PROJECTION OF MAPS AND PRE-PERIODIC VARIETIES

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In [GT09] the authors presented a contradiction to the dynamic Manin-Mumford conjecture using product of CM-elliptic curves. They consider maps $[\omega]\times[\omega'] : E\times E \longrightarrow E\times E$ on the product $E\times E$, where E is an elliptic with complex multiplication by ω, ω'. For $|\omega| = |\omega'| > 1$ the system admits a polarization and therefore a height function $h_{[\omega]\times[\omega']}$. Besides, if ω/ω' is not equal to a root of unity, the diagonal subvariety $\Delta \subsetneq E \times E$ has the following properties:

(i) Δ contains a Zariski dense set of preperiodic points and therefore $h_{[\omega]\times[\omega']}(\Delta) = 0$.
(ii) Δ is not a preperiodic subvariety.

Similar properties can be found for the diagonal subvariety inside the product of projective lines. Any map $[\omega] : E \longrightarrow E$ on an elliptic curve E can be projected to a map $\varphi_{[\omega]}$ on $\mathbb{P}^1$, when we mod out by the hyperelliptic involution $\pi : E \longrightarrow \mathbb{P}^1$. For $[\omega]$ and $[\omega']$ as in [GT09], the diagonal $\Delta' \subsetneq \mathbb{P}^1 \times \mathbb{P}^1$ satisfies:

(i) $h_{\varphi_{[\omega]}\times\varphi_{[\omega']}}(\Delta') = 0$.
(ii) Δ' is not a preperiodic subvariety.

We are interested in the relation between properties of the orbits of Δ and Δ'. In a more general situation, consider the following commutative diagram in the category of projective algebraic varieties:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \tilde{X} & \xrightarrow{\tilde{\varphi}} & \tilde{X} \end{array}$$

and suppose that we have a polarization for $\tilde{\varphi}$, that is, a line bundle $\tilde{\mathcal{L}}$ on $\tilde{X}$ that satisfies $\tilde{\varphi}^*\tilde{\mathcal{L}} \cong \tilde{\mathcal{L}}^{\otimes d}$ for some $d > 1$. Then the line bundle $\mathcal{L} = \pi^*\tilde{\mathcal{L}}$ will satisfy the equation $\varphi^*\mathcal{L} \cong \mathcal{L}^{\otimes d}$ and therefore constitutes a polarization for φ. Suppose now that $X, \tilde{X}, \varphi$ and $\tilde{\varphi}$ are all defined over a number field K. We will have then two canonical

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height functions $\hat{h}_\varphi, \hat{h}_{\tilde{\varphi}}$ on X and $\tilde{X}$ relative to the maps φ and $\tilde{\varphi}$. The height functions $\hat{h}_\varphi, \hat{h}_{\tilde{\varphi}}$ satisfy the following properties:

(i) For $Y \subsetneq X$ a proper subvariety, we have $\hat{h}_\varphi(Y) = \hat{h}_{\tilde{\varphi}}(\pi(Y))$.
(ii) For $Y = \{P\} \in X$, a point with the reduced structure of proper subvariety, P is preperiodic for φ if and only if $\pi(P)$ is preperiodic for $\tilde{\varphi}$.

Question 0.1. *Is it true that $Y \subsetneq X$ is preperiodic for φ if and only if $\pi_* Y \subsetneq \tilde{X}$ is preperiodic for $\tilde{\varphi}$?*

Let E be an elliptic curve with complex multiplication by a ring R and $\omega, \omega' \in R$. Let $\pi : E \to \mathbb{P}^1$ be the map arising when we mod out by the hyperelliptic involution. In this project we propose to investigate the question for the diagram:

$$\begin{array}{ccc} Y \subset E \times E & \xrightarrow{([\omega],[\omega'])} & E \times E \\ {\scriptstyle (\pi,\pi)}\downarrow & & {\scriptstyle (\pi,\pi)}\downarrow \\ \pi(Y) = Y' \subset \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{(\varphi_{[\omega]},\varphi_{[\omega']})} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

In the case of $Y = \Delta$, the diagonal subvariety, we already know:

Theorem 0.2. *Let $\Delta \subsetneq E \times E$ and $\Delta' \subsetneq \mathbb{P}^1 \times \mathbb{P}^1$ denote respectively the diagonal subvarieties. Suppose that $\omega, \omega' \in R$, then Δ is preperiodic for $[\omega'] \times [\omega'] : E \times E \longrightarrow E \times E$ if and only if Δ' is preperiodic for $\varphi_{[\omega']} \times \varphi_{[\omega']} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$.*

which is a consequence of the two lemmas:

Lemma 0.3. *Δ is preperiodic for $[\omega] \times [\omega'] : E \times E \longrightarrow E \times E$ if and only if ω/ω' is a root of unity.*

Proof. We reproduce here the proof in lemma 04 of [GT09]. Suppose that $([\omega]^{n+k}, [\omega']^{n+k})(\Delta) = ([\omega]^n, [\omega']^n)(\Delta)$ for some $n, k > 0$. Consider a non-torsion point $P \in E$, then there exist $Q \in E$ also non-torsion such that $([\omega]^{n+k}, [\omega']^{n+k})(P, P) = ([\omega]^n, [\omega']^n)(Q, Q)$. But then $[\omega]^{n+k}(P) = [\omega]^n(Q)$ and $[\omega']^{n+k}(P) = [\omega']^n(Q)$ or equivalently $[\omega]^n([\omega]^k(P) - Q) = 0$ and $[\omega']^n([\omega']^k(P) - Q) = 0$. These last two equations are saying that there are torsion points P_1, P_2 such that $[\omega]^k(P) - Q = P_1$ and $[\omega']^k(P) - Q = P_2$ and therefore $[\omega]^k(P) - [\omega']^k(P)$ will also be a torsion point, and that cannot be for P non-torsion unless $[\omega]^k - [\omega']^k = 0$ and therefore ω/ω' is a root of unity. Conversely, suppose that $[\omega]^k = [\omega']^k$, then $([\omega]^{n+k}, [\omega']^{n+k})(P) = ([\omega]^n, [\omega']^n)([\omega]^k(P))$ and because $[\omega]$ is surjective $([\omega]^{n+k}, [\omega']^{n+k})(\Delta) = ([\omega]^n, [\omega']^n)(\Delta)$. $\square$

Lemma 0.4. *Δ' is preperiodic for $\varphi_{[\omega']} \times \varphi_{[\omega']} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ if and only if ω/ω' is a root ± 1.*

Proof. The proof is analogous to the case of Δ. Suppose that we have $(\varphi_{[\omega]}, \varphi_{[\omega']})^{n+k}(\Delta') = (\varphi_{[\omega]}, \varphi_{[\omega']})^n(\Delta')$ for some $n \geq 0$ and $k > 0$. Then $(\pi, \pi)([\omega]^{n+k}, [\omega']^{n+k})(\Delta) = (\pi, \pi)([\omega]^n, [\omega']^n)(\Delta)$ and for each $P \in E$ there will be $Q \in E$ with $([\omega]^{n+k}, [\omega']^{n+k})(P, P) = ([\omega]^n, [\omega']^n)(Q, \pm Q)$. But then $[\omega]^{n+k}(P) = [\omega]^n(Q)$ and $[\omega']^{n+k}(P) = [\omega']^n(\pm Q)$ or equivalently $[\omega]^n([\omega]^k(P) - Q) = 0$ and $[\omega']^n([\omega']^k(P) \mp Q) = 0$. These last two equations are saying that there are torsion points P_1, P_2 such that $[\omega]^k(P) - Q = P_1$ and $[\omega']^k(P) \mp Q = P_2$ and therefore $P_1 \mp P_2 = [\omega]^k(P) \mp [\omega']^k(P) = ([\omega]^k \mp [\omega']^k)(P)$ will also be a torsion point, and that cannot be if we choose P non-torsion unless $[\omega]^k \mp [\omega']^k = 0$ and $(\omega/\omega')^k = \pm 1$. □

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