1 Determinant of a matrix

The determinant is a function $\det\colon M_n(\mathbb{R}) \longrightarrow \mathbb{R}$. In the trivial case of a matrix of order 1, the determinant is just equal to that number, this is $\det(a_{1,1}) = a_{1,1}$. In general, we provide a recurrent form of obtaining the determinant of a matrix of any order.

Definition 1. If A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the i-th row and j-th column are deleted from A. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the cofactor of entry a_{ij} .

Proposition 2. *If A is an $n \times n$-matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is independent of the row or column.*

Proof. Let S_n denote the group of bijective maps $\sigma\colon \{1, 2, \dots, n\} \longrightarrow \{1, 2, \dots, n\}$. When computing the determinant, the expression obtained, independent of the column or row that have been chosen, is

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{1,\sigma(1)} \dots a_{n,\sigma(n)}.$$

This expression depend only on the matrix A and not on the column or row selected. $\square$

Definition 3. If A is an $n \times n$-matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the determinant of A, and the sums themselves are called cofactor expansions of A. That is, for example, the expansion using the j-th column is:

$$\det(A) = a_{1,j}C_{1,j} + \dots a_{n,j}C_{n,j}.$$

On the other hand, the expansion using the i-th row is:

$$\det(A) = a_{i,1}C_{i,1} + \dots a_{i,n}C_{i,n}.$$

The determinant of a matrix is also denoted by $|A|$.

Example 4. For the matrix

$$A = \begin{pmatrix} -2 & 10 \\ 2 & -3 \end{pmatrix}.$$

The determinant using the first row expansion is

$$\det(A) = \begin{vmatrix} -2 & 10 \\ 2 & -3 \end{vmatrix} = -2(-3) + 10(-(2)) = 6 - 20 = -14.$$

On the other hand, if we want to do it by the first column expansion instead, we get $\det(A) = -2(-3) + 2(-10) = 6 - 20 = -14$.

Example 5. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}.$$

The determinnt using the first row expansion is

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{vmatrix} = 1\begin{vmatrix} 5 & 3 \\ 0 & 8 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} = 1(40-0) - 2(16-0) + 1(6-15)$$
$$= 40 - 32 - 9 = -1.$$

Theorem 6. *Let A be a square matrix of order n:*

(a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then $\det(B) = k\det(A)$.

(b) If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.

(c) If B is the matrix that results when a multiple of one row of A is added to another or when a multiple of one column is added to another, then $\det(B) = \det(A)$.

Corollary 7. *Let A be a square matrix of order n:*

(a) If the matrix A has two identical rows or two identical column, the determinant is $\det(A) = 0$.

(b) If a matrix A has a row or column consisting only of zeroes, the determinant is $\det(A) = 0$.

Example 8. Let $A = \begin{pmatrix} 2 & 3 & 3 & 3 \\ 1 & 3 & 3 & 3 \\ 2 & -2 & 1 & 5 \\ 1 & 6 & 4 & -4 \end{pmatrix}$. Using (c) we can multiply the second row by -1 and add it to the first row to get the identity

$$\det(A) = \begin{vmatrix} 2 & 3 & 3 & 3 \\ 1 & 3 & 3 & 3 \\ 2 & -2 & 1 & 5 \\ 1 & 6 & 4 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 3 \\ 2 & -2 & 1 & 5 \\ 1 & 6 & 4 & -4 \end{vmatrix} = \begin{vmatrix} 3 & 3 & 3 \\ -2 & 1 & 5 \\ 6 & 4 & -4 \end{vmatrix}$$

$$\det(A) = 3\begin{vmatrix} 1 & 5 \\ 4 & -4 \end{vmatrix} - 3\begin{vmatrix} -2 & 5 \\ 6 & -4 \end{vmatrix} + 3\begin{vmatrix} -2 & 1 \\ 6 & 4 \end{vmatrix} = 3(-24) - 3(-22) + 3(-14) = 3(-16) = -48.$$

Definition 9. If A is any $n \times n$ matrix and $C_{i,j}$ is the cofactor of $a_{i,j}$. then the matrix

$$\begin{pmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n} \\ C_{2,1} & a_{2,2} & \cdots & C_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{m,1} & a_{m,2} & \cdots & C_{m,n} \end{pmatrix}$$

is called the matrix of cofactors of A. The adjunct matrix $\operatorname{adj}(A)$ of A is the transpose of the matrix of cofactors.

Example 10. For $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$, the adjunct matrix $\operatorname{adj}(A)$ is given by:

$$\operatorname{adj}(A) = \begin{pmatrix} 40 & -16 & -9 \\ -13 & 5 & 3 \\ -5 & 2 & 1 \end{pmatrix}.$$

Theorem 11. *A square matrix A is invertible if and only if $\det(A) \neq 0$, in which case, the inverse of A is given by:*

$$A^{-1} = \frac{1}{\det(A)} adj(A).$$

Example 12. For $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$, the inverse A^{-1} is given by:

$$A^{-1} = \frac{1}{-1} \begin{pmatrix} 40 & -16 & -9 \\ -13 & 5 & 3 \\ -5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}.$$

Proposition 13. *Other properties of the determinant:*

(1) $\det(A^t) = \det(A)$.

(2) $\det(AB) = \det(A)\det(B)$.

1.1 Cramer's Rule

Consider a system of equations with the same number of unknowns as equations ($m = n$):

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n,
\end{cases}
$$

with square matrix $A = (a_{i,j})$. This system has a unique solution if and only if $|A| = \det(A) \neq 0$. The solution is then given by

$$x_j = |A_j|/|A|,$$

whee $|A_j| = \det(A_j)$ denotes the determinant of the matrix where the j-column of A is replaced by the column of components $b_1, \ldots, b_n$. This is,

$$
|A_j| = \begin{vmatrix}
a_{1,1} & \cdots a_{1,j-1} & b_1 & a_{1,j+1} \cdots & a_{1,n} \\
a_{2,1} & \cdots a_{2,j-1} & b_2 & a_{2,j+1} \cdots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots & \\
a_{n,1} & \cdots a_{n,j-1} & b_n & a_{n,j+1} \cdots & a_{n,n}
\end{vmatrix}
$$

For homogeneous systems of equations, $b_1 = b_2 = \cdots = b_n = 0$, we have that **there is only the trivial solution if and only if** $|A| \neq 0$.