1 Spectral theorem

1.1 Diagonalization

Definition 1. Two matrices A and B are said to be similar if there exist an invertible matrix P such that $A = P^{-1}BP$.

Example 2. For invertible matrices A and B of order n, the two matrices AB and BA are similar. This is a consequence of the identity

$$AB = B^{-1}(BA)B.$$

Similar matrices have the same eigenvalues. Assume that $A = P^{-1}BP$ and $Ax = \lambda x$ with $x \neq 0$. Then

$$P^{-1}BPx = Ax \Rightarrow P^{-1}BPx = \lambda x \Rightarrow B(Px) = \lambda(Px).$$

Since the matrix P is invertible, the vector $Px \neq 0$ as long as $x \neq 0$. In fact we can prove that the characteristic polynomials of A and B are identical since

$$p_A(\lambda) = |A - \lambda I| = |P^{-1}BP - \lambda I| = |P^{-1}(B - \lambda I)P| = |P^{-1}||B - \lambda I||P| = p_B(\lambda).$$

Example 3. For invertible matrices A and B of order n, the two matrices AB and BA have the same characteristic polynomial.

Definition 4. We say that a square matrix A of order n is **diagonalizable** if there exist an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

Corollary 5. *If a square matrix A of order n is diagonalizable, then $P^{-1}AP = diag(\lambda_1, \ldots, \lambda_n)$, where λ_i are the eigenvalues of A.*

Theorem 6. *A square matrix A of order n is diagonalizable if and only we can find a basis of $\mathbb{R}^n$ made of eigenvectors $x_1, \ldots, x_n$. In this case $P^{-1}AP = diag(\lambda_1, \ldots, \lambda_n)$. where the matrix P is obtained by putting the eigenvectors $x_1, x_2, \ldots, x_n$ as columns.*

1.2 Spectral theorem

Recall that matrix A is said to be diagonalizable if it similar to a diagonal matrix. That is, there exist an invertible matrix P and a diagonal matrix D, such that $A = PDP^{-1}$.

Theorem 7. *Let A be a real symmetric matrix of order n.*

(1) All eigenvalues λ are real numbers and for all eigenvalue λ there exist a real eigenvector $x \in \mathbb{R}^n$.

(2) *Eigenvalues corresponding to different eigenvectors are orthogonal.*

(3) *The matrix is diagonalizable. There exist and orthogonal matrix P (orthogonal in the sense that $P' = P^{-1}$) such that $A = PDP^{-1}$ and D is a diagonal matrix. We refer to this decomposition of A as the spectral decomposition of A.*

Proof. (a)Suppose that λ is an eigenvalue of A and for the vector $x \neq 0$, we have $Ax = \lambda x$. The number λ may, in principle, be complex and the vector x may have some complex components. We have

$$0 \leq (\overline{Ax}')(Ax) = (\bar{x}'A)(Ax) = (\bar{x}'A)(\lambda x) = \lambda(\bar{x}'Ax) = \lambda^2(\bar{x}'x).$$

Since $(\bar{x}'x)$ is a positive real number, hence λ^2 is real and ≥ 0 and λ must be real.
(b) Suppose that $Ax_i = \lambda_i x_i$ and $Ax_j = \lambda_j x_j$ with $x_i \neq x_j$. We have

$$x_j'Ax_i = x_j'\lambda_i x_i \qquad x_i'Ax_j = x_i'\lambda_i x_j.$$

Doing the transpose and using the fact that $A = A'$ we get

$$x_j'\lambda_i x_i = x_j'Ax_i = x_j'\lambda_i x_i \Rightarrow (\lambda_i - \lambda_j)x_i'x_j = 0.$$

Since $\lambda_i \neq \lambda_j$ we obtain $x_i'x_j = 0$ and the vectors x_i and x_j are orthogonal.
(c) Suppose that all eigenvalues are different. The eigenvectors are hence by (b) mutually orthogonal and therefore linearly independent. We can take the eigenvectors to be of length 1 and the matrix P having the x_i as columns will be an orthogonal matrix. The general case where some eigenvalues can be the same, can be solve when we realize that we are able to approximate any symmetric matrix by symmetric matrices with different eigenvalues and do a limit process. □

Corollary 8. *For any matrix A, the matrices $B = A^tA$ and $C = AA^t$ are both diagonalizable with real eigenvalues.*

Remark 9. Linearly independent eigenvectors associated to the same eigenvalue do not need to be orthogonal to each other. We can always, however, use the orthogonalization process to find an orthogonal basis for the eigenspace E_λ associated to the eigenvalue λ. Given a basis $\{v_1, v_2, \ldots, v_r\}$ of E_λ, we construct:

$$\begin{aligned}
u_1 &= v_1 \\
u_2 &= v_2 - \frac{\langle u_1, v_2\rangle}{\langle u_1, u_1\rangle}u_1 \\
u_3 &= v_3 - \frac{\langle u_1, v_3\rangle}{\langle u_1, u_1\rangle}u_1 - \frac{\langle u_2, v_3\rangle}{\langle u_2, u_2\rangle}u_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Practice Questions:

1. Show that for any matrix A, the matrix $B = A^t A$ is symmetric.

2. Find examples of diagonalizable matrix that are not symmetric.

3. Show that the product of two symmetric matrices not need to be symmetric.

4. Given the symmetric matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix}$.

a. Find the eigenvalues of A.
b. Find a basis made of eigenvectors.
c. Find an orthogonal basis made of eigenvectors.
d. Find an orthogonal matrix P such that $P^{-1}AP = D$.