POSITIVE LINEAR RECURRENT RELATIONS

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Abstract. We study necessary and sufficient conditions for some linear recurrent sequences to be positive for all indexes n.

1. Positive linear sequences

1.1. **The question.** We know that if we take the lengths of the sides of a triangle as an unordered triple (a, b, c), we will always have the inequalities

$$\begin{cases} a_1 = b + c - a > 0, \\ b_1 = a + c - b > 0, \\ c_1 = a + b - c > 0. \end{cases}.$$

We would like to see if it is possible to continue the process and form a triangle with sides (a_1, b_1, c_1) and so on. The question can be put into a more general setting in terms of linear recurrent sequences. Given real numbers a, b, c, α and β, we want to study recurrent sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ of real numbers defined in terms of initial values a, b, c and coefficients α, β. More precisely, these sequences $\{a_n\}$, $\{b_n\}, \{c_n\}$ satisfy $a_0 = a, b_0 = b, c_0 = c$ and are defined, for $n \geq 0$, by the recurrent relations

$$\begin{cases} a_{n+1} = \beta(b_n + c_n) + \alpha a_n, \\ b_{n+1} = \beta(a_n + c_n) + \alpha b_n, \\ c_{n+1} = \beta(a_n + b_n) + \alpha c_n. \end{cases} \quad (1)$$

The question we want to address is the following:

Question 1.1. *For what values of (a, b, c, α, β) are the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ positive for all indexes $n \geq 0$?*

Remark 1.2. *For triples (a, b, c) of positive numbers and values of $\beta = 1$ and $\alpha = -1$, the question is equivalent to having a sequence of triangles with sides of lengths (a_n, b_n, c_n) for all $n \geq 0$. For instance, for $(a, b, c) = (5, 5, 6)$ we observe already numbers that are not positive:*

index n	(a_n, b_n, c_n)
0	$(5, 5, 6)$
1	$(6, 6, 4)$
2	$(4, 4, 8)$
3	$(8, 8, 0)$
4	$(0, 0, 16)$
5	$(16, 16, -16)$

The triangle T_2 with sides $(4, 4, 8)$ is already degenerated.
On the other hand, we have the trivial case of the equilateral triangle (a, a, a) for any number $a > 0$ that generates constant sequences $a_n = b_n = c_n = a$ for all $n \geq 0$.

We can pose a second question:

Question 1.3. *For what values (α, β) we get positivity of the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ only in the trivial case $a = b = c$?*

1.2. **Necessary conditions and the trivial cases.** We begin by making a few simple observations that we allow ourselves to focus on the most interested cases.
(1) A necessary condition for all sequences to be positive is that the numbers $a, b, c > 0$.
(2) Also, if $a, b, c > 0$ and $\alpha, \beta \geq 0$ (with $(\alpha, \beta) \neq (0, 0)$), the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are clearly positive for all n.
(3) On the other hand, if $\alpha, \beta \leq 0$ (with $a, b, c > 0$), the first terms will already be $a_1, b_1, c_1 \leq 0$.
(4) The interesting case is then when $a, b, c > 0$ and α, β have different sign.
(5) Adding the three recurrent equations in (1) we obtain another necessary condition given by

$$2\beta + \alpha > 0.$$

We will refer to the following cases as the trivial cases:

(1) When $a = b = c > 0$ is a positive number and $\alpha + 2\beta > 0$. In this situation we obtain the positive sequence

$$a_n = b_n = c_n = (\alpha + 2\beta)^n a, \quad \text{for} \quad n = 0, 1, 2 \ldots$$

A triple (a, b, c) with $a = b = c$ is said to be trivial for (α, β).

(2) When $\alpha = \beta > 0$, $a, b, c > 0$, and we get the positive sequence

$$a_n = b_n = c_n = (3\beta)^n \left(\frac{a + b + c}{3}\right), \quad \text{for} \quad n = 1, 2 \ldots$$

1.3. **The explicit formulas and the notation.** The explicit formulas for the a_n, b_n and c_n can be proven by induction to be:

$$\begin{cases} a_n = (2\beta + \alpha)^n \left(\dfrac{a + b + c}{3}\right) - (\alpha - \beta)^n \left(\dfrac{b + c - 2a}{3}\right), \\ b_n = (2\beta + \alpha)^n \left(\dfrac{a + b + c}{3}\right) - (\alpha - \beta)^n \left(\dfrac{a + c - 2b}{3}\right), \\ c_n = (2\beta + \alpha)^n \left(\dfrac{a + b + c}{3}\right) - (\alpha - \beta)^n \left(\dfrac{a + b - 2c}{3}\right). \end{cases} \tag{2}$$

The positivity of $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ is then equivalent to the following conditions:

$$\begin{cases} (2\beta + \alpha)^n \left(\dfrac{a + b + c}{3}\right) > (\alpha - \beta)^n \left(\dfrac{b + c - 2a}{3}\right), \\ (2\beta + \alpha)^n \left(\dfrac{a + b + c}{3}\right) > (\alpha - \beta)^n \left(\dfrac{a + c - 2b}{3}\right), \\ (2\beta + \alpha)^n \left(\dfrac{a + b + c}{3}\right) > (\alpha - \beta)^n \left(\dfrac{a + b - 2c}{3}\right). \end{cases} \tag{3}$$

Let us assume that $\alpha \neq \beta$ and denote by τ and $\bar{x}$ the values $\tau = \dfrac{2\beta + \alpha}{\alpha - \beta}$ and $\bar{x} = \dfrac{a + b + c}{3}$. The above conditions can be divided in two cases. For $\alpha > \beta$,

$$\begin{cases} \tau^n > \dfrac{\bar{x} - a}{\bar{x}}, \\ \tau^n > \dfrac{\bar{x} - b}{\bar{x}}, \\ \tau^n > \dfrac{\bar{x} - c}{\bar{x}}. \end{cases} \tag{4}$$

For $\beta > \alpha$,

$$\begin{cases} (-\tau)^n > (-1)^n \dfrac{\bar{x}-a}{\bar{x}}, \\ (-\tau)^n > (-1)^n \dfrac{\bar{x}-b}{\bar{x}}, \\ (-\tau)^n > (-1)^n \dfrac{\bar{x}-c}{\bar{x}}. \end{cases} \tag{5}$$

1.4. **The result.** Let us assume that $\alpha \neq \beta$. The study of the sequences $\{(a_n, b_n, c_n)\}$ is going to be carry out with the use of the parameter $\tau = \dfrac{2\beta+\alpha}{\alpha-\beta}$. For the values of $|\tau| < 1$, we have the following remark:

Remark 1.4. *If the absolute value $|\tau| = |-\tau| < 1$, the sequences $\{\tau^n\}$ and $\{(-\tau)^n\}$ satisfy*

$$\tau^n \longrightarrow 0 \qquad \text{and} \qquad (-\tau)^n \longrightarrow 0$$

and we will get, using inequalities (4) and (5), that

$$\bar{x} - a = \bar{x} - b = \bar{x} - c = 0$$

and therefore our trivial case (1), where all initial values must be equal: $a = b = c$. In other words, for values (α, β) with $|\tau| < 1$ and $\alpha \neq \beta$, for the sequences $\{a_n\}, \{b_n\}, \{c_n\}$ to be positive, the triple (a, b, c) must be trivial.

To deal with values of τ with absolute value $t = |\tau|$ greater or equal than one, we prove the following result.

Proposition 1.5. *Suppose that $t \geq 1$ and $L < 1$. The following conditions are equivalent:*
1. *$t^n > (-1)^n L$ for all $n \geq 0$.*
2. *$t > -L$.*
3. *$t > |L|$.*

Proof. We do the implications:
(1) → (2): follows since (2) is just the special case of $n = 1$.
(2) → (3): assuming (2) we have

$$t > -L \Rightarrow -t < L < 1 \leq t \Rightarrow |L| < t.$$

(3) → (1): follows since for $t \geq 1$

$$t^n \geq t > |L| \geq (-1)^n L$$

for all values of n. □

Now, we are ready to put together all our results on several cases.

Proposition 1.6. *Suppose that the initial values a, b and c are positive and the numbers $\alpha \neq \beta$ have different signs and satisfy $2\beta + \alpha > 0$. The possibilities for the sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ to be positive for all values of n are as follows:*
1. *For values $\alpha > -2\beta > 0$, we have $0 \leq \tau < 1$ and the conclusion is*

$$\{a_n\}, \{b_n\}, \{c_n\} > 0 \ \forall n \iff a = b = c.$$

2. *For values $0 < \beta < -2\alpha$, we have $0 \leq -\tau < 1$ and*

$$\{a_n\}, \{b_n\}, \{c_n\} > 0 \ \forall n \iff a = b = c.$$

3. *For values $\beta \geq -2\alpha > 0$, we will get $1 \leq -\tau < 2$ and*

$$\{a_n\}, \{b_n\}, \{c_n\} > 0 \ \forall n \iff a_1, b_1, c_1 > 0.$$

Proof. In (1) we have $\alpha > 0$, $\beta < 0$ and
$$0 \le \frac{2\beta+\alpha}{\alpha-\beta} = \tau = 1 + \frac{3\beta}{\alpha-\beta} < 1.$$
In (2) we have $\beta + 2\alpha < 0$ with $\beta > 0, \alpha < 0$ and
$$0 \le \frac{2\beta+\alpha}{\beta-\alpha} = -\tau = 1 + \frac{\beta+2\alpha}{\beta-\alpha} < 1.$$
The conclusion follows therefore from the inequalities (4) and (5) and remark 1.4. In case (3), since $\alpha < 0$, $\beta > 0$ and $\beta + 2\alpha \ge 0$, we observe that $1 \le -\tau < 2$ from the equalities
$$-\tau = \frac{2\beta+\alpha}{\beta-\alpha} = 2 + \frac{3\alpha}{\beta-\alpha} = 1 + \frac{\beta+2\alpha}{\beta-\alpha}.$$
Then, we use proposition 1.5 for $t = -\tau$ and $L = \frac{\bar{x}-a}{\bar{x}}$, $L = \frac{\bar{x}-b}{\bar{x}}$ and $L = \frac{\bar{x}-c}{\bar{x}}$ respectively, to obtain that the inequalities (5) are equivalent to
$$\begin{cases} -\tau > -\dfrac{\bar{x}-a}{\bar{x}}, \\ -\tau > -\dfrac{\bar{x}-b}{\bar{x}}, \\ -\tau > -\dfrac{\bar{x}-c}{\bar{x}}. \end{cases}$$
These, are at the same time, are equivalent to
$$\begin{cases} a_1 = (2\beta+\alpha)\left(\dfrac{a+b+c}{3}\right) - (\alpha-\beta)\left(\dfrac{b+c-2a}{3}\right) > 0, \\ b_1 = (2\beta+\alpha)\left(\dfrac{a+b+c}{3}\right) - (\alpha-\beta)\left(\dfrac{a+c-2b}{3}\right) > 0, \\ c_1 = (2\beta+\alpha)\left(\dfrac{a+b+c}{3}\right) - (\alpha-\beta)\left(\dfrac{a+b-2c}{3}\right) > 0, \end{cases}$$
which was the conclusion we wanted to reach. □

Corollary 1.7. *Suppose that we have a triple of positive numbers (a, b, c) and the numbers (α, β) satisfy $\beta \ge -2\alpha > 0$, then $\{a_n\}, \{b_n\}, \{c_n\} > 0$ for all $n \ge 0$ if and only if*
$$-\frac{\beta}{\alpha} > \max\left(\frac{a}{b+c}, \frac{b}{a+c}, \frac{c}{a+b}\right).$$
In particular for any (α, β) with $\beta \ge -2\alpha > 0$, we can find infinitely many non-trivial triples (a, b, c) such that the sequences $\{a_n\}, \{b_n\}, \{c_n\} > 0$.

Proof. Since $\alpha < 0$, we can rewrite the inequalities $a_1 > 0$, $b_1 > 0$, $c_1 > 0$ as
$$-\frac{\beta}{\alpha} > \frac{a}{b+c} \qquad -\frac{\beta}{\alpha} > \frac{b}{a+c} \qquad -\frac{\beta}{\alpha} > \frac{c}{a+b}$$
and the first part follows from the theorem 1.6. For the second part, notice that $-\frac{\beta}{\alpha} \ge 2$. Let us choose any positive number c and two positive numbers a, b in the interval $(0, c)$ with the condition $a+b > \frac{c}{2}$. By construction we will have
$$-\frac{\beta}{\alpha} \ge 2 > \frac{c}{a+b}, \qquad -\frac{\beta}{\alpha} \ge 2 > \frac{b}{c+b} \qquad \text{and} \qquad -\frac{\beta}{\alpha} \ge 2 > \frac{a}{c+b}$$
for any (a, b, c) chosen as indicated above. As a consequence (a, b, c) is non-trivial $(a, b < c)$ and determine positive sequences $\{a_n\}, \{b_n\}, \{c_n\}$. □

1.5. **Examples.** We are going to present several numerical examples. We begin by doing the geometric case of constructing the sequences of triangles.

Example 1.8. *Let's go back again to our original motivation. Suppose that (a, b, c) are the sides of a triangle T and we take values of $\beta = 1$ and $\alpha = -1$. The positivity of the sequences*

$$\begin{cases} a_{n+1} = b_n + c_n - a_n, \\ b_{n+1} = a_n + c_n - b_n, \\ c_{n+1} = a_n + b_n - c_n. \end{cases} .$$

is equivalent to being able to construct an infinite sequence of triangles $\{T_{n+1}\}$ using $T_0 = T$ and half the length of the three segments of tangency from the vertices of T_n to the inscribed circle as sides of T_{n+1}. As in this case $0 < \beta < -2\alpha$, by theorem 1.6 (part 2), for the sequences to be positive, the initial values (a, b, c) must be trivial $a = b = c$ and our initial triangle must be equilateral.

Example 1.9. *Suppose that $\beta = 2$ and $\alpha = -1$ which satisfies $\beta \geq -2\alpha > 0$. Following corollary 1.7, we pick a number c, for example $c = 1$ and two numbers a, b in the interval $(0, 1)$ such that $a + b > 0.5$. For example take $a = .3$, $b = .25$ and construct the table for the sequences (a_n, b_n, c_n):*

index n	(a_n, b_n, c_n)
0	$(0.3, 0.25, 1)$
1	$(2.2, 2.31, 0.1)$
2	$(2.7, 2.25, 9.0)$
3	$(19.8, 21.15, 0.9)$
4	$(24.3, 20.25, 81.0)$
5	$(178.2, 190.35, 8.1)$