CSI 35 LECTURE NOTES (Ojakian)

Topic 1: Proofs

OUTLINE

(References: Wells 1 - 4, 15, 27, 28, 30, 80-84, 86; Rosen: 1.6, 1.7, 1.8)

1. Mathematical Proofs

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1. Structured Proofs

   (a) Consider summing two evens. What can you say? Can you prove it? Experiment with numbers.

   (b) Conjecture, counter-example, and proof

   (c) I will use the expression "**Structured Proof**" to refer to a proof which is broken down into simple steps, in which the steps are successively numbered (i.e. 1, 2, 3, etc) and for each step a justifications is written down; the justification can be 1) by definition, or 2) by given, or 3) by a known fact, or 4) by indicating how earlier steps logically imply this step (include the numbers of the steps you use).

   In (3) when I say "known fact" I do *not* mean facts you think are simple about the new definition (ex. you can't just assume things about divisibility). I mean facts from *earlier courses* (ex. distributivity, basic algebra, facts about integers, etc)

   (d)

   PROBLEM 1. *Give a structured proof for the sum of two evens.*

   PROBLEM 2. *Prove that the sum of an odd integer and an even integer is odd.*

2. Divisibility Proofs

   (a) Definition: x divides y. How define?
   (be aware of 0 dividing 0 - to allow or not to allow ...)

   (b) Prove or disprove the following.

      i. Every integer divides itself.
      ii. 103 divides itself (can use "Universal Instantiation").
      iii. Every integer divides 0.
      iv. 0 divides every integer.
      v. If $a|b$ then $a \leq b$

3. Using Cases in Proofs

   (a) Break up the statement you are trying to prove into a number of easier-to-prove statements

   (b) Prove the 100 is not the cube of an integer.

   (c) Prove the triangle inequality: $|x + y| \leq |x| + |y|$ (for x, y integers).

4. Proofs on program correctness

   (a) One kind of question: Does the program always terminate?

   (b) Second kind of question: Given some initial assertion, verify some final assertion.

   PROBLEM 3. *Consider the following program:*

```
y = 5
if y <= 5:
    z = x + 2*y
else:
    z = x - 2*y
```

   Suppose the initial assertion is that $x = 3$. *Then prove the final assertion that* $z = 13$.

   PROBLEM 4. *Consider the following program:*

```
x = 0
for k in range(1, n+1):
    x = x + 2*k
```

   Suppose the initial assertion is that $n = 100$. *Then prove the final assertion that* $x = 10100$.

   PROBLEM 5. *Consider the following program:*

```
while x < 10000:
    if x % 7 == 0:
        x = x + 1
    else:
        x = x + 3
```

   Prove that whatever integer the initial value of x *is, the program terminates.*

5. Proof by Contradiction

   (a) To this point proofs have been Direct. Now Indirect or by contradiction.

   PROBLEM 6. *Prove that among 100 consecutive days, any 51 of these days must contain 2 consecutive days.*

   PROBLEM 7. *Prove that for all positive integers* n, *if* n^2 *is even then so is* n.

6. Proving Equivalences

   (a) Must prove two implications.

   PROBLEM 8. *Prove that for all positive integers* n, n^2 *is even if and only if* n *is even.*

   (Note: part of this proof already done)

   (b) The use of Lemmas:

      i. Lemma: The product of two evens is even.

      ii. Lemma: The product of two odds is odd