# Spaces with simple Picard Group

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# Projective modules

## Definition

We say that an $A$-module $M$ of finite type is locally free for the Zariski topology if we can find elements $\{f_1, \ldots, f_m\}$ such that $\langle f_1, \ldots, f_m \rangle = A$ and $M_{f_i}$ is free over $A_{f_i}$.

A **projective module of finite type** $M$ will be a module over $A$ satisfying any of the following equivalent conditions:

(1) The module $M$ is locally free for the Zariski topology of $\mathrm{Spec}(A)$.

(2) For every prime ideal $\mathfrak{p}$, we have that $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module.

(3) The functor $\hom_A(M, .)$ is exact.

(4) $M$ is a direct summand of a free module

## Example

The free $A$-module $M = A^I$ is a projective $A$-module.

# Projective modules and invertible modules

## Definition

Let $A$ be a ring and $M$ a projective $A$-module of finite type over $A$. The function

$$r(M)\colon \operatorname{Spec}(A) \longrightarrow \mathbb{N}$$

that to a prime ideal $\mathfrak{p}$ associates the dimension of the vector space $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ over the field $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, is locally constant.
One says that $M$ is of **rank** $n$ if the function $r(P)\colon \operatorname{Spec}(A) \longrightarrow \mathbb{N}$ is constant and equal to $n$.

## Definition

Let $A$ be a ring. We say that $M$ is an **invertible** $A$-module if and only if $M$ is **projective of rank one**.

# Invertible modules

## Proposition

Let $M$ be an $A$-module and denote its dual by $M^\vee = \mathrm{Hom}_A(M, A)$. The following are equivalent

1. $M$ is invertible,
2. $M$ is finite locally free of rank one,
3. The canonical map of evaluation $M \otimes_A M^\vee \longrightarrow A$ is an isomorphism,
4. There exist an $A$-module $N$ such that $M \otimes N \cong A$.

## Examples

Let $A$ be an **Dedekind domain** with field of fractions $K$, $0 \neq x \in A$ and $I \subset A$ ideal. We can define an invertible module as the fractional ideal

$$\frac{1}{x} I \subset K.$$

Localization gives Discrete valuation rings and $I$ becomes principal.

# The Picard group of a ring

**Remark**

if $M$ is an invertible $A$-module, then the dual $M^\vee$ is also invertible. If $M_1$ and $M_2$ are invertible modules, the tensor product $M_1 \otimes M_2$ is also invertible.

**Proposition**

The tensor product gives the set of **isomorphism classes of invertible $A$-modules**, the structure of commutative group. The class of $A$ is the neutral element and the class of the dual is the inverse. This group is called the **Picard group of $A$ and denoted** $\mathrm{Pic}(A)$.

**Remark**

We denote a trivial Picard group by $\mathrm{Pic}(A) = 0$.

# The Picard group of a ring

**Example**

If $(A, m)$ is a local ring, then the Picard group is trivial: $\mathrm{Pic}(A) = 0$.

**Remark**

Pic is a contravariant functor $\mathrm{Pic}\colon \mathrm{Rings} \longrightarrow \mathrm{Abelian\ Groups}$ determined by the maps $A \mapsto \mathrm{Pic}(A)$ and $M \mapsto M \otimes_B A$ for any map $f : B \longrightarrow A$.

**Remark**

(Claborn 1966) All abelian groups arises as the Picard group of some Dedekind domain!

# Rings of dimension zero

**Proposition**

For a noetherian ring $A$ of dimension zero we have the following:

- $A$ is artinian.
- $A$ has finite length.
- For a module $M$ over $A$, there is finite list $m_1, m_2, \ldots, m_r$ of maximal ideals of $A$ such that $M_{m_i} \neq 0$. Then the application:

$$M \longrightarrow \prod_{i=1}^{r} M_{m_i}$$

  is an isomorphism. As a consequence, for a noetherian ring of dimension zero, $\mathrm{Pic}(A) = 0$.

# Modules of trivial Picard group

A number of rings have trivial Picard groups. This means that **locally free modules of rank one are in fact free**.

**Proposition**

The following rings have trivial Picard group:

(1) A local ring $A$.

(2) A noetherian ring $A$ of dimension zero.

(3) A unique factorization domain $A$.

(4) A principal ideal domain $A$.

Since UFD and local rings can have any dimension, we know that there are rings of arbitrary dimension with trivial Picard group.

# Number fields

A field $K$ that is a finite extension of the rational numbers is called: **A number field**. The integer $[K : \mathbb{Q}]$ is called the degree of $K$.

**Example**

For example $\mathbb{Q}(i)$, with $i^2 = -1$, is a number field of degree 2 over $\mathbb{Q}$.

**Definition**

Let $K$ be a number field and $A \subset K$ a ring integral over $\mathbb{Z}$ such that the fraction field of $A$ is exactly $K(A) = K$. Such rings $A$ are called **orders** of the number field $K$.

**Example**

We have for example the orders $\mathbb{Z}[\sqrt{5}]$ and $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ in the number field $\mathbb{Q}(\sqrt{5})$.

# Number fields

**Definition**

Let $K$ be a number field. We call ring of integers of $K$ to the integral closure $\mathcal{O}_K$ of $\mathbb{Z}$ in $K$.

**Remark**

Let $K$ be a quadratic extension (degree 2). Then, there exist a $d \in \mathbb{Z}$ such that $K = \mathbb{Q}(\sqrt{d})$. Also, the ring of integers $\mathcal{O}_K$ of $K = \mathbb{Q}(\sqrt{d})$ is as follows:

(a) If $d \equiv 2, 3\, mod(4)$. The ring $\mathcal{O}_K$ is $\mathbb{Z} + \mathbb{Z}\sqrt{d}$.

(b) If $d \equiv 1\, mod(4)$. The ring $\mathcal{O}_K$ is $\mathbb{Z} + \mathbb{Z}(\frac{1+\sqrt{d}}{2})$.

# Number fields

**Corollary**

*Let $A$ be an order in a number field $K$ of degree $n$ over $\mathbb{Q}$ and $M$ be an invertible $A$-module. Then $M$ is a free $Z$-module of rank $n$. An order in a number field is a noetherian ring of dimension one.*

**Definition**

A noetherian ring integrally closed and of dimension one is called a Dedekind ring.

**Example**

The ring of integers $\mathcal{O}_K$ in a number field $K$ is a Dedekind ring. The localization $A_p$ at any prime ideal $p$ will be a discrete valuation ring.

**Proposition**

Let $A$ be a Dedekind ring. Then every ideal of $A$ is invertible.

# Picard group of a number fields

**Proposition**

Let $A$ be an order in a number field $K$. Then $\mathrm{Pic}(A)$ is a finite group.

It be obtained as a consequence of Minkowski Theorem and some properties of the norm function $N\colon \{\text{ ideals } \subset A\} \longrightarrow \mathbb{Z}_+$.
Minkowski's theorem guarantees the existence of a constant $\chi'(A)$ and a surjection

$$N_{\leq}(A) \longrightarrow \mathrm{Pic}(A) \longrightarrow 0,$$

where the set $N_{\leq}(A)$ is made of ideals of bounded norm

$$N_{\leq}(A) = \{\text{ ideals } \mathfrak{a} \subset A \mid N(\mathfrak{a}) \leq \exp(-\chi'(A))\}.$$

The finiteness of the $\mathrm{Pic}(A)$ is a consequence of the finiteness lemma for the norm.

# Picard group of a number fields

## Example

Consider the maximal order (ring of integers) $A = \mathcal{O}_K = \mathbb{Z}[i]$ in the number field $K = \mathbb{Q}(i)$. The ring $A$ is a UFD and $\mathrm{Pic}(A) = 0$.
Again for the number field $K = \mathbb{Q}(\sqrt{-5})$, the ring of integers is $A = \mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$. In this case, the ring $A$ is not a UFD, we can have for example

$$3 \times 3 = (2 + \sqrt{-5})(2 - \sqrt{-5}).$$

The Picard group contains two elements and is

$$\mathrm{Pic}(A) = \mathbb{Z}/2\mathbb{Z} = \langle (2, 1 + \sqrt{-5}) \rangle.$$

We can check that $I = (2, 1 + \sqrt{-5})$ satisfies $I^2 = (2)$.

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# Schemes

Let $A$ be a ring. The **affine scheme** $(\mathrm{Spec}(A), \mathcal{A})$ is the topological space

$$\mathrm{Spec}(A) = \{\mathfrak{p} \mid \mathfrak{p} \subsetneq A \text{ prime ideal }\},$$

together with the **Zariski topology** and a sheaf of rings $\mathcal{A}$, in such a way that for $f \neq 0 \in A$ we have

$$\mathcal{A}(U_f) = \text{ local ring } A_f$$

for the open set $U_f = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$.

## Definition

A **scheme** $(X, \mathcal{O}_X)$ is a locally ringed space that is locally isomorphic to an affine scheme $(\mathrm{Spec}(A), \mathcal{A})$. The sheaf $\mathcal{O}_X$ is called the structural sheaf and $X$.

## Example

The projective space $\mathbb{P}^n_A$ over a ring $A$ is an example of a scheme that is not affine.

# Schemes and invertible sheaves

## Definition

An **invertible sheaf over a scheme** $X$ is a quasi-coherent sheaf $\mathcal{L}$ such that there exist a covering of $X$ by open sets $\{U_i\}_{i \in I}$ such that $\mathcal{L}|U_i \backsimeq \mathcal{O}_{U_i}$. Invertible sheaves are also called **line bundles**.

## Remark

We can check that if $U = \text{Spec}(A)$ is an affine open set of a scheme $X$ and $\mathcal{L}$ is an invertible sheaf on $X$, then $L|U$ is the sheaf associated to a projective $A$-module.

## Example

On the projective space $\mathbb{P}^n_A$, the twisted sheaves $\mathcal{O}(d)$ are invertible sheaves for all $d \in \mathbb{Z}$.

# Picard group of a scheme

## Definition

**The isomorphism classes of invertible sheaves on** $X$ with the tensor product have the natural structure of abelian group. We called the **Picard group of** $X$, denoted $\text{Pic}(X)$.

## Example

The Picard group of an affine scheme $\text{Spec}(A)$ coincide with the Picard group of $A$ as a ring.

## Example

The Picard group of the affine scheme $\mathbb{C}^n$ is trivial $\text{Pic}(\mathbb{C}^n) = 0$.
On the other hand, If a smooth affine curve $X$ over $\mathbb{C}$ is obtained by removing a point from a projective curve $\bar{X}$, then

$$\text{Pic}(X) \cong \text{ Jacobian of } \bar{X} \cong (\mathbb{R}/\mathbb{Z})^{2g}.$$

# Picard group and sheaf cohomology

## Remark

For any short exact sequence of coherent sheaves on the scheme $X$

$$1 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 1,$$

the Sheaf cohomology consider the functor $\Gamma(X, .)$ of global sections. We get a long exact sequence of cohomology groups

$$\to \mathrm{H}^1(X, \mathcal{A}) \longrightarrow \mathrm{H}^1(X, \mathcal{B}) \longrightarrow \mathrm{H}^1(X, \mathcal{C}) \longrightarrow \mathrm{H}^2(X, \mathcal{A}) \longrightarrow \mathrm{H}^2(X, \mathcal{B}) \to$$

## Remark

The group $\mathrm{Pic}(X)$ has a cohomological interpretation as

$$\mathrm{Pic}(X) = \mathrm{H}^1(X, \mathcal{O}_X^*).$$

# Ample line bundles

**Remark**

Let $X$ be a projective scheme and $\mathcal{L}$ a line bundle on $X$. We have the following equivalences defining that $\mathcal{L}$ is ample:

1. There exist a power $\mathcal{L}^m$ and an immersion $\iota \colon X \hookrightarrow \mathbb{P}^n$ with $\mathcal{L}^m = \iota^*\mathcal{O}(1)$.
2. For every quasi-coherent sheaf $\mathcal{F}$ of finite type on $X$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is generated by its global sections for $m >> 0$.
3. For every quasi-coherent sheaf $\mathcal{F}$ of finite type on $X$, the map

$$H^0(\mathcal{F} \otimes \mathcal{L}^{\otimes m}, X) \otimes \mathcal{O}_X \longrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes m},$$

is surjective for $m >> 0$.

# Ample line bundles

## Example

Consider for example the line bundle $\mathcal{O}(d)$ on a projective space $X = \mathbb{P}^n_A$. The sections over an open set $U \subset X$ are

$$\mathcal{O}_X(d)(U) = \{\frac{f}{g} \mid g(x,y) \neq 0 \text{ and } \deg(\frac{f}{g}) = d\}.$$

The global sections $\mathcal{O}_X(d)(X)$ will be polynomials of degree $d$ and only constants for the trivial structural sheaf $\mathcal{O}_X$.
For any $d > 0$, the line bundle $\mathcal{O}(d)$ is ample. For any projective scheme $Y \subset \mathbb{P}^n$, the restriction

$$\mathcal{O}_X(d) \longrightarrow \mathcal{O}_X(d)|_Y$$

will be surjective on global sections for $d$ big enough.

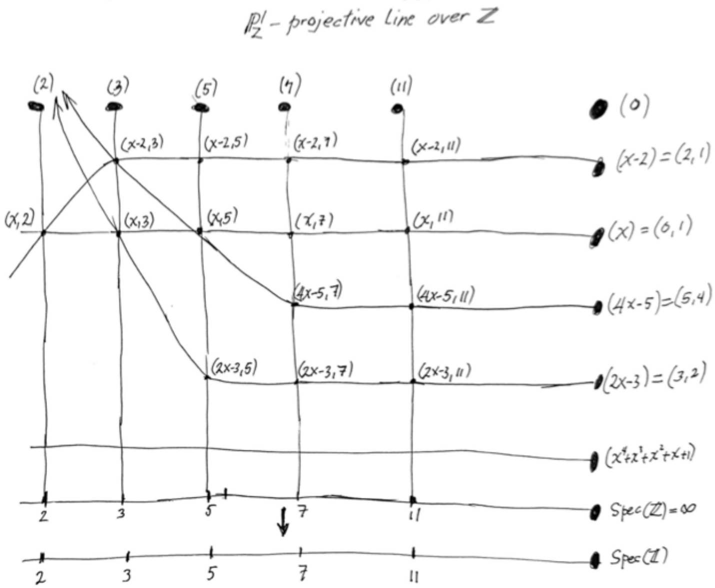
# Projective spaces over $\mathbb{Z}$

## Definition

The scheme $X = \mathbb{P}^n_{\mathbb{Z}}$ is called the projective space over $\mathbb{Z}$ and is defined as $\mathrm{Proj}(\mathbb{Z}[X_0, \dots X_n])$. It has the following properties:

1. It comes equipped with a separated proper map $\mathbb{P}^n_{\mathbb{Z}} \longrightarrow \mathrm{Spec}(\mathbb{Z})$.
2. The generic fibre is $\mathbb{P}^n_{\mathbb{Q}} \longrightarrow \mathrm{Spec}(\mathbb{Q})$.
3. It is locally isomorphic to the affine space $\mathbb{A}^n_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[y_1, \dots y_n])$.
4. The group of global sections $\mathrm{H}^0(X, \mathcal{O}_X) = \mathbb{Z}$.
5. It is an $(n+1)$-dimensional scheme, $n$-dimensional over the base $\mathbb{Z}$.

# Projective line over $\mathbb{Z}$



# Finite schemes have finite Picard group

## Proposition

Let $X \subset \mathbb{P}^r_{\mathbb{Z}}$ be a closed finite scheme. Then $\mathrm{Pic}(X)$ is finite.

## Proof.

Following Bruce and Eman, we are going to do two steps:
(1) Reduction to the reduced case: Consider the nilradical $\mathcal{N}$ with $\mathcal{N}^m = 0$ for some $m > 1$. Take $X' \subset X$ to be the subscheme determined by $\mathcal{N}^{m-1}$. We have the exact sequence

$$1 \longrightarrow \mathcal{N}^{m-1} \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{O}_{X'}^* \longrightarrow 1,$$

and since $\mathrm{H}^1(X, \mathcal{N}^{m-1}) = \mathrm{H}^2(X, \mathcal{N}^{m-1}) = 0$, the long exact sequence of cohomology will give us

$$\mathrm{Pic}(X) = \mathrm{H}^1(X, \mathcal{O}_X^*) \cong \mathrm{H}^1(X', \mathcal{O}_{X'}^*) = \mathrm{Pic}(X').$$

# Finite schemes have finite Picard group

**Proposition**

Let $X \subset \mathbb{P}^r_{\mathbb{Z}}$ be a closed finite scheme. Then $\mathrm{Pic}(X)$ is finite.

**Proof.**

(2) Assume that $X = \mathrm{Spec}(B)$, where $B$ is a reduced and finite $\mathbb{Z}$-algebra. If $p$ is a minimal prime of $B$, we have that $B/p$ is either **zero dimensional or an order in a number field**. In any case the Picard group $\mathrm{Pic}(B/p)$ is finite. Consider now the intersection $a$ of all the other minimal primes other than $p$.

$$\cdots \longrightarrow (B/(p+a))^* \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(B/p) \oplus \mathrm{Pic}(B/a) \longrightarrow \cdots$$

Now, since $(B/(p+a))^*$ is a finite set and $B/a$ has fewer minimal primes than $B$, we can proceed by induction on the number of minimal primes. □

# An application of finite Pic

We know that Bezout's theorem asserts that for a pair $(x, y) \in \mathbb{P}^1_{\mathbb{Z}}$ we can find a linear homogeneous polynomial $f(u, v) = au + bv$ with $\gcd(a, b) = 1$ such that $f(x, y) = 1$.

**Remark**

The following result works instead with finite set of reduced lattice points $S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$.

**Proposition**

If $S$ is a finite set of reduced lattice points $(x, y)$, i.e., with $\gcd(x, y) = 1$, then there is a non-constant homogeneous polynomial $f \in \mathbb{Z}[x, y]$ such that $f(x, y) = 1$ for all $(x, y) \in S$.

A lattice point $(x, y)$ with $x, y \in \mathbb{Z}$ and $\gcd(x, y) = 1$ corresponds to a section of $\mathbb{P}^1_{\mathbb{Z}} \longrightarrow \mathrm{Spec}(\mathbb{Z})$. Consider the set $S$ as a scheme $S \subset \mathbb{P}^1_{\mathbb{Z}}$ via the union of the these sections.

# An application of finite Pic

For any positive integer $d$ we have a restriction map $\mathcal{O}(d)_{\mathbb{P}^1} \longrightarrow \mathcal{O}(d)|_S$.
For any section $(x, y)\colon \operatorname{Spec}(\mathbb{Z}) \longrightarrow S$ we have also the pull-back $\mathcal{O}(d)|_S \longrightarrow \mathcal{O}(d)_{\operatorname{Spec}(\mathbb{Z})}$. If we do the composition, the map

$$\mathcal{O}(d)_{\mathbb{P}^1} \longrightarrow \mathcal{O}(d)|_S \longrightarrow \mathcal{O}(d)_{\operatorname{Spec}(\mathbb{Z})}$$

corresponding to the section $(x, y)$, gives the evaluation map $\mathbb{Z}[x, y]_d \longrightarrow \mathbb{Z}$ when restricted to global sections.
It will be sufficient to show that there exists a global section $f$ of $\mathcal{O}(d)_{\mathbb{P}^1}$ that is nowhere zero. Since $\mathcal{O}(1)_{\mathbb{P}^1_{\mathbb{Z}}}$ is ample, the map $\mathcal{O}(d)_{\mathbb{P}^1} \longrightarrow \mathcal{O}(d)|_S$ will be surjective on global sections for $d >> 0$.
On the other hand, the scheme $S$ made of finitely many points has finite Picard group $\operatorname{Pic}(S)$.
As a consequence, for infinitely many powers $d$, the $\mathcal{O}(d)|S$ will be trivial and the global sections will be just constants. By subjectivity, there will be $f \in \mathbb{Z}[x, y]_d$ for $d$ big enough mapping to a unit.

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*Thanks!*