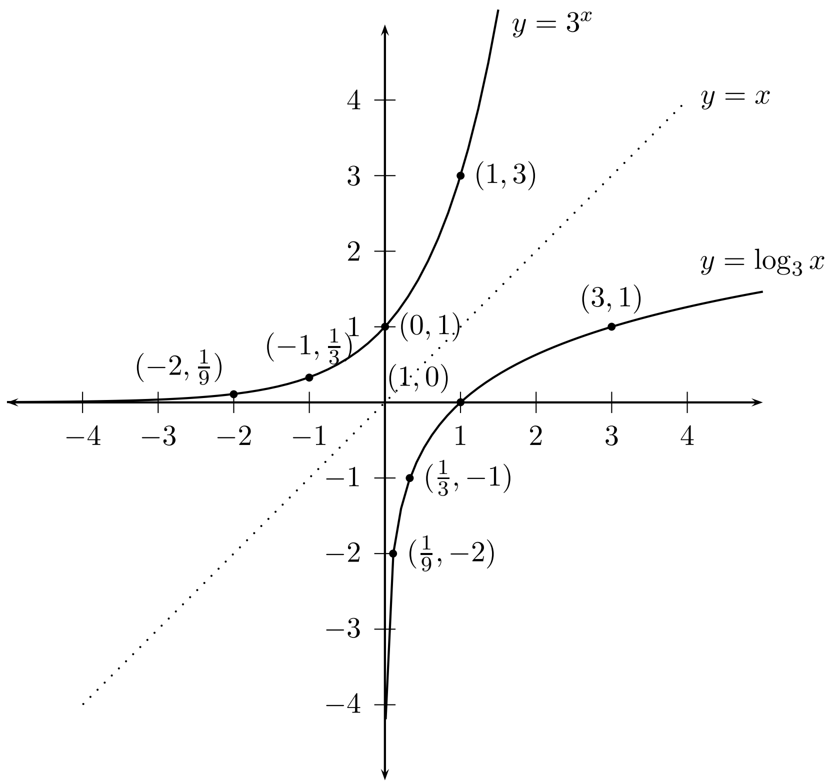
Example 1: Graph $f(x) = \log_3 x$. Since the logarithmic function is the inverse of the exponential function, reverse the table of $y = 3^x$ to get the table for $y = \log_3 x$.



x	3^x
1	3
2	9
0	1
-1	$\frac{1}{3}$
-2	$\frac{1}{9}$

x	$y = \log_3 x$
3	1
9	2
1	0
$\frac{1}{3}$	-1
$\frac{1}{9}$	-2

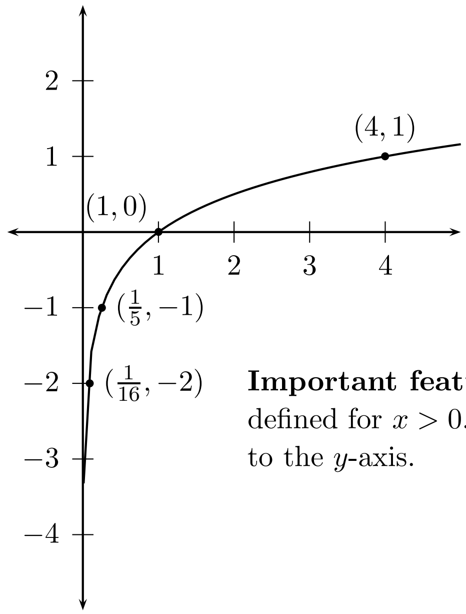
Important features: The function $f(x) = \log_3 x$ is an increasing function defined for $x > 0$. It passes through point $(1, 0)$. It is asymptotic to the y-axis. The graphs $y = 3^x$ and $y = \log_3 x$ are symmetric to each other about the line $y = x$. We see this in the following coordinate plane where we have drawn both these graphs and the line $y = x$ is shown in dotted format.



A note on the side: Let X and Y be two nonempty sets. A function $f : X \to Y$ is said to be injective if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. A function $g : Y \to X$ is said to be the inverse function of f if $g(f(x)) = x$ and $f(g(y)) = y$. Let $b > 0, b \neq 1$ be given. Note that if $f : \mathbb{R} \to (0, \infty)$ is the function $f(x) = b^x$ and $g : (0, \infty) \to \mathbb{R}$ be the function $g(y) = \log_b y$, then, f and g are inverses of each other (check for yourself).

Another example is $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = x^2$ and $g : [0, \infty) : \mathbb{R}$ given by $g(y) = \sqrt{y}$. The reader should check that the graphs of f and g are symmetric about the line $y = x$.

Example 2: Graph $f(x) = \log_4 x$. Since the logarithmic function is the inverse of the exponential function, reverse the table of $y = 4^x$ to get the table for $y = \log_4 x$.

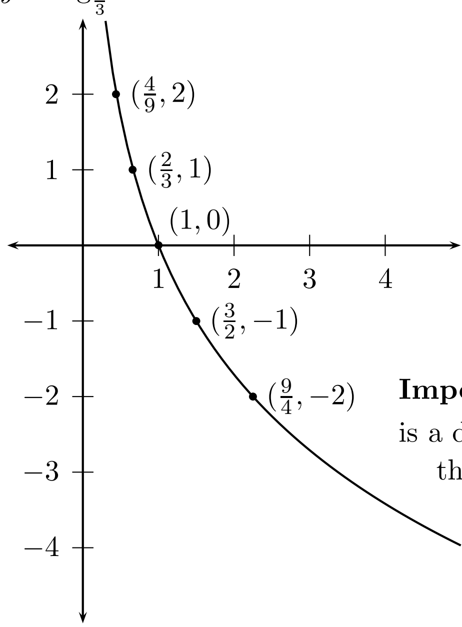


x	4^x
1	4
2	16
0	1
−1	$\frac{1}{4}$
−2	$\frac{1}{16}$

x	$y = \log_4 x$
4	1
16	2
1	0
$\frac{1}{9}$	−1
$\frac{1}{16}$	−2

Important features: The function $f(x) = \log_4 x$ is an increasing function defined for $x > 0$. It passes through point $(1, 0)$. It is asymptotic to the y-axis.

Example 3: Graph $f(x) = \log_{\frac{2}{3}} x$. Here, reverse the table of $y = \left(\frac{2}{3}\right)^x$ to get the table for $y = \log_{\frac{2}{3}} x$.



x	$\left(\frac{2}{3}\right)^x$
1	$\frac{2}{3}$
2	$\frac{4}{9}$
0	1
−1	$\frac{3}{2}$
−2	$\frac{9}{4}$

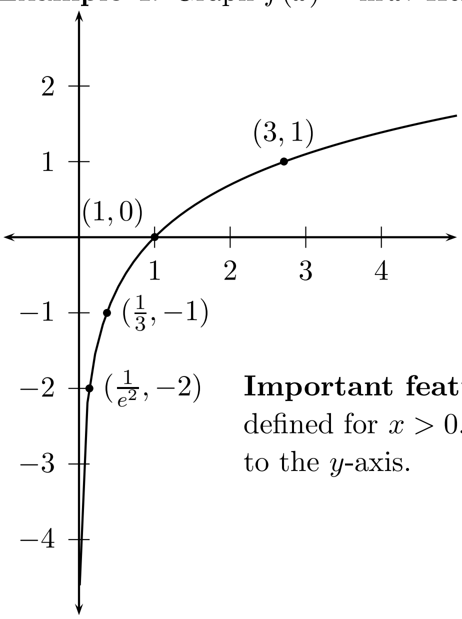
x	$y = \log_{\frac{2}{3}} x$
$\frac{2}{3}$	1
$\frac{4}{9}$	2
1	0
$\frac{3}{2}$	−1
$\frac{9}{4}$	−2

Important features: The function $f(x) = \log_{\frac{2}{3}} x$ is a decreasing function defined for $x > 0$. It passes through point $(1, 0)$. It is asymptotic to the y-axis.

Special Logarithmic functions: Recall the Euler's number e, which is an irrational number approximately equal to 2.71828. The logarithm with base e is called **natural logarithm** and $\log_e$ is denoted by ln. The logarithm with base 10 is called **common logarithm** or **decadic**

logarithm and $\log_{10}$ is simply denoted by log. In other words, if the base is not specified, the base is assumed to be 10.

Example 4: Graph $f(x) = \ln x$. Here, reverse the table of $y = e^x$ to get the table for $y = \ln x$.

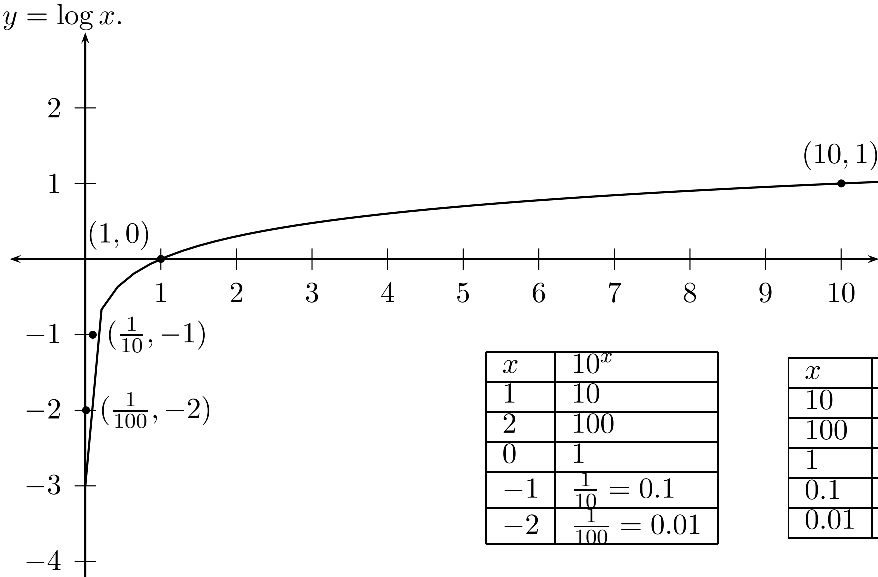


x	e^x
1	e
2	e^2
0	1
-1	$\frac{1}{e}$
-2	$\frac{1}{e^2}$

x	$y = \ln x$
e	1
e^2	2
1	0
$\frac{1}{e}$	-1
$\frac{1}{e^2}$	-2

Important features: The function $f(x) = \ln x$ is an increasing function defined for $x > 0$. It passes through point $(1, 0)$. It is asymptotic to the y-axis.

Example 5: Graph $f(x) = \log x$. Here, reverse the table of $y = 10^x$ to get the table for $y = \log x$.

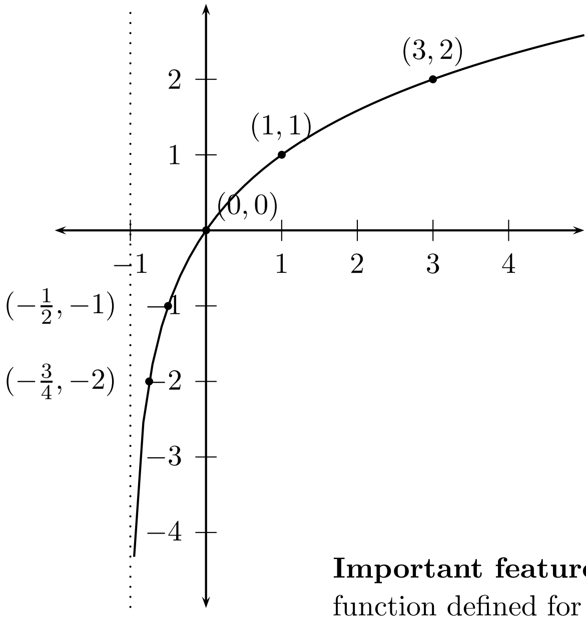


x	10^x
1	10
2	100
0	1
-1	$\frac{1}{10} = 0.1$
-2	$\frac{1}{100} = 0.01$

x	$y = \log x$
10	1
100	2
1	0
0.1	-1
0.01	-2

Important features: The function $f(x) = \log x$ is an increasing function defined for $x > 0$. It passes through point $(1, 0)$. It is asymptotic to the y-axis.

Example 7: Graph $f(x) = \log_2(x+1)$. This is a tricky example. We need to reverse the table for $y = 2^x$ and then **shift** the x values to obtain the table for $y = \log_2(x+1)$.



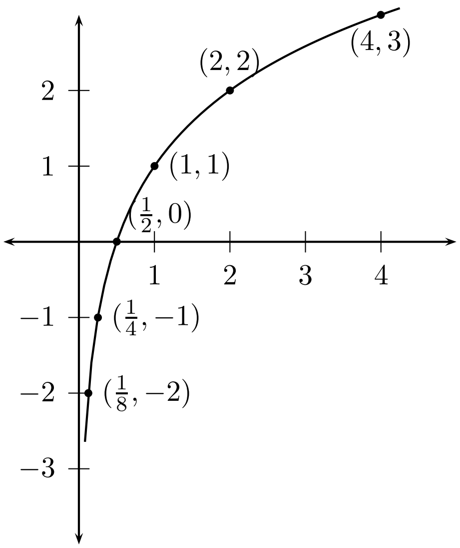
x	2^x
1	2
2	4
0	1
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$

x	$y = \log_2 x$
2	1
4	2
1	0
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2

x	$x+1$	$y = \log_2(x+1)$
1	2	1
3	4	2
0	1	0
$-\frac{1}{2}$	$\frac{1}{2}$	-1
$-\frac{3}{4}$	$\frac{1}{4}$	-2

Important features: The function $f(x) = \log_2(x+1)$ is an increasing function defined for $x > -1$. It passes through point $(0, 0)$.
It is asymptotic to the the line $x = -1$.

Example 8: Graph $f(x) = \log_2 x + 1$. This is different from the previous example. We need to reverse the table for $y = 2^x$ and then **shift** the y values to obtain the table for $y = \log_2 x + 1$.



x	2^x
1	2
2	4
0	1
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$
-3	$\frac{1}{8}$

x	$y = \log_2 x$
2	1
4	2
1	0
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2
$\frac{1}{8}$	-3

x	$\log_2 x$	$y = \log_2 x + 1$
2	1	2
4	2	3
1	0	1
$\frac{1}{2}$	-1	0
$\frac{1}{4}$	-2	-1
$\frac{1}{8}$	-3	-2

Important features: The function $f(x) = \log_2 x + 1$ is an increasing function defined for $x > 0$.

It passes through point $(\frac{1}{2}, 0)$. It is asymptotic to the the y-axis.
Classroom Exercises: Graph the following functions:

- $f(x) = \log_2 x$
- $f(x) = \log_7 x$
- $f(x) = \log_{\frac{3}{5}} x$
- $f(x) = \log_4(x+1)$
- $f(x) = \log_4 x - 1$
- $f(x) = \ln x - 1$
- $f(x) = \log x + 1$.

4.2.1 Homework Exercises:

(1) Complete the following table:

Exponential equations	Logarithmic equations	Exponential equations	Logarithmic equations
$2^4 = 16$		$3^5 = 243$	
$4^3 = 64$		$5^2 = 25$	
$6^0 = 1$		$6^{-2} = \dfrac{1}{36}$	
$7^{-3} = \dfrac{1}{343}$		$3^{-1} = \dfrac{1}{3}$	
$10^{-5} = \dfrac{1}{10,000}$		$3 = \log_2 8$	
	$2 = \log_3 9$		$7 = \log_2 128$
	$1 = \log_5 5$		$0 = \log_9 1$
	$-2 = \log_{11}\left(\dfrac{1}{121}\right)$		$-3 = \log_9\left(\dfrac{1}{729}\right)$

(2) Solve for x:

- $x = \log_4 16$
- $2x - 7 = \log_3 9$
- $2 = \log_5(3x-7)$
- $1 = \log_4(4x+3)$
- $-2 = \log_5(3x-7)$
- $-1 = \log_4(4x+3)$

(3) Graph the following functions:

- $f(x) = \log_5 x$
- $f(x) = \log_6 x$
- $f(x) = \log_{\frac{2}{3}} x$
- $f(x) = \log_3(x-1)$
- $f(x) = \log_2 x - 1$
- $f(x) = \ln x + 1$
- $f(x) = \log x - 1$.

Appendix A

Polynomials

A.1 Multiplication of polynomials

Here we will first learn to multiply two polynomials. In this process we will use the distributive laws, and then combine like terms, if any.

Examples :

- $3xyz \cdot 5x^2y^3z = 15x^3y^4z^2$ *(Multiply the coefficients and the variables separately)*
- $\frac{2}{3}yz \cdot 5x^2y^7 = \frac{10}{3}x^2y^8z = 3\frac{1}{3}x^2y^8z$ *(Multiply the coefficients and the variables separately)*
- $4(3x^2 + 5x - 7)$ *(Use distributive law)*
  $= 12x^2 + 20x - 28$ *(This is the final answer.)*
- $\frac{1}{4}x^3(3x^2 + 5y - 7)$ *(Use distributive law)*
  $= \left(\frac{1}{4}x^3 \cdot 3x^2\right) + \left(\frac{1}{4}x^3 \cdot 5y\right) - \left(\frac{1}{4}x^3 \cdot 7\right)$
  $= \frac{3}{4}x^5 + \frac{5}{4}x^3y - \frac{7}{4}x^3$
  $= \frac{3}{4}x^5 + \left(1\frac{1}{4}\right)x^3y - \left(1\frac{3}{4}\right)x^3$ *(This is the final answer.)*
- $(4x^2 - 3y)(3x^2 + 5y)$ *(Use distributive law)*
  $= (4x^2 - 3y) \cdot 3x^2 + (4x^2 - 3y) \cdot 5y$ *(Use distributive law again)*
  $= 12x^4 - 9x^2y + 20x^2y - 15y^2$ *(Combine the like terms $-9x^2y + 20x^2y$)*
  $= 12x^4 + 11x^2y - 15y^2$ *(This is the final answer.)*

- $(4x^2+3xy-5y^2)(5x^2-y^2+3)$ *(Use distributive law)*
$= (4x^2+3xy-5y^2)\cdot 5x^2 - (4x^2+3xy-5y^2)\cdot y^2 + (4x^2+3xy-5y^2)\cdot 3$
(Use distributive law again)
$= 20x^4+15x^3y-25x^2y^2-(4x^2y^2+3xy^3-5y^4)+12x^2+9xy-15y^2$
(We pay careful attention to the negative)
$= 20x^4+15x^3y-25x^2y^2-4x^2y^2-3xy^3+5y^4+12x^2+9xy-15y^2$
(Now combine like terms)
$= 20x^4+15x^3y-29x^2y^2-3xy^3+5y^4+12x^2+9xy-15y^2$
(This is the final answer.)

- $\left(\frac{2}{5}x^2-\frac{1}{3}x+y\right)(-2y-3x+4)$ *(Use distributive law)*
$= -\left(\frac{2}{5}x^2-\frac{1}{3}x+y\right)\cdot 2y - \left(\frac{2}{5}x^2-\frac{1}{3}x+y\right)\cdot 3x + \left(\frac{2}{5}x^2-\frac{1}{3}x+y\right)\cdot 4$
(Use distributive law again)
$= -\left(\frac{4}{5}x^2y-\frac{2}{3}xy+2y^2\right) - \left(\frac{6}{5}x^3-\frac{3}{3}x^2+3xy\right) + \frac{8}{5}x^2-\frac{4}{3}x+4y$
(We pay careful attention to the negatives)
$= -\frac{4}{5}x^2y+\frac{2}{3}xy-2y^2-\frac{6}{5}x^3+\frac{3}{3}x^2-3xy+\frac{8}{5}x^2-\frac{4}{3}x+y$ *(Combine like terms)*
$= -\frac{4}{5}x^2y+\left(\frac{2}{3}-3\right)xy-2y^2-\frac{6}{5}x^3+\left(\frac{3}{3}+\frac{8}{5}\right)x^2-\frac{4}{3}x+4y$
$= -\frac{4}{5}x^2y-\left(2\frac{1}{3}\right)xy-2y^2-\left(1\frac{1}{5}\right)x^3+\left(2\frac{3}{5}\right)x^2-\left(1\frac{1}{3}\right)x+4y$ *(The answer.)*

Note, $\left(\frac{2}{3}-3\right)=-2\frac{1}{3}$, $\left(\frac{3}{3}+\frac{8}{5}\right)=\left(1+1\frac{3}{5}\right)=2\frac{3}{5}$, *and* $\frac{4}{3}=1\frac{1}{3}$.

Some important formulae :

- Square of a sum

$(x+y)^2 = (x+y)(x+y)$
$= (x+y)\cdot x+(x+y)\cdot y$ *(Use distributive law)*
$= x^2+xy+xy+y^2$ *(Use distributive law again)*
$= x^2+2xy+y^2$ *(Combining like terms gets us the final answer.)*

$(x+y)^2 = x^2+2xy+y^2$

- Square of a difference

$$\begin{aligned}(x-y)^2 &= (x-y)(x-y) \\ &= (x-y)\cdot x-(x-y)\cdot y && \textit{(Use distributive law)} \\ &= x^2-xy-(xy-y^2) && \textit{(Use distributive law again)} \\ &= x^2-xy-xy+y^2 && \textit{(Pay careful attention to the negative)} \\ &= x^2-2xy+y^2 && \textit{(Combining like terms gets us the final answer.)} \\ (x-y)^2 &= x^2-2xy+y^2\end{aligned}$$

- Difference of squares

$$\begin{aligned}(x+y)(x-y) &= (x+y)\cdot x-(x+y)\cdot y && \textit{(Use distributive law)} \\ &= x^2+xy-(xy+y^2) && \textit{(Use distributive law again)} \\ &= x^2+xy-xy-y^2 && \textit{(Pay careful attention to the negative)} \\ &= x^2-y^2 && \textit{(Combining like terms gets us the final answer.)} \\ (x+y)(x-y) &= x^2-y^2\end{aligned}$$

Classroom Exercises : Perform the following multiplications:

- $2x^3y^2\cdot 5xyz$
- $\frac{3}{5}abc\cdot\frac{2}{7}a^2bcd$
- $2(5x^2-6x+2)$
- $-4(3x^2+7x-5)$
- $-2x(3x^2+7x-5)$
- $\frac{1}{3}(2y^2+5y-3)$
- $\frac{2}{3}y^3(2y^2+5y-3)$
- $(3x-2y)(x^2-2xy+3y)$
- $\left(\frac{2}{3}y+5\right)(4xy+x+y)$
- $(x-y)(x^2+xy+y^2)$ (Difference of cubes formula)
- $(x+y)(x^2-xy+y^2)$ (Sum of cubes formula)

A.1.1 Homework Exercises

Find the following products:

1. $3x^2y^9z^3 \cdot 5x^2y^2z^8$
2. $-7a^2b^7 \cdot 4a^7b^9$
3. $-3(5y^2 + 3y - 7)$
4. $\frac{2}{5}\left(2x^2 - 3x + 5\right)$
5. $-\frac{1}{2}n(4m^2 + 7mn)$
6. $\frac{2}{3}y(5y^3 + 4y^2 - 6y + 3)$
7. $\left(\frac{2}{3}x + y\right)(3x^2 + 4xy - y^2)$
8. $\left(\frac{1}{5}x + \frac{2}{3}y\right)(2x + 7y)$
9. (Square of a sum) $(a + b)^2$
10. (Square of a difference) $(a - b)^2$
11. (Difference of squares) $(a + b)(a - b)$
12. (Difference of cubes) $(a - b)(a^2 + ab + b^2)$
13. (Sum of cubes) $(a + b)(a^2 - ab + b^2)$
14. (Cube of a sum) $(a + b)^3$
15. (Cube of a difference) $(a - b)^3$

A.2 Division by a monomial

Here we learn to divide a polynomial by a monomial. Recall that division by a number is multiplication by its reciprocal. In the same way, division by a monomial is multiplication by its reciprocal.
Examples :

- $$\begin{aligned} 12a^4b^3c^5 \div (3a^2b^3c) &= 12a^4b^3c^5 \times \frac{1}{3a^2b^3c} && \textit{(Division is multiplication by the reciprocal)} \\ &= \frac{12a^4b^3c^5}{3a^2b^3c} \\ &= 4a^2c^4 && \textit{(This is the final answer.)} \end{aligned}$$

- $$\begin{aligned} (a+b+c) \div (3abc) &= (a+b+c) \times \frac{1}{3abc} && \textit{(Division is multiplication by the reciprocal)} \\ &= a \cdot \frac{1}{3abc} + b \cdot \frac{1}{3abc} + c \cdot \frac{1}{3abc} && \textit{(Use distributive law)} \\ &= \frac{a}{3abc} + \frac{b}{3abc} + \frac{c}{3abc} \\ &= \frac{1}{3bc} + \frac{1}{3ac} + \frac{1}{3ab} && \textit{(This is the final answer.)} \end{aligned}$$

- $$\begin{aligned} (2x^2y + 3xy^2 - 4xy) \div (6xy) &= (2x^2y + 3xy^2 - 4xy) \times \frac{1}{6xy} \\ & \qquad \textit{(Division is multiplication by the reciprocal)} \\ &= 2x^2y \cdot \frac{1}{6xy} + 3xy^2 \cdot \frac{1}{6xy} - 4xy \cdot \frac{1}{6xy} && \textit{(Use distributive law)} \\ &= \frac{2x^2y}{6xy} + \frac{3xy^2}{6xy} - \frac{4xy}{6xy} \\ &= \frac{x}{3} + \frac{y}{2} - \frac{2}{3} && \textit{(This is the final answer.)} \end{aligned}$$

Classroom Exercises : Perform the following divisions:

- $25a^9b^3c^4 \div (10a^7b^5c^6)$
- $(4x^2y^3z + 3x^3y^3z - 7xyz) \div (5xyz)$
- $(28r^3s^2t - 49r^5s^4t^2 - 2r^2s^2t^2) \div (14r^2st)$

A.2.1 Homework Exercises

Perform the following divisions:

1. $75a^6b^5c^4 \div (20a^7b^4c^6)$
2. $44x^8y^3r^9 \div (26x^3y^9r^{10})$
3. $(4x^2y^3z + 3x^3y^3z - 7xyz) \div (5xyz)$

4. $(8a^2b^5c^3 + 6a^2b^3c^2 - 3abc^2) \div (6ab)$

5. $(3r^4s^3t^2 - 4r^2s^3t^2 + 5rs^4t) \div (12s^2t)$

6. $(18x^3y^4a^5 - 33x^3ya^2 + 12x^3ya) \div (15x^2y)$

7. $(26a^2b^3 + 24a^5b^2c^3 - 8a^3b^2c) \div (13a^2)$

8. $(22x^2 - 18x^2y) \div (6x)$

A.3 Factoring polynomials

When we write $2 \times 3 = 6$, we say that the product of 2 and 3 is 6, and that 2 and 3 are factors of 6. A prime number is a natural number other than 1 whose only factors are itself and 1.

Analogously, when $(x + y)(x - y) = x^2 - y^2$, we say that the product of $(x + y)$ and $(x - y)$ is $x^2 - y^2$, and that $(x + y)$ and $(x - y)$ are factors of $x^2 - y^2$. An irreducible polynomial is a polynomial other than the constant polynomial 1, whose only factors are itself and 1.

To factor a polynomial is to write it as a product of irreducible polynomials. For this we go through several steps. The reader is urged to check every step before moving on to the next one.

A.3.1 The Greatest Common Factor (GCF)

Recall from your previous classes, the concept of the Greatest Common Factor, or GCF, of a list of natural numbers. It is the greatest natural number which is a factor of every number in the list.

Examples :

- GCF(12,33,18) = 3.
- GCF(15,10,20) = 5.
- GCF(44,6,12) = 2.
- GCF(12,18,30)= 6.

Classroom Exercises : What is the GCF in each of the cases?

- $\{12, 15, 18\}$
- $\{15, 30, 50\}$
- $\{12, 25\}$

The **Greatest Common Factor (GCF)** of two or more monomials with integer coefficients is the product of

- the GCF of the coefficients, and
- the highest power of every variable that is a factor of every monomial.

Examples :

- GCF$(12x^2y^3, 33x^4y^2, 18x^3y^3) = 3x^2y^2$. Note that GCF$(12, 33, 18) = 3$. Further, x^2 is the highest power of x that is a factor of x^2y^3, x^4y^2, x^3y^3; similarly, y^2 is the highest power of y that is a factor of x^2y^3, x^4y^2, x^3y^3.
- GCF$(15a^4b^7, 10a^3b^9, 20a^6b^4) = 5a^3b^4$. Again, the GCF$(15, 10, 20) = 5$, while a^3 is the highest power of a that is a factor of every monomial, and b^4 is the highest power of b that is a factor of every monomial.
- GCF$(44r^3s^4t^2, 6r^5s^3t^7, 12r^4s^3t^5) = 2r^3s^3t^2$.

Classroom Exercises : What is the GCF in each of the cases?

- $\{12a^3b^4c^2, 15a^5b^3c^5, 18a^3b^2c^7\}$
- $\{15x^3, 30x^7, 50x^4\}$
- $\{12r^9s^4t^4, 25r^4t\}$

Now we are ready to factor. The very first step in factoring any polynomial is to factor out the GCF of its terms using the distributive law of multiplication over addition/subtraction.

Examples : Factor the following:

- $12x^2y^3+33x^4y^2+18x^3y^3 = 3x^2y^2(4y+11x^2+6xy)$. Note that we have used the distributive law of multiplication over addition. To factor $(4y + 11x^2 + 6xy)$, if possible, is beyond the scope of our book.
- $15a^4b^7 - 10a^3b^9 - 20a^6b^4 = 5a^3b^4(3ab^3 - 2ab^5 - 4a^3)$.
- $-44r^3s^4t^2 + 6r^5s^3t^7 - 12r^4s^3t^5 = 2r^3s^3t^2(-22rs + 3r^2t^5 - 6rt^3)$.
- $2x^2 + 10x - 12 = 2(x^2 + 5x - 6)$. In this case, the second factor $(x^2 + 5x - 6)$ can be factored further, which we will see in a short while.

Classroom Exercises : Factor the following:

- $12a^3b^4c^2 + 15a^5b^3c^5 - 18a^3b^2c^7$
- $-15x^3 + 30x^7 - 50x^4$
- $12r^9s^4t^4 + 25r^4t$
- $27x^2 - 3y^2$

A.3.2 The Grouping method

This method is typically applied when the polynomial to be factored has four terms. Given any polynomial, we first check whether any GCF can be factored out. Next, if it is a four-term polynomial, we check the grouping method.

Examples :

- $2ab + 6ac + b^2 + 3bc$ *(The GCF here is 1. So we proceed.)*
  $= \underbrace{2ab + 6ac} + \underbrace{b^2 + 3bc}$ *(Group the first two terms, and the last two terms.)*
  $= 2a(b + 3c) + b(b + 3c)$ *(Factor out the GCF from each group.)*
  $= (2a + b)(b + 3c)$ *(Factor out $(b + 3c)$ to get the final answer.)*

- $18px^2 - 9x^2q + 6xyp - 3xyq$ *(The GCF here is $3x$, which we first factor.)*
  $= 3x(6px - 3xq + 2yp - yq)$
  $= 3x(\underbrace{6px - 3xq} + \underbrace{2yp - yq})$ *(Group the first two terms, and the last two terms.)*
  $= 3x\,(3x(2p - q) + y(2p - q))$ *(Factor out the GCF from each group.)*
  $= 3x(3x + y)(2p - q)$ *(Factor out $(2p - q)$ to get the final answer.)*

- $12x^2ab + 9xab - 8xyab - 6yab$ *(The GCF is ab, which we factor.)*
  $= ab(12x^2 + 9x - 8xy - 6y)$
  $= ab(\underbrace{12x^2 + 9x} \underbrace{-8xy - 6y})$ *(Group the relevant terms.)*
  (Keep track of the negatives.)
  $= ab\,(3x(4x + 3) - 2y(4x + 3))$ *(Factor out the GCF from each group.)*
  $= ab(3x - 2y)(4x + 3)$ *(Factor out $(4x + 3)$ to get the final answer.)*

- $20x^4y^2 - 70x^3y^2 - 12x^3y^3 + 42x^2y^3$ *(The GCF is $2x^2y^2$, which we factor.)*
  $= 2x^2y^2(10x^2 - 35x - 6xy + 21y)$
  $= 2x^2y^2(\underbrace{10x^2 - 35x} \underbrace{-6xy + 21y})$ *(Group the relevant terms.)*
  (Keep track of the negatives.)
  $= 2x^2y^2\,(5x(2x - 7) - 3y(2x - 7))$ *(Factor out the GCF from each group.)*
  $= 2x^2y^2(5x - 3y)(2x - 7)$ *(Factor out $(2x - 7)$ to get the final answer.)*

In some cases, we will have to rearrange the terms to be able to use this method. The following example illustrates this.

- $28x^2 - 3y + 7xy - 12x$ *(The GCF here is 1. So we proceed.)*
  $= \underbrace{28x^2 - 3y} + \underbrace{7xy - 12x}$ *(Notice that these groups do not lead to a factorization.)* *(We need to rearrange the terms.)*
  $= 28x^2 - 12x + 7xy - 3y$
  $= \underbrace{28x^2 - 12x} + \underbrace{7xy - 3y}$
  $= 4x(7x - 3) + y(7x - 3)$
  $= (4x + y)(7x - 3)$ *(The final answer.)*

The grouping method may not work. The following example illustrates this.

- $28x^2 - 4x + 7xy - 3y$ *(The GCF here is 1. So we proceed.)*
  $= \underbrace{28x^2 - 4x} + \underbrace{7xy - 3y}$
  $= 4x(28x^2 - 4x) + 7xy - 3y$ *(This does not lead us to a factorization.)* *(Rearranging also does not lead us to a factorization.)*

Classroom Exercises : Factor the following by first checking for the GCF, and then using the grouping procedure:

- $2pr - ps + 2qr - qs$
- $6ac + 18a - 3bc - 9b$
- $5x^2 - 15x + 4xy - 12y$
- $15r^4s^3 - 45r^3s^2 - 6r^3s^4 + 18r^2s^3$
- $16abxy + 24axy - 4b^2xy - 6bxy$
- $4xy - 15x + 6x^2 - 10y$ (This will require a rearrangement of the terms).

A.3.3 The Standard Formulae

Given a polynomial, so far, we have learnt to factor out the GCF, and then use the grouping method whenever relevant. Now we use the standard formulae we had encountered earlier in the chaper.

Difference of squares :

$(a + b)(a - b) = (a + b) \cdot a - (a + b) \cdot b$ *(Use distributive law)*
$= a^2 + ab - (ab + b^2)$ *(Keep track of the negative)*
$= a^2 + ab - ab - b^2$
$= a^2 - b^2$
$(a + b)(a - b) = a^2 - b^2$ *(The difference of squares)*

Examples : Factor the following:

- $x^2 - y^2$ *(GCF =1; grouping is not relevant.)*
  (Proceed with standard formulae.)
  $= (x+y)(x-y)$ *(The difference of squares.)*

- $4r^2 - 9$ *(GCF =1; grouping is not relevant.)*
  (Proceed with standard formulae.)
  $= (2r)^2 - (3)^2$
  $= (2r+3)(2r-3)$ *(The difference of squares.)*

- $25x^2y^2 - 36a^2b^2$ *(GCF =1; grouping is not relevant.)*
  (Proceed with standard formulae.)
  $= (5xy)^2 - (6ab)^2$
  $= (5xy+6ab)(5xy-6ab)$ *(The difference of squares.)*

- $12a^3b - 27ab^3$ *(Factor out the GCF = 3ab.)*
  $= 3ab(4a^2 - 9b^2)$ *(Grouping is not relevant.)*
  $= 3ab\left((2a)^2 - (3b)^2\right)$ *(Proceed with standard formulae.)*
  $= 3ab(2a+3b)(2a-3b)$ *(The difference of squares.)*

- $4x^4 - 400y^6$ *(Factor out the GCF = 4.)*
  $= 4(x^4 - 100y^6)$ *(Grouping is not relevant.)*
  $= 4\left((x^2)^2 - (10y^3)^2\right)$ *(Proceed with standard formulae.)*
  $= 4(x^2+10y^3)(x^2-10y^3)$ *(The difference of squares.)*

- $3x^2 - y^2$ *(GCF =1; grouping is not relevant.)*
  (Proceed with standard formulae.)
  $= (\sqrt{3}x)^2 - (y)^2$
  $= (\sqrt{3}x+y)(\sqrt{3}x-y)$ *(The difference of squares.)*

- $7m^2 - 5n^2$ *(GCF =1; grouping is not relevant.)*
  (Proceed with standard formulae.)
  $= (\sqrt{7}m)^2 - (\sqrt{5}n)^2$
  $= (\sqrt{7}m+\sqrt{5}n)(\sqrt{7}m-\sqrt{5}n)$ *(The difference of squares.)*

Classroom Exercises : Factor the following, by first checking for the GCF, then checking for the relevance of grouping method, and lastly, the standard formulae:

- $p^2 - 9$
- $9m^2 - 16$
- $49p^2q^2 - 25r^2t^2$
- $8x^2 - 18y^2$
- $3a^4 - 12a^2b^2$
- $m^2 - 2n^2$
- $6p^2 - 10q^2$

Difference of cubes Consider the following product:

$$\begin{aligned}
(a-b)(a^2+ab+b^2) &= (a-b)\cdot a^2 + (a-b)\cdot ab + (a-b)\cdot b^2 && \textit{(Use distributive law)}\\
&= a^3 - a^2b + a^2b - ab^2 + ab^2 - b^3 && \textit{(Use distributive law again)}\\
&= a^3 - b^3 && \textit{(Combine like terms)}\\
(a-b)(a^2+ab+b^2) &= a^3 - b^3 && \textit{(Difference of cubes.)}
\end{aligned}$$

Sum of cubes Consider the following product:

$$\begin{aligned}
(a+b)(a^2-ab+b^2) &= (a+b)\cdot a^2 - (a+b)\cdot ab + (a+b)\cdot b^2 && \textit{(Use distributive law)}\\
&= a^3 + a^2b - (a^2b + ab^2) + ab^2 + b^3 && \textit{(Use distributive law again)}\\
&= a^3 + a^2b - a^2b - ab^2 + ab^2 + b^3 && \textit{(Keep track of the negative)}\\
&= a^3 + b^3 && \textit{(Combine like terms)}\\
(a+b)(a^2-ab+b^2) &= a^3 + b^3 && \textit{(Sum of cubes.)}
\end{aligned}$$

Examples :

- $t^3 - 1$ *(GCF=1; grouping is irrelevant;)*
  (This is not a difference of squares.)

  $= t^3 - 1^3$
  $= (t-1)(t^2+t+1)$ *(Difference of cubes.)*

- $t^3 + 1$ *(GCF=1; grouping is irrelevant;)*
  (This is not a difference of squares or cubes.)

  $= t^3 + 1^3$
  $= (t+1)(t^2-t+1)$ *(Sum of cubes.)*

- $2a^3 - 16$ *(GCF=2, factor it; grouping is irrelevant;)*
  $= 2(a^3 - 8)$ *(This is not a difference of squares.)*
  $= 2(a^3 - 2^3)$
  $= 2(a - 2)(a^2 + 2a + 4)$ *(Difference of cubes.)*

- $x^3 + 27y^3$ *(GCF=1; grouping is irrelevant;)*
  (This is not a difference of squares or cubes.)
  $= x^3 + (3y)^3$
  $= (x + 3y)(x^2 - 3xy + 9y^2)$ *(Sum of cubes.)*

Classroom Exercises : Factor, by checking for the GCF, then grouping method, then standard formulae, in that order.

- $x^3 - 27$
- $a^3 + 8$
- $2r^3 - 2$
- $2r^3 + 16$

A.3.4 Monic Quadratics in one variable

Recall that a trinomial is a polynomial with three nonzero terms. A quadratic polynomial in one variable x is a trinomial and can be written in the form

$$ax^2 + bx + c \qquad \textit{for real numbers } a, b, c \textit{ with } a \neq 0.$$

When the real number $a = 1$, the trinomial is called monic. That is, a monic quadratic polynomial takes the form

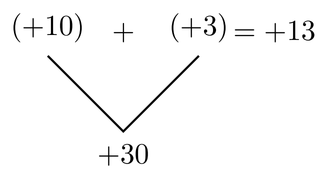
$$x^2 + bx + c \qquad \textit{for real numbers } b, c.$$

In this section we will assume that b and c are integers, and factor them. In some cases the factorization is possible only with complex numbers. We will not address such problems.
Factoring a monic quadratic in one variable $x^2 + bx + c$ requires finding two factors of c, such that their sum is b. That is, we look for numbers whose product is c and their sum is c. We will now see some examples.

The reader is **urged** to provide detailed work. Do not merely write down the final answer. In this method we convert the given trinomial to a four-term polynomial and then use the grouping procedure.

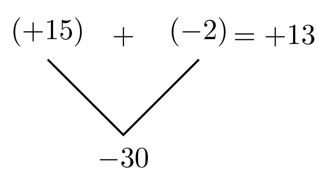
Examples :

- $x^2 + 13x + 30$ *(GCF=1; standard formulae are irrelevant.)*
  $= x^2 + 10x + 3x + 30$ *(Grouping method is now used.)*
  $= \underbrace{x^2 + 10x} + \underbrace{3x + 30}$
  $= x(x + 10) + 3(x + 10)$
  $= (x + 3)(x + 10)$ *(The final answer.)*

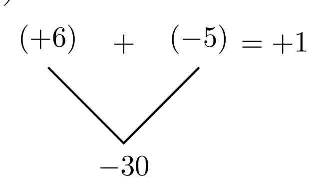


Note that the signs of the coefficients are extremely important.

- $x^2 + 13x - 30$ *(GCF=1; standard formulae are irrelevant.)*
  $= x^2 + 15x - 2x - 30$ *(Grouping method is now used.)*
  $= \underbrace{x^2 + 15x}\underbrace{-2x - 30}$
  $= x(x + 15) - 2(x + 15)$
  $= (x - 2)(x + 15)$ *(The final answer.)*

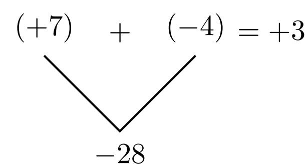


- $x^2 + x - 30$ *(GCF=1; standard formulae are irrelevant.)*
  $= x^2 + 6x - 5x - 30$ *(Grouping method is now used.)*
  $= \underbrace{x^2 + 6x}\underbrace{-5x - 30}$
  $= x(x + 6) - 5(x + 6)$
  $= (x - 5)(x + 6)$ *(The final answer.)*

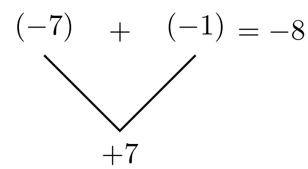


Here is a generalization for factoring a monic trinomial in two variables.

- $a^2 + 3ab - 28b^2$ *(GCF=1; standard formulae are irrelevant.)*
  $= a^2 + 7ab - 4ab - 28b^2$ *(Grouping method is now used.)*
  $= \underbrace{a^2 + 7ab}\underbrace{-4ab - 28b^2}$
  $= a(a + 7b) - 4b(a + 7b)$
  $= (a - 4b)(a + 7b)$ *(The final answer.)*

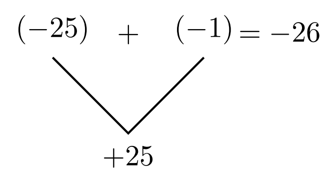


- $3x^2 - 24x + 21$ *(Factor the GCF = 3.)*
  $= 3(x^2 - 8x + 7)$ *(Standard formulae are irrelevant.)*
  $= 3(x^2 - 7x - 1x + 7)$ *(Grouping method is now used.)*
  $= 3(\underbrace{x^2 - 7x}\underbrace{-1x + 7})$
  $= 3(x(x - 7) - 1(x - 7))$
  $= 3(x - 1)(x - 7)$ *(The final answer.)*



Here is a generalization for factoring a monic trinomial of higher degree.

- $x^4 - 26x^2 + 25$ *(GCF=1; standard formulae are irrelevant.)*
  $= x^4 - 25x^2 - 1x^2 + 25$ *(Grouping method is now used.)*
  $= \underbrace{x^4 - 25x^2}\underbrace{-1x^2 + 25}$
  $= x^2(x^2 - 25) - 1(x^2 - 25)$
  $= (x^2 - 1)(x^2 - 25)$ *(Now, difference of squares.)*
  $= (x + 1)(x - 1)(x + 5)(x - 5)$ *(The final answer.)*



Classroom Exercises : Factor the following trinomials. Provide detailed work.

- $x^2 + 2x + 1$
- $x^2 - 4x - 21$
- $x^2 + 22x + 21$
- $p^2 + 4pq - 96q^2$
- $2a^2 - 10ab - 72b^2$
- $x^4 - 5x^2 + 4$

A.3.5 Non-monic Quadratics in one variable

A non-monic quadratic in one variable, x, can be written in the form

$$ax^2 + bx + c \qquad \textit{for real numbers } a, b, c, a \neq 0, a \neq 1.$$

In this section, we will consider non-monic quadratics of the form

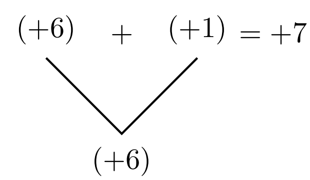
$$ax^2 + bx + c \qquad \textit{for integers } a, b, c, a \neq 0, a \neq 1.$$

To factor a non-monic quadratic, we use a procedure called the ac-**method**. Sometimes the coefficients are very large. In such cases, we use the ac-method along with a small trick.

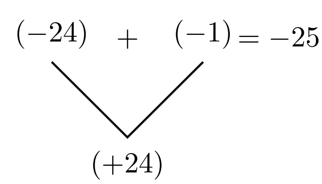
Small coefficients : First we learn the ac-method for small coefficients. We look for factors of ac whose sum is b. The student is **cautioned** against using guessing as a procedure. A carefully detailed procedure is guaranteed to give you correct answers, which guessing cannot.

Examples :

- $2x^2 + 7x + 3$ *(GCF=1; standard formulae are irrelevant.)*
  $a = 2, b = 7, c = 3$ ($ac = 2 \cdot 3 = 6$. *Factor* 6.)
  $= 2x^2 + 6x + 1x + 3$ *(Use grouping method.)*
  $= \underbrace{2x^2 + 6x} + \underbrace{1x + 3}$
  $= 2x(x + 3) + 1(x + 3)$
  $= (2x + 1)(x + 3)$ *(The final answer.)*

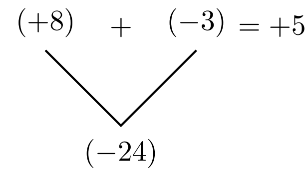


- $3x^2 - 25x + 8$ *(GCF=1; standard formulae are irrelevant.)*
  $a = 3, b = -25, c = 8$ ($ac = 3 \cdot 8 = 24$. *Factor* 24.)
  $= 3x^2 - 24x - 1x + 8$ *(Use grouping method.)*
  $= \underbrace{3x^2 - 24x} \underbrace{-1x + 8}$
  $= 3x(x - 8) - 1(x - 8)$
  $= (3x - 1)(x - 8)$ *(The final answer.)*



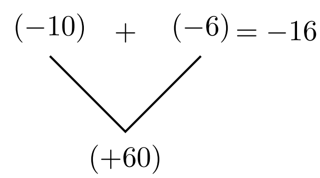
Here is a generalization for factoring a trinomial in higher degree.

- $12x^4 + 15x^3 - 18x^2$ *(Factor the GCF=$3x^2$; standard formulae are irrelevant.)*
  $= 3x^2(4x^2 + 5x - 6)$ $a = 4, b = 5, c = -6$ $(ac = 4 \cdot (-6) = -24.$ *Factor* $-24.)$
  $= 3x^2(4x^2 + 8x - 3x - 6)$ *(Use grouping method.)*
  $= 3x^2(\underbrace{4x^2 + 8x}\underbrace{-3x - 6})$
  $= 3x^2\left(4x(x+2) - 3(x+2)\right)$
  $= 3x^2(4x - 3)(x + 2)$ *(The final answer.)*



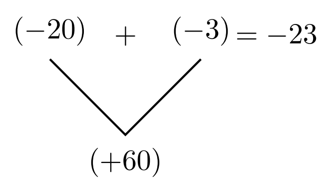
Here is a generalization for factoring a trinomial in two variables.

- $8x^3y - 32x^2y^2 + 30xy^3$ *(Factor out GCF=$2xy$; standard formulae are irrelevant.)*
  $= 2xy(4x^2 - 16xy + 15y^2)$ $a = 4, b = -16, c = 15$ $(ac = 4 \cdot 15 = 60.$ *Factor* $60.)$
  $= 2xy(4x^2 - 10xy - 6xy + 15y^2)$ *(Use grouping method.)*
  $= 2xy(\underbrace{4x^2 - 10xy}\underbrace{-6xy + 15y^2})$
  $= 2xy\left(2x(2x - 5y) - 3y(2x - 5y)\right)$
  $= 2xy(2x - 3y)(2x - 5y)$ *(The final answer.)*



Here is a generalization of factoring a trinomial in higher degree.

- $36x^7 - 69x^5 + 15x^3$ *(Factor out the GCF=$3x^3$; standard formulae are irrelevant.)*
  $= 3x^3(12x^4 - 23x^2 + 5)$ $a = 12, b = -23, c = 5$ $(ac = 12 \cdot 5 = 60.$ *Factor* $60.)$
  $= 3x^3(12x^4 - 20x^2 - 3x^2 + 5)$ *(Use grouping method.)*
  $= 3x^3(\underbrace{12x^4 - 20x^2}\underbrace{-3x^2 + 5})$
  $= 3x^3\left(4x^2(3x^2 - 5) - 1(3x^2 - 5)\right)$
  $= 3x^2(4x^2 - 1)(3x^2 - 5)$ *(Differences of squares.)*
  $= 3x^2(2x + 1)(2x - 1)(\sqrt{3}x + \sqrt{5})(\sqrt{3}x - \sqrt{5})$
  (The final answer.)



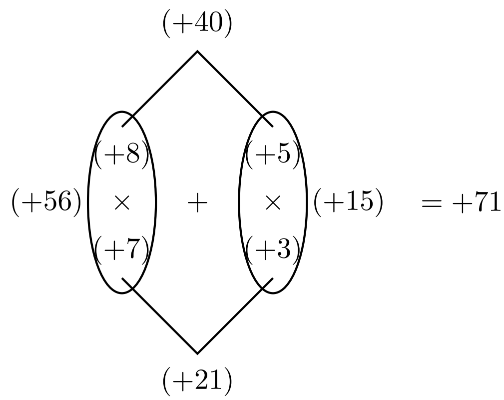
Classroom Exercises : Factor the following trinomials.

- $2x^2 + x - 6$ • $7x^2 + 20x - 3$
- $15x^2 + x - 2$ • $24a^4 + 10a^3 - 4a^2$
- $18p^4q + 21p^3q^2 - 9p^2q^3$ • $5x^9 - 47x^7 + 18x^5$

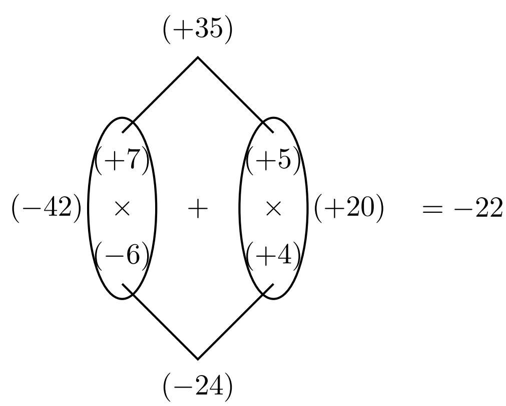
Large coefficients : Sometimes, the coefficients in the trinomials are so large that the *ac*-method becomes quite difficult. In this case, instead of factoring the product *ac*, we factor the numbers *a* and *c* separately in a meaningful way. It is best explained through these examples.

Examples :

- $40x^2 + 71x + 21$ *(GCF =1; standard formulae are irrelevant.)*
  $= 40x^2 + 56x + 15x + 21$ $a = 40, b = 71, c = 21$. *Factor* 40 *and* 21.
  $= \underbrace{40x^2 + 56x} + \underbrace{15x + 21}$
  $= 8x(5x + 7) + 3(5x + 7)$
  $= (8x + 3)(5x + 7)$
  (The final answer.)



- $35x^2 - 22x - 24$ *(GCF =1; standard formulae are irrelevant.)*
  $= 35x^2 - 42x + 20x - 24$ $a = 35, b = -22, c = -24$. *Factor* 35 *and* -24.
  $= \underbrace{35x^2 - 42x} + \underbrace{20x - 24}$
  $= 7x(5x - 6) + 4(5x - 6)$
  $= (7x + 4)(5x - 6)$
  (The final answer.)



Classroom Exercises : Factor the following trinomials.
• $40x^2 + 122x + 33$ • $21x^2 - 83x + 72$

A.3.6 Summary

To factorize a polynomial, follow every step:

Step 1 Factor out the GCF.

Step 2 Check whether the grouping method can be used.

Step 3 Check for any of the standard formulae (Difference of squares, difference of cubes, sum of cubes).

Step 4 Convert the monic trinomial to a four-term polynomial and apply the grouping procedure.

Step 5 Convert the non- monic trinomial to a four-term polynomial and apply the grouping procedure.

A.3.7 Homework Exercises

1. What is the GCF in each of the cases?
   - $\{14, 49, 56\}$
   - $\{30, 45, 60\}$
   - $\{18, 24, 36\}$
   - $\{14a^5b^7c^4, 49a^3b^4c^9, 56abc\}$
   - $\{30x^8, 45x^4, 60x^5\}$
   - $\{18t^4s^3, 24ts, 36t^9\}$

2. Factor the following:

   (a) $14a^5b^7c^4 - 49a^3b^4c^9 + 56abc$

   (b) $30x^8 + 45x^4 + 60x^5$

   (c) $-18t^4s^3 - 24ts + 36t^9$

   (d) $12x^5 - 16x^4 - 5x^3$

   (e) $-25x^7 - 30x^9 + 15x^{15}$

   (f) $2pm + 3qm + 2pn + 3qn$

   (g) $40ma + 32m - 60na - 48n$

   (h) $32y^2 + 3x + 4y + 24xy$

   (i) $20y^2 + 4y + 24xy + 3x$

   (j) $15a^5b^4 - 35a^4b^4 - 6a^4b^5 + 14a^3b^5$

   (k) $18p^2rs + 24prs - 6nprs - 8nrs$

   (l) $a^2 - b^2$

   (m) $x^2 - 4$

   (n) $9r^2 - 1$

   (o) $4x^2 - 25y^2$

   (p) $100m^2n^2 - 9p^2q^2$

   (q) $16p^4 - 9q^8$ (This will require several factoring steps)

   (r) $48r^2t^4 - 12$

(s) $7r^2 - 3s^2$

(t) $5x^2 - 1$

(u) $x^3 - 1$

(v) $x^3 + 1$

(w) $8a^3 - 27$

(x) $8a^3 + 27$

(y) $3r^3 - 24$

(z) $3r^3 + 24$

3. Factor the following trinomials:

(a) $x^2 + x - 20$

(b) $x^2 + 10x + 16$

(c) $x^2 - 17x + 16$

(d) $3p^2 + 36pq + 33q^2$

(e) $3a^2 - 21ab - 54b^2$

(f) $2x^2 + 28xy + 48y^2$

(g) $x^4 - 13x^2 + 36$ (This will require several factoring steps)

(h) $3x^2 + 11x + 10$

(i) $2x^2 - 3x - 14$

(j) $6x^2 - 11x + 4$

(k) $36x^4 - 39x^3 + 9x^2$

(l) $12p^4q^2 - 26p^3q^3 + 12p^2q^4$

(m) $3x^{10} - 16x^8 + 16x^6$ (This will require several factoring steps)

(n) $21x^2 + 59x + 40$

(o) $30x^2 + x - 20$