REPORT ON THE ESSENTIAL MINIMUM

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Abstract. We collect recent results on the essential minimum of height functions on arithmetic varieties.

1. Essential minimum of the height

The notion of height $h(\xi)$ of a point $\xi \in \bar{\mathbb{Q}}$ or in general, the height $h(P)$ of a point P in an algebraic variety over a number field K, should provide an idea of the arithmetic complexity of the point.

1.1. **Places on number fields.** Suppose that we are working with a number field K. A place of K is identified with an absolute value $|.|_v : K \longrightarrow \mathbb{R}^+$ and a weight n_v. The weights n_v are being normalized in such a way that $n_v = 1$ for all absolute values $|.|_v : \mathbb{Q} \longrightarrow \mathbb{R}^+$ and for any extension of number fields K/K_0,

$$n_v = \frac{[K_v : K_{v_0}]}{[K : K_0]} n_{v_0},$$

where K_v is denoting the completion K with respect to $|.|_v$ and $|.|_v$ is extending $|.|_{v_0}$. We will denote by $\mathcal{M}_K$ the set of places of K.

Example 1.1. The places of $\mathbb{Q}$ are of two kinds (all of them with $n_v = 1$):

(1) Usual absolute value: $|x|_\infty = \max(x, -x)$.
(2) P-adic absolute value $|.|_p$: suppose that $p \in \mathbb{Q}$ is a rational prime and $\xi = p^{m_p} \dfrac{a}{b}$ with $p \nmid a, b$ then $|\xi|_p = p^{-m_p}$.

Example 1.2. For a number field K, the places $|.|_\sigma$ extending the ordinary absolute value $|.|_\infty$ can be obtained as $|x|_\sigma = |\sigma(x)|_\infty$ for an embedding $K \hookrightarrow \mathbb{C}$. The weight is $n_v = 1$ or $n_v = 2$ depending if σ is a real or a complex embedding.

Remark 1.3. Basic properties of the places are:

(1) If K/K_0 is a finite extension: $\sum_{v \in \mathcal{M}_K, v/v_0} n_v = n_{v_0}$.
(2) (product formula) $\forall\, \alpha \in K^\times$ we have $\sum_{v \in \mathcal{M}_K} n_v \log |\alpha|_v = 0$.

1.2. **Heights associated to metrized line bundles.** Suppose, as before, that K is a number field. Let X be an n-dimensional projective algebraic variety over K and $\mathcal{L}$ a line bundle on X. For each place $v \in \mathcal{M}_K$ we consider:

(1) X_v the v-adic analytic space, that is, $X(\mathbb{C})$ for $v \mid \infty$ and the Berkovich analytic space X_v^{an} over the completion of the algebraic closure $\mathbb{C}_v$ of K_v for finite places.
(2) A metric $\|.\|_v$ on the line bundle $\mathcal{L}_v = \mathcal{L} \otimes_K \mathbb{C}_v$ on the v-adic analytic space X_v.

Berkovich analytic spaces, introduced in [1], are locally compact spaces X^{an} associated to algebraic varieties X over non-Archimedean field with a continuous map $\pi : X^{an} \longrightarrow X$. We refer to section 1.2 and section 1.3 of [2] for properties of these analytic spaces and analytification $\mathcal{L}^{an}$ of line bundles on X. On the other hand:

Definition 1.4. *A metric on a line bundle $\mathcal{L}$ is an assignment that to each open set $U \subset X$ and every section s of $\mathcal{L}$ on U associates a continuous function:*

$$\|s(.)\| : U \longrightarrow \mathbb{R}^+$$

that is compatible with the restriction on open sets, and defines a metric on the fibres:

(1) *$\|s(P)\| = 0$ if and only if $s(P) = 0$.*
(2) *For λ a regular section of $\mathcal{O}_X(U)$, $\|\lambda s(P)\| = |\lambda(P)|\|s(P)\|$.*

Example 1.5. (Canonical metric on $\mathcal{O}(1)$) In the case $X = \mathbb{P}^1$ and $L = \mathcal{O}(1)$ we have the metric that, if $(x_0 : x_1)$ represent coordinates on $\mathbb{P}^1$, is given by

$$\|(\lambda_0 x_0 + \lambda_1 x_1)(a_0 : a_1)\|_v = \frac{|\lambda_0 a_0 + \lambda_1 a_1|_v}{\sup(|a_0|_v, |a_1|_v)}.$$

Using $s = x_1$ we recover the Weil height of a point $\xi = (\xi, 1) \in K^*$. The metric so defined is called the canonical metric on $\mathcal{L} = \mathcal{O}(1)$.

Definition 1.6. *Let F be a field that is complete with respect to a non-Archimedean absolute value an denote by F^0 its valuation ring. A model of $(X, \mathcal{L})$ is a triple $(\tilde{X}, \tilde{\mathcal{L}}, e)$, where $\tilde{X}$ is a flat model over* $\operatorname{Spec} F^0$ *of X, $\tilde{\mathcal{L}}$ is a line bundle on $\tilde{X}$ and $e \geq 1$ is an integer such that $\tilde{\mathcal{L}}|X \cong \mathcal{L}^e$.*

Remark 1.7. A proper model $\tilde{X}$ of a proper variety X admits a surjective reduction map red : $X^{an} \longrightarrow \tilde{X}$ as explained in section 2.3 of [1].

Definition 1.8. *(Algebraic metric induced by a model on the associated analytic space over non-Archimedean fields) Let* $(\tilde{X}, \tilde{\mathcal{L}}, e)$ *be a model of* $(X, \mathcal{L})$*. Let* s *be a local section of the analytification* $\mathcal{L}^{an}$ *defined at the point* $P \in X^{an}$*. Let* $\tilde{U} \subset \tilde{X}$ *be a trivializing open neighbourhood of* $red(P)$ *and* σ *a generator of* $\tilde{\mathcal{L}}|\tilde{U}$*. Let* $U = \tilde{U} \cap X$ *and* $\lambda \in \mathcal{O}(U^{an})$ *such that* $s^e = \lambda\sigma$ *on* U^{an}*. Then, the metric induced by the proper model* $(\tilde{X}, \tilde{\mathcal{L}}, e)$ *on* $\mathcal{L}^{an}$*, denoted* $\|.\|_{\tilde{X},\tilde{\mathcal{L}},e}$ *is given by*

$$\|s(P)\|_{\tilde{X},\tilde{\mathcal{L}},e} = |\lambda(P)|^{1/e}.$$

Equivalently the norm of the local frame $\|\sigma(P)\| \equiv 1$*.*

Definition 1.9. *A metrized line bundle* $\bar{\mathcal{L}}$ *is a collection*

$$\bar{\mathcal{L}} = (\mathcal{L}, (\|.\|_v)_{v\in\mathcal{M}})$$

with notation as before. Such collection is defined to be quasi-algebraic if there exist an integral model which defines the metric $\|.\|_v$ *for all except maybe a finite number of* v*.*

Definition 1.10. *Let* X *be an algebraic variety defined over a number field* K *and* $\bar{\mathcal{L}} = (\mathcal{L}, (\|.\|_v)_{v\in\mathcal{M}})$ *a quasi-algebraic metrized line bundle on* X*. The height* $h_{\bar{\mathcal{L}}}(P)$ *of a point* $P \in X(\bar{K})$ *can be expressed by the intrinsic formula*

$$h_{\bar{\mathcal{L}}}(P) = -\sum_{v\in\mathcal{M}_K} \frac{1}{\#\,\mathrm{Gal}(P)_v} \sum_{Q\in\mathrm{Gal}(P)_v} \log \|s(Q)\|_v,$$

where $\mathrm{Gal}(P)_v$ *is denoting the* v*-Galois orbit of* P*, i.e. the image of* $\mathrm{Gal}(\bar{K} : K)P$ *under the map* $i_v : X(\bar{K}) \longrightarrow X_v$*.*

Remark 1.11. Let K be a number field and suppose that for each algebraic extension L/K and for each $w \in \mathcal{M}_F$ extending the place $v \in \mathcal{M}_K$ we denote by i_w the map $i_w : X(L) \longrightarrow X_v^{an}$ sending the algebraic points over F into the v-adic analytic space. An equivalent definition for the height of a point $P \in X(F)$ with respect to quasi-algebraic metrized line bundle $\bar{\mathcal{L}}$ is the sum

$$h_{\bar{\mathcal{L}}}(P) = -\sum_{w\in\mathcal{M}_L} n_v \log \|s \circ i_w(P)\|_v,$$

for any rational regular section s such that $P \notin |\operatorname{div}(s)|$.

Example 1.12. Let $\xi \in \bar{\mathbb{Q}}^*$ of degree $d \geq 1$ with minimal polynomial over $\mathbb{Z}$

$$P_\xi = \alpha_0 x^d + \cdots + \alpha_{d-1}x + \alpha_d = \alpha_0 \prod_{\eta\in G_\xi} (x - \eta) \in \mathbb{Z}[x],$$

where G_ξ is denoting the Galois orbit of ξ. The Weil height of ξ is

$$h(\xi) = \frac{1}{d}(\sum_{\eta\in G_\xi} \log\max(1,|\eta|) + \log|\alpha_0|).$$

Definition 1.13. *The essential minimum of X with respect to $\bar{\mathcal{L}}$ is defined as*

$$\mu^{ess}_{\bar{\mathcal{L}}}(X) = \sup_{Y\subseteq X, Y \text{ closed}} \inf_{P\in(X\setminus Y)(\bar{K})} h_{\bar{\mathcal{L}}}(P).$$

Remark 1.14. The essential minimum is the generic infimum for the function $h_{\bar{\mathcal{L}}}$. An equivalent definition will be

$$\mu^{ess}_{\bar{\mathcal{L}}}(X) = \inf\{\theta\in\mathbb{R}\,|\,\{P\in X(\bar{K})\,|\,h_{\bar{\mathcal{L}}}(P)\le\theta\} \text{ is Zariski dense}\}$$

1.3. **Semi-positive metrics.** A very special type of metric is the case of a semi-positive metrized line bundle $\bar{\mathcal{L}}$. In this situation the metrics $\|.\|_v$ on $\mathcal{L}_v$ are limits of smooth metrics in the Archimedean case ($v\mid\infty$) and limits of algebraic metrics (induced by models $(\tilde{X},\tilde{L})$ of $(X,\mathcal{L})$) in the non-Archimedean case.

Example 1.15. The canonical metric is semi-positive on the line bundle $\mathcal{L}=\mathcal{O}(1)$ on $\mathbb{P}^1$!

We can extend to notion of height to subvarieties of $Y\subset X$ and in particular define $h_{\bar{\mathcal{L}}}(X)$. An important result of Shou-Wu Zhang (Theorem 5.2 in [12]) states that for $\bar{\mathcal{L}}$ semi-positive and ample, the essential minimum can be bounded below:

$$\mu^{ess}_{\bar{\mathcal{L}}}(X) \ge \frac{h_{\bar{\mathcal{L}}}(X)}{(n+1)\deg_{\bar{\mathcal{L}}}(X)}.$$

1.4. **Falting height.** Let $\mathcal{X} := \mathbb{P}^1_{\mathbb{Z}}$ and consider the section s_∞ : $\mathrm{Spec}(\mathbb{Z}) \longrightarrow \mathcal{X}$ defined by $[1,0]$. We denoted by D_∞ the divisor induced by this section and by $\mathcal{L}=\mathcal{O}_{\mathcal{X}}(D_\infty)$ the associated line bundle. The complex points $\mathbb{P}^1(\mathbb{C})$ of the surface $\mathcal{X}$ are in holomorphic bijection with the modular curve

$$X = (\mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H})\cup\infty \xrightarrow{i} \mathbb{P}^1(C),$$

where the map i is induced by the j-invariant map $j:\mathbb{H}\longrightarrow\mathbb{C}$ given by

$$j(\tau) = \frac{1}{q} + 744 + 196884q + \dots, \qquad q = e^{2\pi i\tau}.$$

The line bundle $\mathcal{L}(\mathbb{C})$ is isomorphic to the modular forms $\mathcal{M}_{12}(\mathrm{Sl}_2(\mathbb{Z}))$ of weight 12 and level 1 over X and this isomorphism carries the Peterson scalar product defined for $f\in\mathcal{M}_{12}(\mathrm{Sl}_2(\mathbb{Z}))$ as

$$\|f\|_{Pet} = (4\pi\,\mathrm{Im}(\tau))^6|f(\tau)|$$

to sections of $\mathcal{L}(\mathbb{C})$ on X. We have then a metrized line bundle $(\mathcal{L}, \|.\|_{Pet})$ in the sense of Arakelov that is singular at $(1,0)$. To be able to define a height function we put the canonical metric $\|.\|_{can}$ over the finite places to have $(\mathcal{L}, \|.\|_v) = (\mathcal{L}, \|.\|_{v,can} \cup \|.\|_{v|\infty,Pet})$
Suppose that for each prime number p we have fixed an extension of the p-adic norm $|.|_p$ on $\bar{\mathbb{Q}}$. Also denote by $\mathcal{O}(\alpha)$ the orbit of $\alpha \in \bar{\mathbb{Q}}$ under the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Then, as an application of definition 1.10 for the section Δ, the Falting height can be expressed as:

$$h_{Fal}(\alpha) = \frac{1}{12}\left(\frac{1}{\#\mathcal{O}(\alpha)} \sum_{\alpha' \in \mathcal{O}(\alpha)} g_{hyp}(\alpha') + \frac{1}{\#\mathcal{O}(\alpha)} \sum_{p \text{ prime}} \sum_{\alpha' \in \mathcal{O}(\alpha)} \log^+ |\alpha'|_p\right),$$

where the function $g_{hyp} : \mathbb{C} \longrightarrow \mathbb{R}$ is defined to be the function such that $g_\infty = g_{hyp} \circ j$ and the hyperbolic Green function $g_\infty : \mathbb{H} \longrightarrow \mathbb{R}$ is defined by

$$g_\infty(\tau) = -\log(\|\Delta(\tau)\|_{Pet}) = -\log((4\pi \,\mathrm{Im}(\tau))^6 |\Delta(\tau)|).$$

We want to consider not only line bundle but also real line bundles, or at least, the notion of real global sections with positive coefficients

$$s = s_1^{\otimes a_1} \otimes \cdots \otimes s_l^{\otimes a_l} \in \bigsqcup_{n \geq 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes n}) \otimes \mathbb{R}^+,$$

where $s_i \in \bigsqcup_{n\geq 0} \Gamma(\mathcal{X}, \mathcal{L}^{\otimes n})$ and $a_1, \ldots, a_l$ are positive real numbers. The support of s is given by $|\,\mathrm{div}(s)| = \bigcup_k |\,\mathrm{div}(s_k)|$ and the Green function g_s associated to the section s will be

$$g_s(x) = -\log \|s(x)\|_{Pet} = -\log \prod_{j=1}^{l} \|s_j(x)\|_{Pet}^{a_j}.$$

Proposition 1.16. *Let s be a real section of weight one and $x \in \mathcal{X}(\mathbb{C}) \setminus |\,\mathrm{div}(s)|$ and algebraic point then we have the inequality:*

$$h_{Fal}(x) \geq \inf_{y \in \mathcal{X}(\mathbb{C})} g_s(y) = -\log \sup_{y \in \mathcal{X}(\mathbb{C})} \|s(y)\|_{Pet}.$$

In particular we obtain $\mu_{Fal}^{ess} \geq \inf_{y \in \mathcal{X}(\mathbb{C})} g_s(y)$.

Proof. The proof is based on the fact that the finite places have a non-negative contribution to the height. Choose a representation of s as $s = s_1^{\otimes a_1} \otimes \cdots \otimes s_l^{\otimes a_l}$ and $K = \mathbb{Q}(x)$. Also denote by Σ the set of places over infinity.

$$\begin{aligned} h_{Fal}(x) &= \sum_{i=1}^{k} \frac{a_i}{[K:\mathbb{Q}]}\left(\sum_{v \text{ finite}} \log^+ \|s(\xi)\|_{can,v} - \sum_{v|\infty} \log \|s(\xi)\|_{Pet,v}\right) \\ &\geq \sum_{i=1}^{k} \frac{a_i}{[K:\mathbb{Q}]}\left(-\sum_{v|\infty} \log \|s(\xi)\|_{Pet,v}\right) \\ &= \frac{1}{[K:\mathbb{Q}]}\sum_{\sigma\in\Sigma} g_s(\sigma(x)) \\ &\geq \inf_{y\in\mathcal{X}(\mathbb{C})} g_s(y). \end{aligned}$$

Therefore for a real section s of $\mathcal{L}$ and $x \in \mathcal{X}(\mathbb{C}) \setminus |\operatorname{div}(s)|$ we have $h_{Fal}(x) \geq \inf_{y\in\mathcal{X}(\mathbb{C})} g_s(y)$ and as a consequence $\mu_{Fal}^{ess} \geq \inf_{y\in\mathcal{X}(\mathbb{C})} g_s(y)$. □

Example 1.17. For $\alpha = j(E_\alpha) \in \{0,1\}$, like for example $\tau = \frac{1+\sqrt{3}i}{2}$ that has j-invariant zero, the equation above gives $h_{Fal}(\alpha) = \dfrac{1}{12} g_{hyp}(\alpha)$.

In the following we consider $\rho = e^{\pi i/3}$ and denote by $\mathcal{T}$ the fundamental domain

$$\mathcal{T} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0,\ |z| > 1 \text{ and } \operatorname{Im}(z) < 1/2\}.$$

Some properties of the functions g_∞ and g_{hyp} are:

Lemma 1.18. *For every $\tau \in \mathcal{T}$ we have $g_\infty(\tau) \geq g_\infty(.5 + i\operatorname{Im}(\tau))$ with equality if and only if $\Re(\tau) = .5$. Moreover the function $t \mapsto g_\infty(.5+it)$ is strictly increasing on $[\frac{\sqrt{3}}{2}, \infty)$ and in particular attains its minimum at $\xi = 0$.*

Proof. This is lemma in 3.1 in [4]. It is a consequence of the vanishing properties of the normalized Eisenstein series $E_2^\star$ on the orbits of i and ρ under $\mathrm{Sl}_2(\mathbb{Z})$. Consider the real valued function $l : \mathbb{R} \longrightarrow \mathbb{R}$ given by $l(x) = g_\infty(s + i\operatorname{Im}(\tau))$. The derivative

$$l'(s) = 2\Re(\delta g_\infty(s + i\operatorname{Im}(\tau))) = 2\pi \operatorname{Im}(E_2^*(s + i\operatorname{Im}(\tau))),$$

and this last one is zero only if $\Re(s) = 0$ or $\Re(s) = .5$. Now from the product formula for

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

is clear that $|\Delta(i\operatorname{Im}(\tau))| \leq |\Delta(.5 + \operatorname{Im}(\tau))|$. For the second part, consider the function $h : (0,\infty) \longrightarrow \mathbb{R}$ defined by $h(t) = g_\infty(.5 + it)$.

The function h satisfies

$$h'(t) = -2\,\mathrm{Im}(\delta g_\infty(.5+it)) = 2\pi\Re(E_2^*(.5+2it)) = 2\pi E_2^*(.5+it).$$

Now, using that the function E_2^* vanishes only at orbit of i and ρ, we get that h' does not vanishes in $(\frac{\sqrt{3}}{2}, \infty)$. $\square$

Lemma 1.19. *Let $g_{hyp} : \mathbb{C} \longrightarrow \mathbb{R}$ be the function defined by the expression $g_\infty = g_{hyp} \circ j$. Then we have $0 < \delta_x g_{hyp}(1) < 1$ and the function $g_1 : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{R}$ defined by $g_1(\xi) = g_{hyp}(\xi) - \delta_x g_{hyp}(1) \log|\xi|$, attains its minimum value at, and only at, $\xi = 1$.*

This is proposition A in [4]. The idea of the proof is to translate the analysis from the upper half-plane $\mathbb{H}$ to the unit disk $\mathbb{D}$ using the map

$$\psi(w) = \frac{\bar{\rho}w + \rho}{w+1} : \mathbb{H} \longrightarrow \mathbb{D}.$$

Now we can define the functions

$$j_{\mathbb{D}} = j \circ \psi : \mathbb{D} \longrightarrow \mathbb{C}, \qquad g_{\mathbb{D}} = g_\infty \circ \psi : \mathbb{D} \longrightarrow \mathbb{R},$$

and

$$f : \mathbb{D} \longrightarrow \mathbb{C} \quad \text{defined by} \quad f(w^3) = j_{\mathbb{D}}(w).$$

Using these function an estimate for $\delta_x g_{hyp}(1)$ stronger that the needed inequality $0 < \delta_x g_{hyp}(1) < 1$ can be actually proved. It can be proved

$$\frac{1}{1032} \leq \delta_x g_{hyp}(1) \leq \frac{1}{1025}.$$

For the second part of the lemma, the proof of $g_1(\xi) \geq g_1(1)$ for all $\xi \neq 0$ is divided in three cases according to the value of $\mathrm{Im}(\tau)$, where $\tau \in T$ and $j(\tau) = \xi$:

case 1. $\mathrm{Im}(\tau) \geq 1$. For $\tau \in \mathbb{H}$ satisfying $\mathrm{Im}(\tau) \geq 1$.
case 2. $\frac{1}{\pi}\log(19) \leq \mathrm{Im}(\tau) \leq 1$.
case 3. $\mathrm{Im}(\tau) \leq \frac{1}{\pi}\log(19)$.

Theorem 1.20. *The first and second minima for the Falting height are $h_{Fal}(0)$ and $h_{Fal}(1)$. We have the inequality*

$$h_{Fal}(0) < h_{Fal}(1) < \mu_{Fal}^{ess}.$$

Proof. This is theorem 1 in [4]. It is obtained as a consequence of lemma 1.18 and lemma 1.19 in the same paper. By lemma 1.18 we know that

$$h_{Fal}(1) = \frac{1}{12}h_{hyp}(1) > \frac{1}{12}h_{hyp}(0) = h_{Fal}(0).$$

To prove the rest of the result it is enough to find $\kappa > 0$ such that for every algebraic number $\alpha \neq 0, 1$ we have $h_{Fal}(\alpha) \geq h_{Fal}(1) + \kappa$.

We are going to use proposition 1.16 for a different section of weight 12, namely $s = (j-1)^{\epsilon} j^{\partial_x g_{hyp}(1)} \Delta$ where ϵ sufficiently small. We will actually consider $\epsilon \in (0, 1 - \partial_x g_{hyp}(1))$ (this last interval is non-empty by lemma 1.19). We construct, for each prime p, the non-negative function $G_{\epsilon,p} : \mathbb{C}_p \setminus \{0,1\} \longrightarrow \mathbb{R}$ defined by

$$G_{\epsilon,p}(z) = \log^+ |z|_p - \partial_x g_{hyp}(1) \log |z|_p - \epsilon \log |z-1|_p.$$

On the other hand for places at infinity, for each $\epsilon \in (0, 1 - \partial_x g_{hyp}(1))$ consider the function $G_\epsilon : \mathbb{C} \setminus \{0,1\} \longrightarrow \mathbb{R}$ defined by the formula

$$G_\epsilon(z) = g_1(z) - \epsilon \log |z-1| = g_{hyp}(z) - \partial_x g_{hyp}(1) \log |z| - \epsilon \log |z-1|.$$

The formula for the Falting height in terms of the Galois orbits, can be expressed using the functions G_ϵ and $G_{\epsilon,p}$ with the help of the product formula. We obtain:

$$12 h_{Fal}(\alpha) = \frac{1}{\#\mathcal{O}(\alpha)} \sum_{\alpha' \in \mathcal{O}(\alpha)} G_\epsilon(\alpha') + \frac{1}{\#\mathcal{O}(\alpha)} \sum_{p \text{ prime}} \sum_{\alpha' \in \mathcal{O}(\alpha)} G_{\epsilon,p}(\alpha'),$$

and we need to show that $\inf_{\mathbb{C}\setminus\{0\}} G_\epsilon(z) > g_{hyp}(1)$. But the asymptotic of g_{hyp} coming from g_∞ tell us that

$$g_{hyp}(z) = \log |z| - 6 \log(\log |z|) + O(1) \quad \text{as } z \longrightarrow \infty.$$

Therefore for $\epsilon_0 > 0$ satisfying $\epsilon_0 + \partial_x g_{hyp}(1) < 1$ and $|z| > R_0$ we have that $G_\epsilon(z) \to \infty$ in z and then for any $C > 0$ fixed:

$$G_\epsilon(z) \geq g_{hyp}(1) + C.$$

By proposition 1.19 there is an $\epsilon \in (0, \epsilon_0)$ such that for some $\delta > 0$ and every z satisfying $|z-1| \geq 1/2$ and $|z| \leq R_0$ we have the bound

$$G_\epsilon(z) \geq g_{hyp}(1) + \delta.$$

Again using proposition 1.19 for z satisfying $|z-1| \leq 1/2$ we have

$$G_\epsilon(z) = g_1(z) - \epsilon \log |z-1| \geq g_{hyp}(z) + \epsilon \log(2)$$

which completes the proof of the theorem. □

1.5. **Toric Varieties.** Toric varieties are algebraic varieties that admit a torus action. Let $\mathbb{T}^n = \mathbb{G}_m^n$ be the split algebraic torus over a field K. We clearly have an action $\mu : \mathbb{T}^n \times \mathbb{T}^n \longrightarrow \mathbb{T}^n$.

Definition 1.21. *A toric variety with torus $\mathbb{T}^n$ is a normal variety X such that $\mathbb{T}^n \subset X$ and the natural action μ extends to an action of $\mathbb{T}^n$ on the whole X.*

One possible construction of toric varieties uses rational polyhedral cones and fans. Let $N \cong \mathbb{Z}^n$ be a lattice of dimension n and let us denote $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. A strongly convex rational polyhedral cone is a set $\sigma \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ such that:

(1) It is convex, i.e. $\lambda x + (1-\lambda)y \in \sigma$ for $x, y \in \sigma$ and $\lambda \in [0,1]$.
(2) It is a cone, i.e. $\lambda x \in \sigma$ for $x \in \sigma$ and $\lambda \in \mathbb{R}^+$.
(3) It is polyhedral, meaning that it is defined as intersection of semi-spaces $\sigma = \cap_i H^+_{u_i}$, where $u_i \in N_{\mathbb{R}}$ and

$$H^+_{u_i} = \{v \in N_{\mathbb{R}} \,|\, (v, u_i) \geq 0\}.$$

(4) It is rational, i.e. $u_i \in N$.
(5) It is strongly convex, meaning that it does not contain a linear subspace other 0.

Definition 1.22. *A face τ of σ is given by the intersection $\sigma \cap H_u$ with a semiplane, where $\sigma \subset H^+_u$. A one dimensional face is called a ray. A $(n-1)$-dimensional face is called a facet.*

Definition 1.23. *Let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual of N. The dual cone $\sigma^\vee \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ is given by*

$$\sigma^\vee = \{u \in M_{\mathbb{R}} \,|\, (u, v) \geq 0 \text{ for all } v \in N_{\mathbb{R}}\}.$$

Remark 1.24. A cone σ is strongly convex if and only if the dual cone $\sigma^\vee$ is of maximal dimension. If σ is of maximal dimension and strongly convex, then the dual is given by $\sigma^\vee = \cap_i H^+_{u_i}$ where the set $\{u_1, \ldots, u_n\}$ generates σ.

Definition 1.25. *Let σ be a strongly convex rational polyhedral cone. The affine toric variety X_σ associated to the cone σ is given by $X_\sigma = \mathrm{Spec}(K[M_\sigma])$, where $K[M_\sigma]$ is the semi-group algebra generated by the integral points $M_\sigma = M \cap \sigma^\vee$ of the dual cone $\sigma^\vee$. To each element $m \in M_\sigma$ we will associate the character χ^m which can be identified with $t^m = t_1^{m_1} t_2^{m_2} \ldots t_n^{m_n}$ via the map $\beta : M \longrightarrow \mathbb{Z}^n$. By Gordan's lemma, the ring $K[M_\sigma] = K[\chi_{M_\sigma}]$ is generated as K-algebra by the finitely many integral points in the unit cube of $\sigma^\vee$.*

Definition 1.26. *A fan Σ is a set or strongly convex rational polyhedral cones, such that if $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma' \in \Sigma$ and for all $\sigma \in \Sigma$, if $\tau \subset \sigma$ is a face of σ, then $\tau \in \Sigma$.*

If τ is a face of σ, the map $\tau \hookrightarrow \sigma$ induces an open immersion $X_\tau \hookrightarrow X_\sigma$ and we can glue together X_σ and $X_{\sigma'}$ along $X_{\sigma \cap \sigma'}$ associated to their common face $\sigma \cap \sigma'$, to form the toric variety X_Σ.

Example 1.27. Let Σ be a fan. (0) is a face of every $\sigma \in \Sigma$, thus $X_0 = \mathbb{T}^n$ is an open set in all X_σ and therefore in X_Σ. The action of X_0 on X_σ is defined as corresponding to the map $m \mapsto m \otimes m$ from M_σ to $M \otimes M_\sigma$.

Example 1.28. If we take the fan in $\mathbb{R}$ made out of the three cones $\{0, \sigma_+ = \mathbb{R}^+ e_1, \sigma_- = \mathbb{R}^+(-e_1)\}$ we get

$$X_+ = \mathrm{Spec}(K[x]), \qquad X_- = \mathrm{Spec}(K[x^{-1}]),$$

glued along $X_0 = \mathrm{Spec}(K[x, x^{-1}])$ by the map $x \to x^{-1}$. This gives $\mathbb{P}^1_K$ as a toric variety.

Example 1.29. In a similar way as we did to obtain $\mathbb{P}^1$, we can consider the simplex $\Delta^n = conv(0, e_1, \ldots, e_n)$ and the fan Σ_{Δ^n} generated by

$$\{e_1, e_2, \ldots, e_n, e_0 = -(e_1 + \cdots + e_n)\},$$

in the sense that the cones $\sigma \in \Sigma_{\Delta^n}$ are generated by a strict subset of the above set of vectors. The toric variety obtained is $X_\Sigma = \mathbb{P}^n$.

Let $v \in \mathcal{M}_\mathbb{Q}$ be a place of $\mathbb{Q}$ and let $\mathbb{S}_v \subset \mathbb{T}^n_v$ be the compact invariant torus. In general $\mathbb{S}_v$ is a compact analytic subgroup of $\mathbb{T}^n_v$ defined as

$$\mathbb{S}_v = \{P \in \mathbb{T}^n_v \,|\, \chi^m(P)| = 1\,,\, \forall m \in M\}.$$

Example 1.30. $\mathbb{S}_\infty = (\mathcal{S}^1_\mathbb{C})^n = \{(t_1, \ldots, t_n) \in (\mathbb{C}^*)^n \,|\, |t_i| = 1 \text{ for all } i\}$.

Definition 1.31. *Let $\bar{\mathcal{L}} = (L, \|.\|) = (\mathcal{L}, (\|.\|_v)_{v \in \mathcal{M}_\mathbb{Q}})$ be a metrized toric line bundle on the toric variety X. The metric $\|.\|$ is called toric if $\|.\|_v$ is $\mathbb{S}_v$-invariant for all v of $\mathcal{M}_\mathbb{Q}$.*

Suppose that the fan Σ is complete. We have a valuation map

$$val_v : \mathbb{T}_v \subset X_v \longrightarrow N_\mathbb{R} \cong \mathbb{R}^n$$

given by $val_v(x_1, \ldots, x_n) = (-\log |x_1|_v, \ldots, -\log |x_n|)$. The fibre of val_v over $0 \in N_\mathbb{R}$ coincides with the compact invariant torus $\mathbb{S}_v$.
We will define now a family of functions $\varphi_v : N_\mathbb{R} \longrightarrow \mathbb{R}$ called the metric functions associated to the metrized toric line bundle $\bar{\mathcal{L}} = (\mathcal{L}, \{\|.\|_v\}_{v \in \mathbb{M}_\mathbb{Q}})$.

Definition 1.32. *Let $\{\|.\|_v\}_{v \in \mathbb{M}_\mathbb{Q}}$ be a toric metric on the toric line bundle $\mathcal{L}$. We define the metric function $\varphi_v : N_\mathbb{R} \cong \mathbb{R}^n \longrightarrow \mathbb{R}$ by*

$$\varphi_v(u) = -\log \|s(P)\|_v,$$

where $u = val_v(P)$. It is well defined because a toric metric is $\mathbb{S}_v$-invariant.

Example 1.33. Consider the line bundle $\mathcal{L} = \mathcal{O}(D_\Psi)$ associated to a divisor with support function Ψ. If $\|.\|$ is any toric metric on $\mathcal{L}$ then the function $|\varphi_v - \Psi|$ is bounded.

Definition 1.34. *A metric $\{\|.\|_v\}_{v\in\mathbb{M}_\mathbb{Q}}$ on $\mathcal{O}(D_\Psi)$ is adelic if $\varphi_v = \Psi$ for all v except maybe a finite number.*

Theorem 1.35. *There is a bijection between the set of semi-positive adelic toric metrics on $\mathcal{L}$ and the set of continuous concave functions $\{\psi_v\}_{v\in\mathbb{Q}}$ on $N_\mathbb{R}$ such that $|\psi_v - \Psi|$ is bounded and $\psi_v = \Psi$ for all v except maybe a finite number.*

This is Theorem in 4.8.1 in [2].

Example 1.36. We can define a adelic metric on $\mathcal{O}(D_\Psi)$, called the canonical metric $\|.\|_{\mathrm{can},v}$, by the equation

$$\log \|s_D(P)\|_{\mathrm{can},v} = \Psi_D(val_v(P)),$$

where $v \in \mathbb{M}_\mathbb{Q}$. The canonical metric is semi-positive if and only if D_Ψ is nef.

Remark 1.37. For a toric metric $\{\|.\|_v\}_{v\in\mathbb{M}_\mathbb{Q}}$ on a toric line bundle $\mathcal{L} = \mathcal{O}(D_\Psi)$ and $P \in X_0(\mathbb{Q})$ we have $\log \|s_D(P)\|_v = \varphi_{v,D}(val_v(P))$ for all places. The formula for the height of P becomes

$$h_{\bar{\mathcal{L}}}(P) = -\sum_{v\in\mathcal{M}_\mathbb{Q}} \log \|s(P)\|_v = -\sum_{v\in\mathcal{M}_\mathbb{Q}} \varphi_{v,D}(val_v(P)).$$

Definition 1.38. *The v-adic roof function ϑ_v is given by the function $\vartheta_v : \Delta_\mathcal{L} \longrightarrow \mathbb{R}$ defined by the formula:*

$$\vartheta_v(x) = \inf_{y\in\mathbb{R}^n} (x, y) - \varphi_v(y).$$

The global v-adic roof function is defined as $\vartheta = \sum_v \vartheta_v$. In the case of a semi-positive metric, the v-adic roof function coincide with the Legendre-Frechnel dual $\varphi_v^\vee$ of the concave function φ_v.

Example 1.39. Suppose that X is the projective space as in example 1.29. The Legendre-Frechnel dual $\varphi_\infty^\vee$ of the function

$$\varphi_\infty = \Psi_{\Delta^n}(u_1, \dots, u_n) = \min(0, u_1, u_2, \dots, u_n)$$

is the indicator function $\varphi_\infty^\vee = i_{\Delta^n}$ of the associated polytope Δ^n.

Theorem 1.40. *For a semi-positive metrized line bundle $\bar{\mathcal{L}} = \overline{\mathcal{O}(D_\Psi)}$, the roof functions satisfy $\vartheta_v = 0$ for all v except maybe a final number. In fact we have a bijection between the set of semi-positive adelic toric metrics on $\mathcal{L}$ and the set of continuous concave functions $\{\psi_v^\vee\}_{v\in\mathbb{Q}}$ on Δ_Ψ such that $\psi_v^\vee = 0$ for all v except maybe a finite number.*

This is Theorem 4.9.2 in [2].

Remark 1.41. For a toric metric $\{\|.\|_v\}_{v\in\mathbb{M}_{\mathbb{Q}}}$ on a toric line bundle $\mathcal{L} = \mathcal{O}(D_\Psi)$ and $P \in X_0(\mathbb{Q})$ we have $\sum_{v\in\mathbb{M}_{\mathbb{Q}}} val_v(P) = 0$ and the height of P in the connected component X_0 of 0 satisfies

$$h_{\bar{\mathcal{L}}}(P) = \sum_{v\in\mathcal{M}_{\mathbb{Q}}} (x, val_v(P)) - \varphi_{v,D}(val_v(P)) \geq \sum_{v\in\mathcal{M}_{\mathbb{Q}}} \vartheta_v(x) = \vartheta(x),$$

for any $x \in \Delta_\Psi$. In particular, that absolute minimum $\mu^{abs}_{\bar{\mathcal{L}}}(X_0) \geq \max_{x\in\Delta_\Psi} \vartheta(x)$. A sharper result is obtained in the following theorem.

Theorem 1.42. *Let X be a proper toric variety over $\mathbb{Q}$ and $\bar{\mathcal{L}} = \overline{\mathcal{O}(D)}$ a toric metrized $\mathbb{R}$-line bundle. Then*

$$\mu^{ess}_{\overline{\mathcal{O}(D)}}(X) = \max_{x\in\Delta} \vartheta(x).$$

This is Theorem A (Corollary 3.10) in [3].

1.6. **Curves on Products of the multiplicative group.** Let G_m be the multiplicative group of dimension 1. Let us define a distance function by

$$d_\infty : (G_m^n)^2 \longrightarrow \mathbb{R}^+, \qquad d_\infty(x, y) = h(xy^{-1}).$$

Proposition 1.43. *Let $C \subset (G_m^n)^2(\mathbb{Q})$, there exist a constant κ such that for all $x, y \in G_m^n(\bar{\mathbb{Q}})$ except at most a finite number,*

$$d_\infty(x, y) \geq K.$$

This is Theorem 6.2 in [11], obtained as a consequence of positive self-intersection and the inequality between successive minima.

Theorem 1.44. *For all algebraic numbers $\alpha \neq 0, 1, \dfrac{1 \pm \sqrt{3}i}{2}$, we have*

$$h(\alpha) + h(1 - \alpha) \geq \frac{1}{2} \log \frac{1 + \sqrt{5}}{2} = .2406059...,$$

with equality if and only if alpha or $1 - \alpha$ is a primitive 10-th root of unity.

This is Theorem 1 in [10] and the proof is based on the following lemma in the same paper.

Lemma 1.45. *For all $z \in \mathbb{C}$ and for all places v, if $n_v = 1, 2$ or 0 depending on the place v being real Archimedean, complex Archimedean* *or non-Archimedean, we have:*

$$\max(0, \log|z|_v) + \max(0, \log|1-z|_v) \geq$$
$$\geq \frac{\sqrt{5}-1}{2\sqrt{5}} \log|z^2 - z|_v + \frac{1}{2\sqrt{5}} \log|z^2 - z + 1|_v + \frac{n_v}{2} \log \frac{1+\sqrt{5}}{2}.$$

Variety	**Metric/height**	**L. bd. for μ^{ess}**	**Formula for μ^{ess}**
$(X,\mathcal{L})$ Gen.Alg.Var	Semi-pos. smooth. met.	$\mu^{ess}_{\bar{\mathcal{L}}}(X) \geq \frac{h_{\bar{\mathcal{L}}}(X)}{(n+1)\deg_{\bar{\mathcal{L}}}(X)}$	—-
(X,φ) Dyn. Systems.	Canonical metric	$\mu^{ess}_{\bar{\mathcal{L}}}(X) \geq 0$	$\mu^{ess}_{\bar{\mathcal{L}}}(X) = 0$
Toric varieties	Toric metric	$\mu^{ess}_{\bar{\mathcal{L}}}(X) \geq \mu^{abs}_{\bar{\mathcal{L}}}(X_0)$	$\mu^{ess}_{\bar{\mathcal{L}}}(X) = \max_{x\in\Delta} \vartheta(x)$
Mod. Space of E.C.	Faltings height	$h_{Fal}(1) = -.74862817$	—-
$\{x+y=1\} \subset (G_m)^2$	"Product height"	$\mu^{ess}_{\bar{\mathcal{L}}}(X) \geq \frac{1}{2}\log\frac{1+\sqrt{5}}{2}$	—-
$X \subset$ toric variety	Toric metric	—-	—-

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