**11.** Show that the relation $R$ consisting of all pairs $(x, y)$ such that $x$ and $y$ are bit strings of length three or more that agree in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

**12.** Show that the relation $R$ consisting of all pairs $(x, y)$ such that $x$ and $y$ are bit strings of length three or more that agree except perhaps in their first three bits is an equivalence relation on the set of all bit strings of length three or more.

**13.** Show that the relation $R$ consisting of all pairs $(x, y)$ such that $x$ and $y$ are bit strings that agree in their first and third bits is an equivalence relation on the set of all bit strings of length three or more.

**14.** Let $R$ be the relation consisting of all pairs $(x, y)$ such that $x$ and $y$ are strings of uppercase and lowercase English letters with the property that for every positive integer $n$, the $n$th characters in $x$ and $y$ are the same letter, either uppercase or lowercase. Show that $R$ is an equivalence relation.

**15.** Let $R$ be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $a + d = b + c$. Show that $R$ is an equivalence relation.

**16.** Let $R$ be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that $R$ is an equivalence relation.

**17.** (*Requires calculus*)
   **a)** Show that the relation $R$ on the set of all differentiable functions from **R** to **R** consisting of all pairs $(f, g)$ such that $f'(x) = g'(x)$ for all real numbers $x$ is an equivalence relation.
   **b)** Which functions are in the same equivalence class as the function $f(x) = x^2$?

**18.** (*Requires calculus*)
   **a)** Let $n$ be a positive integer. Show that the relation $R$ on the set of all polynomials with real-valued coefficients consisting of all pairs $(f, g)$ such that $f^{(n)}(x) = g^{(n)}(x)$ is an equivalence relation. [Here $f^{(n)}(x)$ is the $n$th derivative of $f(x)$.]
   **b)** Which functions are in the same equivalence class as the function $f(x) = x^4$, where $n = 3$?

**19.** Let $R$ be the relation on the set of all URLs (or Web addresses) such that $x R y$ if and only if the Web page at $x$ is the same as the Web page at $y$. Show that $R$ is an equivalence relation.

**20.** Let $R$ be the relation on the set of all people who have visited a particular Web page such that $x R y$ if and only if person $x$ and person $y$ have followed the same set of links starting at this Web page (going from Web page to Web page until they stop using the Web). Show that $R$ is an equivalence relation.

In Exercises 21–23 determine whether the relation with the directed graph shown is an equivalence relation.
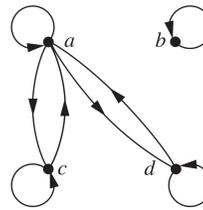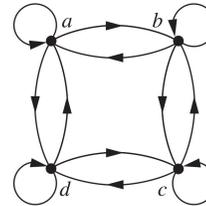
**21.**   **22.**
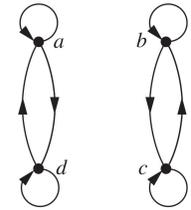


**23.**



**24.** Determine whether the relations represented by these zero–one matrices are equivalence relations.

**a)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$   **b)** $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$   **c)** $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

**25.** Show that the relation $R$ on the set of all bit strings such that $s R t$ if and only if $s$ and $t$ contain the same number of 1s is an equivalence relation.

**26.** What are the equivalence classes of the equivalence relations in Exercise 1?

**27.** What are the equivalence classes of the equivalence relations in Exercise 2?

**28.** What are the equivalence classes of the equivalence relations in Exercise 3?

**29.** What is the equivalence class of the bit string 011 for the equivalence relation in Exercise 25?

**30.** What are the equivalence classes of these bit strings for the equivalence relation in Exercise 11?
   **a)** 010   **b)** 1011   **c)** 11111   **d)** 01010101

**31.** What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 12?

**32.** What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation from Exercise 13?

**33.** What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation $R_4$ from Example 5 on the set of all bit strings? (Recall that bit strings $s$ and $t$ are equivalent under $R_4$ if and only if they are equal or they are both at least four bits long and agree in their first four bits.)

**34.** What are the equivalence classes of the bit strings in Exercise 30 for the equivalence relation $R_5$ from Example 5 on the set of all bit strings? (Recall that bit strings $s$ and $t$ are equivalent under $R_5$ if and only if they are equal or they are both at least five bits long and agree in their first five bits.)

**35.** What is the congruence class $[n]_5$ (that is, the equivalence class of $n$ with respect to congruence modulo 5) when $n$ is

   **a)** 2?    **b)** 3?    **c)** 6?    **d)** −3?

**36.** What is the congruence class $[4]_m$ when $m$ is

   **a)** 2?    **b)** 3?    **c)** 6?    **d)** 8?

**37.** Give a description of each of the congruence classes modulo 6.

**38.** What is the equivalence class of each of these strings with respect to the equivalence relation in Exercise 14?

   **a)** *No*        **b)** *Yes*        **c)** *Help*

**39. a)** What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 15?

   **b)** Give an interpretation of the equivalence classes for the equivalence relation $R$ in Exercise 15. [*Hint:* Look at the difference $a - b$ corresponding to $(a, b)$.]

**40. a)** What is the equivalence class of $(1, 2)$ with respect to the equivalence relation in Exercise 16?

   **b)** Give an interpretation of the equivalence classes for the equivalence relation $R$ in Exercise 16. [*Hint:* Look at the ratio $a/b$ corresponding to $(a, b)$.]

**41.** Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

   **a)** $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$

   **b)** $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$

   **c)** $\{2, 4, 6\}, \{1, 3, 5\}$    **d)** $\{1, 4, 5\}, \{2, 6\}$

**42.** Which of these collections of subsets are partitions of $\{-3, -2, -1, 0, 1, 2, 3\}$?

   **a)** $\{-3, -1, 1, 3\}, \{-2, 0, 2\}$

   **b)** $\{-3, -2, -1, 0\}, \{0, 1, 2, 3\}$

   **c)** $\{-3, 3\}, \{-2, 2\}, \{-1, 1\}, \{0\}$

   **d)** $\{-3, -2, 2, 3\}, \{-1, 1\}$

**43.** Which of these collections of subsets are partitions of the set of bit strings of length 8?

   **a)** the set of bit strings that begin with 1, the set of bit strings that begin with 00, and the set of bit strings that begin with 01

   **b)** the set of bit strings that contain the string 00, the set of bit strings that contain the string 01, the set of bit strings that contain the string 10, and the set of bit strings that contain the string 11

   **c)** the set of bit strings that end with 00, the set of bit strings that end with 01, the set of bit strings that end with 10, and the set of bit strings that end with 11

   **d)** the set of bit strings that end with 111, the set of bit strings that end with 011, and the set of bit strings that end with 00

   **e)** the set of bit strings that contain $3k$ ones for some nonnegative integer $k$, the set of bit strings that contain $3k + 1$ ones for some nonnegative integer $k$, and the set of bit strings that contain $3k + 2$ ones for some nonnegative integer $k$.

**44.** Which of these collections of subsets are partitions of the set of integers?

   **a)** the set of even integers and the set of odd integers

   **b)** the set of positive integers and the set of negative integers

   **c)** the set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remainder of 2 when divided by 3

   **d)** the set of integers less than −100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100

   **e)** the set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6

**45.** Which of these are partitions of the set $\mathbf{Z} \times \mathbf{Z}$ of ordered pairs of integers?

   **a)** the set of pairs $(x, y)$, where $x$ or $y$ is odd; the set of pairs $(x, y)$, where $x$ is even; and the set of pairs $(x, y)$, where $y$ is even

   **b)** the set of pairs $(x, y)$, where both $x$ and $y$ are odd; the set of pairs $(x, y)$, where exactly one of $x$ and $y$ is odd; and the set of pairs $(x, y)$, where both $x$ and $y$ are even

   **c)** the set of pairs $(x, y)$, where $x$ is positive; the set of pairs $(x, y)$, where $y$ is positive; and the set of pairs $(x, y)$, where both $x$ and $y$ are negative

   **d)** the set of pairs $(x, y)$, where $3 \mid x$ and $3 \mid y$; the set of pairs $(x, y)$, where $3 \mid x$ and $3 \nmid y$; the set of pairs $(x, y)$, where $3 \nmid x$ and $3 \mid y$; and the set of pairs $(x, y)$, where $3 \nmid x$ and $3 \nmid y$

   **e)** the set of pairs $(x, y)$, where $x > 0$ and $y > 0$; the set of pairs $(x, y)$, where $x > 0$ and $y \leq 0$; the set of pairs $(x, y)$, where $x \leq 0$ and $y > 0$; and the set of pairs $(x, y)$, where $x \leq 0$ and $y \leq 0$

   **f)** the set of pairs $(x, y)$, where $x \neq 0$ and $y \neq 0$; the set of pairs $(x, y)$, where $x = 0$ and $y \neq 0$; and the set of pairs $(x, y)$, where $x \neq 0$ and $y = 0$

**46.** Which of these are partitions of the set of real numbers?

   **a)** the negative real numbers, $\{0\}$, the positive real numbers

   **b)** the set of irrational numbers, the set of rational numbers

   **c)** the set of intervals $[k, k + 1]$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

   **d)** the set of intervals $(k, k + 1)$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

   **e)** the set of intervals $(k, k + 1]$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$

   **f)** the sets $\{x + n \mid n \in \mathbf{Z}\}$ for all $x \in [0, 1)$

**47.** List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

   **a)** $\{0\}, \{1, 2\}, \{3, 4, 5\}$

   **b)** $\{0, 1\}, \{2, 3\}, \{4, 5\}$

   **c)** $\{0, 1, 2\}, \{3, 4, 5\}$

   **d)** $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$