1. A presentation of the étale fundamental group

1.1. **Preliminaries.** A group G is pro-finite if it can be obtained as inverse limit of a system G_i of finite groups with maps $G_i \to G_j$ for all $i > j$, satisfying the necessary properties. The pro-finite completion of a group G is the inverse limit of the system consisting of finite quotients G/H, where H is a normal subgroup of finite index. A pro-finite group comes equipped with the initial topology that makes continuous all the projection maps $p_i : G \to G_i$. The topology so defined is called the pro-finite topology. One can take a basis for open sets made of the co-sets of the normal subgroups of finite index.

Definition 1.1. *Let $(X, \mathcal{O}(X))$ and $(X', \mathcal{O}(X'))$ be schemes. A finite morphism $\varphi : X' \to X$ is locally free if the direct image $\varphi_* \mathcal{O}(X')$ is locally free as an $\mathcal{O}(X)$-module. If moreover each fibre X'_P is the spectrum of a finite étale $\kappa(P)$-algebra, we have a finite étale morphism.*

Remark 1.2. A finite morphism $X' \to X$ amounts to define a quasi-coherent $\mathcal{O}(X)$-algebra $\mathcal{A}$, that is a locally of finite rank as an $\mathcal{O}(X)$-module. The image of a finite étale morphism is both: open and closed. It is open because $\varphi_* \mathcal{O}(X')$ being locally free, forces non-zero stalks over every open set. It is closed because a finite map is stable under base change and therefore proper.

1.2. **Anabelian Geometry.** For a family of Schemes F (anabelian varieties) over certain types of fields, we try to recover the isomorphism class of a scheme $X \in F$ from the isomorphism class of a pro-finite group $\pi_1^{etale}(X)$ associated to X. The construction of the étale fundamental group $\pi_1^{etale}(X)$ is inspired by the known fact that the subgroups of the topological fundamental group $\pi_1^{top}(X)$ correspond to covering maps $X' \to X$ of a connected topological space X. Also, in the topological situation proper curves of genus $g > 1$ over $\mathbb{C}$ are determined by their fundamental groups. When considering fields k that are not algebraically closed we loose the machinery that we have over $\mathbb{C}$ and try to replace it with the action of some absolute Galois group $\mathrm{Gal}(\bar{k}|k)$; at the same time covering maps will be replaced with finite étale covers.

The main example of anabelian varieties are hyperbolic curves over finitely generated extensions of $\mathbb{Q}$. Hyperbolic curves are curves obtained by taken smooth projective curves of genus g and removing $n > 2 - 2g$ points. The geometric étale group $\pi_1^{etale}(X \times \mathrm{Spec}(\bar{k}))$ of a hyperbolic curve will have no center. Later we will see how center freeness plays a role when we try to recuperate the X from its fundamental group.

For hyperbolic curves defined over finitely generated extensions k of $\mathbb{Q}$, Mochizuki was able to prove a conjecture of Grothendieck that allows to recover the hyperbolic curve X from its étale fundamental group $\pi_1^{etale}(X)$. In mathematical terms the conjecture says that the functor $\pi_1 = \pi_1^{etale}$:

$$\pi_1 : \mathrm{Hyp}_k \rightsquigarrow \mathrm{Prof}^{\mathrm{ext,open}}_{\mathrm{Gal}(k)},$$

is fully faithful from the category Hyp_k of hyperbolic curve with dominant morphisms to the subcategory $\mathrm{Prof}^{\mathrm{ext,open}}_{\mathrm{Gal}(k)}$ of pro-finite with extra Galois action. We need to explain the nature of the étale fundamental group as well as the meaning of the category $\mathrm{Prof}^{\mathrm{ext,open}}_{\mathrm{Gal}(k)}$.

1.3. **The étale fundamental group.** Let X be a connected scheme. Following the results on Galois theory for coverings of topological spaces, we are going to consider the family $\mathrm{Fét}(X)$ of finite étale covers of X. In the case of connected étale covers we have the following rigidity principle.

Proposition 1.3. *Let $X' \xrightarrow{\varphi} X$ a connected étale cover, then the non-trivial elements of* $\mathrm{Aut}(X'|X)$ *act without fix points on the fibres and therefore* $\mathrm{Aut}(X'|X)$ *is a finite group.*

Proof. When we have maps $\varphi_1, \varphi_2 : X' \to X''$ over X that coincide on a geometric point $\bar{x}'$, then we will have $\varphi_1 \equiv \varphi_2$. In effect by passing to $X' \times X''$, it will be enough to prove it for sections $s_1, s_2 : X \to X''$ that coincide on a geometric point $\bar{x}$. And this last case is true because the sections $s_1, s_2 : X \to X''$, will be étale maps with image being simultaneously open and closed. They will represent then isomorphisms onto a whole connected component and therefore $s_1 = s_2$ on X. By taking $\varphi_1 = \varphi$ and $\varphi_2 = \varphi \circ \lambda$ for a non-trivial element $\lambda \in \mathrm{Aut}(X'|X)$ we obtain the proposition. □

Remark 1.4. A section of an étale covering is an étale map. In general if $X' \xrightarrow{\varphi} X$ and $X'' \xrightarrow{\psi} X'$ are morphism of schemes and $\varphi \circ \psi$ and φ are finite étale maps with φ separated, then the map ψ is also finite étale. This is consequence of the fact that, when we denote the graph of ψ by $\Gamma_\psi : X'' \to X'' \times_X X'$ and $p_2 : X'' \times_X X' \to X'$, we can obtain $\psi = p_2 \circ \gamma$ as a composition of finite étale maps. The graph of ψ is finite étale without any assumptions because is the base change of the finite map diagonal $\Delta : X \to X \times X$.

Definition 1.5. *A finite étale connected cover is Galois if the group of automorphism* $\mathrm{Aut}(X'|X)$ *acts transitively on the fibres, and therefore is a finite group.*

Let X be a connected scheme and $\bar{x} : \mathrm{Spec}(\Omega) \to X$ a geometric point, for Ω algebraically closed field. We want to defined a functor on the family $\mathrm{Fét}(X)$ of finite étale covers of X. The functor is the fibre functor $\mathrm{Fib}_{\bar{x}} : \mathrm{Fét} \to \mathrm{Sets}$, that assigns to every element $X' \xrightarrow{p} X$ of $\mathrm{Fét}(X)$, the underlying set $\mathrm{Fib}_{\bar{x}}(X')$ of the fibre $X' \times_X \mathrm{Spec}(\Omega)$ over $\mathrm{Spec}(\Omega)$.

Definition 1.6. *The group* $\pi_1(X, \bar{x})$ *is by definition the group of automorphism* $f : \mathrm{Fib}_{\bar{x}} \simeq \mathrm{Fib}_{\bar{x}}$ *of the functor* $\mathrm{Fib}_{\bar{x}}$.

Remark 1.7. The group $\pi_1(X, \bar{x})$ is a pro-finite group that can be express as inverse limit of $\mathrm{Aut}(P_\alpha|X)^{op}$ for the inverse system of Galois covers $P_\alpha \to X$. Once we have well-defined maps $\phi_{\alpha,\beta} : P_\alpha \to P_\beta$ for $\alpha > \beta$ we can make the system of Galois covers into a directed system. The functor $\mathrm{Fib}_{\bar{x}}$ is pro-representable by the family of Galois covers in the sense that for all $X' \to X$

$$\varinjlim \mathrm{Hom}(P_\alpha, X') \xrightarrow{\sim} \mathrm{Fib}_{\bar{x}}(X'),$$

and as a consequence

$$\varprojlim \mathrm{Aut}(P_\alpha|X)^{op} \xrightarrow{\sim} \mathrm{Aut}(\mathrm{Fib}_{\bar{x}}) \xrightarrow{\sim} \pi_1(X, \bar{x}).$$

Example 1.8. In the case of X be a scheme over $\mathrm{Spec}(\mathbb{C})$ the $\pi_1(\bar{X}, \bar{x})$ is nothing but the pro-finite completion $\widehat{\pi_1^{top}(X, \bar{x})}$ of the topological fundamental group $\pi_1^{top}(X, \bar{x})$. Pro-finite groups appear for the first time when consider finite covering maps because whenever we have an action $G \times X \to X$ and X is finite, we can extend the action to a continuous action $\hat{G} \times X \to X$ of the pro-finite completion of $\hat{G}$.

Remark 1.9. By definition $\pi_1(X, \bar{x})$ acts on the fibre $\mathrm{Fib}_{\bar{x}}(X')$ for all $X' \xrightarrow{p} X$ in $\mathrm{Fét}(X)$. This action is continuous when we have the discrete topology on the fibre and the pro-finite topology on $\pi_1(X, \bar{x})$.

Theorem 1.10. *(Grothendieck) The functor* $\mathrm{Fib}_{\bar{x}}$ *actually defines an equivalence of categories between* $\mathrm{Fét}(X)$ *and finite sets with a continuous action by* $\pi_1(X, \bar{x})$*. In this correspondence connected covers will correspond to finite sets with a transitive action by* $\pi_1(X, \bar{x})$ *and Galois covers will correspond to finite quotients of* $\pi_1(X, \bar{x})$.

Example 1.11. Let $X = A$ an abelian variety defined over a algebraically closed field k, then $\pi_1(A)$ is commutative and $\pi_1(A) \xrightarrow{\sim} T(A) \xrightarrow{\sim} \prod_l T_l(A) \xrightarrow{\sim} \prod_l \lim_r A(k)_{l^r}$.

Remark 1.12. Connected covers correspond to transitive action of the fundamental group G on finite sets and therefore to the fibre being a coset space $\{gU\}$ for some open subgroups $U \subset G$. In this way trivial covers correspond to trivial action on the fibre or disjoint union of points G. The inclusion of open sets $V_1 \subset V_2 \subset G$ is equivalent to a map of covers $X_1 \to X_2$.

1.4. **Properties of the fundamental group.** Suppose that we have a morphism $\phi : X_2 \to X_1$ of connected schemes together with geometric points $\bar{x}_1 : \mathrm{Spec}(\Omega) \to X_1$ and $\bar{x}_2 : \mathrm{Spec}(\Omega) \to X_2$, such that $\bar{x}_1 = \phi \circ \bar{x}_2$. The base change will induce a continuous map

$$\phi_* : \pi_1(X_2, \bar{x}_2) \to \pi_1(X_1, \bar{x}_1).$$

The base change BC gives a composition of functors $\mathrm{Fib}_{\bar{x}_1} = \mathrm{Fib}_{\bar{x}_2} \circ BC$ and in this sense $\pi_1(X_2, \bar{x}_2)$ acts on $\mathrm{Fib}_{\bar{x}_1}$ via ϕ_*. On the other hand we can recover the base change $X' \to X' \times_{X_1} X_2$ as the functor sending the $\pi_1(X_1, \bar{x}_1)$-set $\mathrm{Fib}_{\bar{x}_1}(X')$ to the $\pi_1(X_2, \bar{x}_2)$-set obtained by composing with ϕ_*. In general the map ϕ_* is neither injective nor surjective.

Lemma 1.13. *The map ϕ_* is surjective if for every connected étale cover $X' \to X_1$ we obtain by base change a connected étale cover $X_1' \times_{X_1} X_2 \to X_2$,*

Proof. When the map is onto, a transitive action of $\pi_1(X_1, \bar{x}_1)$ in $\mathrm{Fib}_{\bar{x}_1}(X')$ pulls back via ϕ_* to a transitive action of $\pi_1(X_2, \bar{x}_2)$ on the fibre of the base change $\mathrm{Fib}_{\bar{x}_2}(X' \times X_2)$. On the other hand if $\mathrm{Im}(\phi_*)$ is not the whole $\pi_1(X_1, \bar{x}_1)$, there will be an open set U such that $\mathrm{Im}(\phi_*) \subset U \subset \pi_1(X_1, \bar{x}_1)$. The pull back of the connected cover associated to U will have a trivial action by $\pi_1(X_2, \bar{x}_2)$ and will be therefore a trivial cover not isomorphic to X_2, hence not connected. □

Lemma 1.14. *An open set U satisfies $\mathrm{Im}(\phi_*) \subset U \subset \pi_1(X_1, \bar{x}_1)$ if and only if the pull back of the connected cover associated to U admits a section over X_2.*

Proof. $\mathrm{Im}(\phi_*) \subset U \subset \pi_1(X_1, \bar{x}_1)$ if and only if the connected étale cover associated to U pulls back to the trivial cover and therefore $\pi_1(X_2, \bar{x}_2)$ leaves the point $s \in \mathrm{Fib}_{\bar{x}_2}(X' \times_{X_1} X_2)$ corresponding to U fixed and hence, by remark 1.2, it fixes a whole connected component $\bar{s}$. This provides a section $s : X_2 \to X' \times X_2$. □

Lemma 1.15. *An open set U satisfies $\mathrm{Ker}(\phi_*) \subset U \subset \pi_1(X_2, \bar{x}_2)$ if and only if for the finite étale connected cover $X'' \to X_2$ associated to U there exist a morphism $X_i \to X''$, where X_i is a connected component of $X' \times_{X_1} X_2$ for some cover $X' \to X_1$.*

Proof. A connected component X_i of $X' \times_{X_1} X_2$ can be identified with an open set U_i of $\pi_1(X_2, \bar{x}_2)$ and the connected cover $X'' \to X_1$ with an open set $U'' \subset \pi_1(X_2, \bar{x}_2)$, the map $X_i \to X'$ exist if and only if $U_i \subset U''$. Now the choice of X_i fixes a geometric point in the fibre where $\mathrm{Ker}(\phi)$ acts trivially, therefore $\mathrm{Ker}(\phi_*) \subset U_i$ and $\mathrm{Ker}(\phi_*) \subset U''$. The other direction needs a topological lemma:

Lemma 1.16. *(topological lemma) Let $H \subset G$ be a closed subgroup and G pro-finite, then: the intersection of open subgroups containing H is precisely H and for every open subgroup $V' \subset H$ there exist V open in G such that $V' = H \cap V$.*

With the topological lemma in our hands we continue the proof of the lemma characterizing injectivity. Suppose that $\mathrm{Ker}(\phi_*) \subset U''$, as $H = \phi_* \pi_1(X_2, \bar{x}_2)$ is closed because is compact and $V'' = \phi_* U''$ is open in H (which is compact of finite index). By the topological lemma we can find an open set $V \subset \pi_1(X_1, \bar{x}_1)$ such that $V \cap H = V''$ giving rise to a connected étale cover $X' \to X$. A connected component X_i of $X' \times_{X_1} X_2$ corresponds to some open set $U_i \subset \pi_1(X_2, \bar{x}_2)$ and because $\phi_* U_i \subset V''$ we have $U_i \subset U''$ and therefore there is a map $X_i \to X'$. □

Lemma 1.17. *The map ϕ_* is injective if and only if for every connected étale cover $X'' \to X_2$ there exist a connected finite étale cover $X' \to X_1$ and a map $X_i \to X''$ where X_i is a connected component of $X' \times X_2$.*

Proof. It follows from the characterization of the $\mathrm{Im}(\phi_*)$ because the intersection of all open subgroups is trivial. □

Lemma 1.18. *Given $X_2 \xrightarrow{\varphi} X_1 \xrightarrow{\psi} X_0$ and geometric points $\bar{x}_2, \bar{x}_1, \bar{x}_0$ respectively, the sequence*

$$\pi_1(X_2, \bar{x}_2) \xrightarrow{\varphi_*} \pi_1(X_1, \bar{x}_1) \xrightarrow{\psi_*} \pi_1(X_0, \bar{x}_0)$$

is exact if it satisfies the two conditions:

(i) *For every finite connected étale cover* $X' \to X$ *the base change* $X' \times_X X_2$ *is trivial.*
(ii) *For every finite connected étale cover* $X' \to X_1$*, such that* $X' \times_{X_1} X_2 \to X_2$ *has a section, there exist a finite étale cover* $X'_0 \to X$ *and a morphism* $X^i_0 \to X'$ *over* X_1*, where* X^i_0 *is a connected component of* $X_0 \times_X X_1$*.*

Proof. The first condition says that $\psi_* \circ \varphi_*$ is trivial. The second condition stays that if U is an open set

$$\mathrm{Im}(\psi_*) \subset U \Rightarrow \mathrm{Ker}(\varphi_*) \subset U,$$

which is equivalent to $\mathrm{Im}(\psi_*) \subset \mathrm{Ker}(\varphi_*)$. □

Theorem 1.19. *Let* X *be a quasi-compact, integral and geometrically integral scheme over* k*. There is an exact sequence of fundamental groups coming from the maps* $\varphi : \bar{X} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}\,\bar{k} \to X$ *and* $\psi : X \to \mathrm{Spec}(k)$*:*

$$1 \to \pi_1(\bar{X}, \bar{x}) \xrightarrow{\varphi_*} \pi_1(X, \bar{x}) \xrightarrow{\psi_*} \pi_1(\mathrm{Spec}(k), \bar{x}) \to 1$$

Proof. A finite étale cover $\bar{Y} \to \bar{X}$ can be obtained as $\bar{Y} \xrightarrow{\sim} Y_L \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k_s)$, for a finite cover Y_L of $X_L = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$. This should take care of injectivity. The map ψ_* is onto because X is geometrically integral and therefore geometrically connected. The composition $\psi_* \circ \varphi_*$ is trivial. The only other condition to check is condition (ii) for exactness in the middle. Suppose that $Y \to X$ is finite Galois cover such that $Y_{k_s} \to \bar{X}$ has a section. As X is integral, the generic fibre is $\mathrm{Spec}(K)$, where K is a finite Galois extension of $K(X)$ (separable comes from being étale and normal from integrality or X). When tensoring with k_s, this becomes $\mathrm{Spec}(K) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k_s) \xrightarrow{\sim} k_s(X) \times \cdots \times k_s(X)$ and therefore $K = K(X) \otimes L$ for some finite Galois extension. Considering the associated Galois cover $X_L \to X$, we have for some open U, $X_L(U) \xrightarrow{\sim} Y(U)$ and because they are locally free, this forces $X_L \xrightarrow{\sim} Y$. □

Now, if $X' \to \mathrm{Spec}(k)$ is a finite étale cover of X, then $X' = \mathrm{Spec}(L)$ where L is a finite étale algebra and therefore a finite direct product of finite separable extensions of k . The fibre over a geometric point $X_{\bar{x}}$ can be identified with the set of homomorphisms $\mathrm{Hom}(L, k_s)$ for a separable closure $k_s \subset \bar{k}$ of k. In this way the action of $\pi_1(\mathrm{Spec}(x), \bar{x})$ on the fibre can be identified with the action of $\mathrm{Gal}(k_s/k)$ on k_s and the group $\pi_1(\mathrm{Spec}(x), \bar{x})$ becomes naturally isomorphic to $\mathrm{Gal}(k_s/k)$. So, the sequence becomes:

$$1 \to \pi_1(\bar{X}, \bar{x}) \xrightarrow{\varphi_*} \pi_1(X, \bar{x}) \xrightarrow{\psi_*} \mathrm{Gal}(k_s/k) \to 1$$

There is a natural action of $\pi_1(X, \bar{x})$ on it normal subgroup $\pi_1(\bar{X}, \bar{x})$, when we consider for every element $g \in \pi_1(X, \bar{x})$ the automorphism of $\pi_1(\bar{X}, \bar{x})$ defined by $g \to gxg^{-1}$. In particular when g is in $\pi_1(\bar{X}, \bar{x})$ we will get inner automorphisms. We have the commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\bar{X}, \bar{x}) & \xrightarrow{\varphi_*} & \pi_1(X, \bar{x}) & \xrightarrow{\psi_*} & \mathrm{Gal}(k_s/k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \rho_X & & \\
1 & \longrightarrow & \mathrm{Inn}(\pi_1(\bar{X}, \bar{x})) & \longrightarrow & \mathrm{Aut}(\pi_1(\bar{X}, \bar{x})) & \longrightarrow & \mathrm{Out}(\pi_1(\bar{X}, \bar{x})) & \longrightarrow & 1
\end{array}$$

The map ρ_X is called the outer Galois representation. It is continuous group homomorphism, because so are the maps of pro-finite groups in the quotient.
For a group G, let us denote by $Z(G)$ its center. We can extend the diagram to:

$$
\begin{array}{ccccccccc}
 & & 1 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & Z(\pi_1(\bar{X},\bar{x})) & & & & & & \\
 & & \downarrow & & & & & & \\
1 & \longrightarrow & \pi_1(\bar{X},\bar{x}) & \xrightarrow{\varphi_*} & \pi_1(X,\bar{x}) & \xrightarrow{\psi_*} & \mathrm{Gal}(k_s/k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \rho_X} & & \\
1 & \longrightarrow & \mathrm{Inn}(\pi_1(\bar{X},\bar{x})) & \longrightarrow & \mathrm{Aut}(\pi_1(\bar{X},\bar{x})) & \longrightarrow & \mathrm{Out}(\pi_1(\bar{X},\bar{x})) & \longrightarrow & 1 \\
 & & \downarrow & & & & & & \\
 & & 1 & & & & & &
\end{array}
$$

As a consequence of this, we get that, when $\pi_1(\bar{X},\bar{x})$ is center free, the group $\pi_1(X,\bar{x})$ is simply the fibre product of $\mathrm{Aut}(\pi_1(\bar{X},\bar{x}))$ and $\mathrm{Gal}(k_s/k)$ via ρ_X. So in this case the group $\pi_1(X,\bar{x})$ is determined by the geometric object $\pi_1(\bar{X},\bar{x})$ plus an action of $\mathrm{Gal}(k_s/k)$.

1.5. **The exterior category of pro-finite groups over G.** In this section we are going to be work with schemes over k and consider pro-finite groups over $G = Gal(k_s|k)$. When k is a perfect field this last group will just be $G = Gal(\bar{k}|k)$.
Let $G_i \xrightarrow{p_i} G$, for $i = 1, 2$, be pro-finite groups over G. Let $G_1 \xrightarrow{\varphi} G_2$ a map over G, in such a way that the diagram commutes up to conjugation by $g : G \to G$. The set $\mathrm{Hom}(G_1, G_2)$ carries an action from the left and the right by G_1 and G_2 respectively. By properties of group homomorphisms, the equivalent relation that we obtain when we mod out by the right action of G_2 is finer. We want to consider such action to build a new category with pro-finite groups as objects and this new type of morphisms:

$$\mathrm{Hom}_G^{ext}(G_1, G_2) = \mathrm{Hom}_G^{*}(G_1, G_2)/(G_2 - \text{action})$$

$$\mathrm{Prof}_G^{ext,open} = \{ \text{ object are pro-finite groups with morphisms as above and open images } \}$$

The following conjecture was stated by Grothendieck in a letter to Faltings:

Conjecture 1.20. *(Relative Grothendieck) Let k be a finitely generated extension of $\mathbb{Q}$. Let Hyp_k be the category of hyperbolic curves over k equipped with dominant k-morphisms. The functor*

$$\pi_1 : \mathrm{Hyp}_k \rightsquigarrow \mathrm{Prof}_G^{ext,open}$$

is fully faithful.

A birrational analog will read:

Conjecture 1.21. *(Birrational Grothendieck) Let k be a finitely generated extension of $\mathbb{Q}$. Let Bir_k^{dom} denote the category of fields finitely generated over k together with k-morphisms. The contravariant functor:*

$$\mathrm{Gal} : \mathrm{Bir}_k^{dom} \rightsquigarrow \mathrm{Prof}_G^{ext,open}$$

is fully faithful.

Recall that hyperbolic curves are smooth projective curves of genus g with n points removed in such a way that $2g-2+n > 0$. The isomorphism version of the first conjecture is true for hyperbolic curves even for finite extensions k of the p-adic fields $\mathbb{Q}_p$. That is, for X and X' hyperbolic k-curves:

$$\mathrm{Isom}_k^{\mathrm{dom}}(X, X') \xrightarrow{\sim} \mathrm{Isom}_G^{\mathrm{ext,open}}(\pi_1(X), \pi_1(X')).$$

The isomorphism version of the second conjecture is true also for k finitely generated over $\mathbb{Q}$ or $\mathbb{Q}_p$. If K, K' are finitely generated over k we have:

$$\mathrm{Isom}_k(K, K') \xrightarrow{\sim} \mathrm{Isom}_G^{\mathrm{ext}}(\mathrm{Gal}(K), \mathrm{Gal}(K')).$$

The Neukirch-Ushida theorem for number fields is saying that a topological map $\mathrm{Gal}(k) \to \mathrm{Gal}(k')$ between pro-finite topological groups, can be extended to an inner automorphism $g \to \tau g \tau^{-1}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ conjugating the number field k to k'.

Theorem 1.22. *(NU)For k, k' finitely generated extensions of $\mathbb{Q}$:*

$$\mathrm{Isom}(\bar{k}/k, \bar{k}'/k') \xrightarrow{\sim} \mathrm{Isom}(\mathrm{Gal}(k), \mathrm{Gal}(k')).$$

The Neukirch-Ushida theorem allows us to pass from relative results to absolute results:

Theorem 1.23. *(Mochizuki 2007) Let X, Y be hyperbolic curves defined over a finite extensions (possible distinct) of $\mathbb{Q}$. The natural map*

$$\mathrm{Isom}(X, Y) \to \mathrm{Isom}^{ext}(\pi(X), \pi(Y))$$

is bijective.

Over finite extensions of $\mathbb{Q}_p$ we do not have a theorem analogous to NU, and therefore we do not expect to have an absolute result like the previous theorem.

1.6. **Applications.** In the following result we use Belyi's theorem and the outer Galois representation to embed the arithmetic group $\mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q})$ into the group $\mathrm{Out}(\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}))$ of a topological nature.

Theorem 1.24. *The outer Galois representation*

$$\rho_{\mathbb{P}^1 \setminus \{0,1,\infty\}} : \mathrm{Gal}(\bar{\mathbb{Q}}|\mathbb{Q}) \to \mathrm{Out}(\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0, 1, \infty\}))$$

is injective.

Proof. Let us denote $U = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $\bar{U} = U_{\bar{\mathbb{Q}}}$. Suppose that the map is not injective, then, by use of Galois theory and continuity of the map ρ, there exist an extension $L/\mathbb{Q}$ (maybe infinite) such that the representation $\rho_{U_L} : \mathrm{Gal}(L) \to \mathrm{Out}(\pi_1(\bar{U}))$ is trivial. The representation ρ_{U_L} comes from the exact sequence

$$1 \to \pi_1(\bar{U}) \xrightarrow{\varphi_*} \pi_1(U_L) \xrightarrow{\psi_*} \mathrm{Gal}(\bar{\mathbb{Q}}|L) \to 1.$$

To say that ρ_{U_L} is trivial means that, conjugation by an element $y \in \pi_1(U_L)$ is the same as the conjugation by an element $x \in \pi_1(\bar{U})$, and this means that yx^{-1} belong to the centralizer Z of $\pi_1(\bar{U})$ in $\pi_1(U_L)$. The group $\pi_1(\bar{U})$ is free in two generators and therefore has trivial center. Because of this, Z and $\pi_1(\bar{U})$ has no intersection and $\pi_1(U_L)$ is the direct product of $\pi_1(\bar{U})$ and Z and we have a retraction map $\pi_1(U_L) \to \pi_1(\bar{U})$. This implies that finite covers of $\bar{U}$ come from finite covers of U_L. By Belyi's theorem this will implies that any integral proper curve defined over $\bar{\mathbb{Q}}$ can be defined over L, which is a contradiction with the following example:

Example 1.25. Take an elliptic curve E with j-invariant $j(E) \notin L$. It is not possible to find a proper integral normal curve defined over L such that $X_{\bar{\mathbb{Q}}} \xrightarrow{\sim} E$. Because if that were the case, the X will have genus 1 and the $\mathrm{Jac}(X)$ will be an elliptic curve over L with $\mathrm{Jac}(X) \xrightarrow{\sim} E$ contradicting the fact that $j(E) \notin L$.

□

Proposition 1.26. *Let X be an smooth projective integral scheme over $\mathbb{C}$. The fundamental group $\pi_1(X, \bar{x})$ is topologically finitely generated (contains a finitely generated dense subset) for every base point.*

Proof. For X a smooth, projective, integral scheme with $\dim(X) \geq 2$ and $X' \to X$ a finite étale cover, we can find a hyperplane section H such that $X \cap H$ is smooth, connected and $X' \times_X (H \cap X) \to H \cap X$ is a connected étale cover. We have therefore a surjection $\pi_1(X \cap H, \bar{x}) \twoheadrightarrow \pi_1(X, \bar{x})$, where $X \cap H$ is of dimension strictly smaller than X. When we repeat this argument we get

$$\pi_1(C, \bar{x}) \to \pi_1(X, \bar{x}) \to 1$$

for a smooth projective curve C over $\mathbb{C}$ and the results follows from the fact that the topological fundamental group of a projective smooth curve has $2g$ generators. □

In dimension one we have a very important type of étale covers. The integral closure $A \subset B$ of $\mathbb{Z}$ inside two fields $K \subset L$ where the field extension L/K is unramified. An an example of unramified extension we can take $K = \mathbb{Q}(\sqrt{-5})$ and $L = \mathbb{Q}(\sqrt{-1}, \sqrt{-5})$. In this case the ring of integers $\mathcal{O}_L = \mathbb{Z}(\sqrt{-1}, \frac{1+\sqrt{-5}}{2})$ and the relative discriminant $d_{L/K}$ is a unit.

Proposition 1.27. *The affine scheme* Spec($\mathbb{Z}$) *is simply connected.*

Proof. Suppose that $X \overset{\varphi}{\to} \mathrm{Spec}(\mathbb{Z})$ is a finite connected étale cover, then $X = \mathrm{Spec}(A)$ where A is an order in a number field K and, by a theorem of Minkowski [Szp], there will be rational primes $p = u_p(p_1)^{n_1} \dots (p_r)^{n_r}$ with some $n_i > 1$, namely the primes dividing the discriminant $d_A > 1$ of A over $\mathbb{Z}$. So, every finite étale cover is trivial and therefore the group $\pi_1(\mathrm{Spec}(\mathbb{Z}))$ is trivial. □

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