Kerry Ojakian's GRASCan 2024 Exercises

Sources for some exercises:
An Invitation to Pursuit-Evasion Games and Graph Theory by Anthony Bonato (2020).

Cop Number.
$c(G)$ = cop number of G.

1. What is the cop number of the path?
2. What is the cop number of the k disjoint paths?
3. Suppose $c(G) = 3$ and $c(H) = 5$. Then what is $c(G \cup H)$?
4. Find the cop numbers of each of the following families (be aware, that your answers may need to depend on the value of the parameter ...):
   (a) K_n (complete graph on n vertices)
   (b) C_n (cycle on n vertices)
   (c) $K_{s,t}$ (complete bipartite graph with a size s independent set and a size t independent set)
5. What is the cop number of the 3 by 3 grid (i.e. with 9 vertices)?
6. What is the cop number of the a by b grid.

Cop-win Graphs.

7. What are the cop-win graphs with no cycles?
8. Give an example of a cop-win graph that contains a cycle.
9. Find all the cop-win graphs, up to isomorphism, of order at most 4. [Bonato 2020]
10. Give two different cop-win orderings of the path $P_4 = (1, 2, 3, 4)$.
11. How many cop-win orderings does K_n have?
12. How many cop-win orderings does C_n have?

More on Cops and Robber.

13. What is the capture time of the path P_n? (i.e. path with n vertices)
14. Give examples of graphs H and G so that H is an *induced* subgraph of G, and such that $c(H) < c(H)$.

15. Give examples of graphs H and G so that H is an *induced* subgraph of G, and such that $c(H) = c(H)$.

    Ok, if you were cheap, now find an H which is a *proper* induced subgraph!

16. Give examples of graphs H and G so that H is a subgraph (not necessarily induced!) of G, and such that $c(H) > c(H)$.

17. Give examples of graphs H and G so that H is an *induced* subgraph of G, and such that $c(H) > c(H)$. [Bonato 2020]

Cops with attacking robbers.
$cc(G)$ = the minimum number of cops need to catch an attacking robber, on G.

18. What is $cc(P_n)$?

19. Is there a graph with attacking cop number 1?

20. What is $cc(C_6)$ and $cc(C_7)$?

21. Consider $cc(C_6)$ again. What is a more general upper bound on $cc(G)$?

22. What is an easy bound on $cc(G)$ in terms of $c(G)$?

Graph Burning.
$b(G)$ = burning number of G.

23. Find the burning number of the following paths: P_n, for $3 \leq n \leq 10$

24. Find the burning number of K_n (don't miss the cheap case!).

25. Find $b(K_{a,b})$.

26. What is the worst possible burning number and a graph that achieves it?

    (this question is easy as long as you don't assume what you may be assuming ...)

27. Find the burning number of P_9. Then find the burning number of the disjoint union of P_4 and P_5 (i.e. a *path-forest* with 2 paths).

28. Considering the last question, do you think that if $n = a + b$, then the burning number of P_n is the same as the burning number of the disjoint union of P_a and P_b?

    (Consider an example with $n = 9$, but a different a and b from the last exercise.)

29. Find an example of a graph G with a subgraph H such such that $b(H) < b(G)$.

30. Can you do the last question if the subgraph H is required to be spanning? (i.e. you keep all the vertices, but may lose some edges)
31. Give an example of a graph G with an *induced* subgraph H such such that $b(H) > b(G)$. [Bonato 2020]

Burning Hypergraphs.

32. What is the **lazy burning number** of the following hypergraph: 3-uniform sunflower (single intersection) with 4 petals? k-uniform hypergraph with 4 petals?
33. What is the **burning number** of the following hypergraph: 3-uniform sunflower (single intersection) with 4 petals? k-uniform hypergraph with 4 petals?
34. What is the *lazy burning number* and the *burning number* of the trivial hypergraph: one hyperedge containing all the vertices?
35. Find a hypergraph on n vertices such that the the *lazy burning number* and the *burning number* are both n.
36. What is the *lazy burning number* of any ordinary graph? (i.e. ordinary means 2-uniform).

    Consider disconnected graphs ...

Cleaning.
$brush(G)$ = minimum number of brushes needed to clean the graph.

37. What is the brush number of a path? of a cycle?
38. What is $brush(K_n)$?
39. What is the maximum possible brush number of a graph on n vertices? (connected or disconnected)

    Watch out ...

Cat Herding.
$cat(G)$ = number of edges that must be removed from G.

40. What is the cat-herding number P_4?

    Find a bad sequence of edge-deletions which takes more time than $cat(P_4)$.
41. What is the smallest possible value of the herding number for a graph on n vertices?

42. What is a very cheap upper bound on the herding number?

43. What is cat-herding number for for K_2, K_3, K_4, and K_5?

Zero Forcing.
$z(G)$ = minimum number of colored start vertices in G, to force everything colored.

44. What is the zero-forcing number of C_n?

45. Consider C_6. Color a set of vertices which does **not** zero-force C_6, even though the number of colored vertices is larger than $z(C_6)$.

46. What is the zero-forcing number of a path?

47. What is $z(K_n)$?

"Contrained" Variations.
Each piece can move/propagate/etc only once!

48. What is the *constrained* cop number of P_5? Of P_7? Of P_n?

49. What is another name (a very well known graph parameter!) for the constrained cop number?

50. What is the *constrained* zero forcing number of P_5? Of P_7? Of P_n?

"Invisible" Variations.
The piece being pursued is invisible!

51. What is the cop number of K_n if the robber is invisible?

52. What is the invisible cop number of the 4 by 5 grid?

53. Find some graphs where the invisible cop number is the same as the usual cop number.

54. In cops and robber, we might consider the case in which the robber is invisible. Why might we **not** consider the case in which the cops are invisible?

55. Suppose the robber is invisible and the cops are constrained (also called the "deduction number"). How many cops are needed to win on P_5?

    (watch out - note that it is not just the domination number!)

56. Referring to the last problem, what is the deduction number of P_n? Look back at contrained zero-forcing; what is the connection?