Hyperbolic polarizations, pairs of inverse maps and the Dirichlet property on arithmetic varieties

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The polarization property of self-maps

Definition

An algebraic dynamical system will be given by a projective, normal, geometrically integral algebraic variety X defined over a field K and a finite, surjective self-map $\varphi\colon X \longrightarrow X$, also defined over K.

Suppose that E is a nonzero $\mathbb{R}$-divisor on X and for some real number $\alpha > 1$, we have the linear equivalence $\varphi^* E \sim \alpha E$. A situation like this will be called a polarized dynamical system (X, φ, E, α) on X.

Example

A map $\varphi\colon \mathbb{P}^d \longrightarrow \mathbb{P}^d$, given by homogeneous polynomials of degree $\alpha > 1$ on the projective space $\mathbb{P}^d$, admits a polarization $(\mathbb{P}^d, \varphi, H, \alpha)$, where H is a hyperplane section.

Intersection number and the polarization property

Let X be of dimension d. We have on X a multilinear intersection product $(D_1, D_2, \ldots, D_d)$, that depends only on the linear equivalence class of the D_i. From the polarization property $\varphi^* E \sim \alpha E$ and the multilinearity of the intersection number, we obtain

$$\alpha^d (E^d) = ((\alpha E)^d) = ((\varphi^* E)^d) = \deg(\varphi)(E^d).$$

As a consequence we have the following properties:

(1) If the polarizing divisor E is ample, then $\deg(\varphi) = \alpha^d$.

(2) If $\deg(\varphi) \neq \alpha^d$, then the self-intersection $(E^d) = 0$.

In the particular case of automorphisms:

Fact:

If $\varphi \colon X \longrightarrow X$ is an automorphism on X and (X, φ, E, α) is a polarized dynamical system, then the self-intersection $(E^d) = 0$.

The family $S_{a,b}$ or $K3$ surfaces in $\mathbb{P}^2 \times \mathbb{P}^2$

Consider the family of $K3$ surfaces $S_{a,b} \subset \mathbb{P}^2 \times \mathbb{P}^2$ studied by Silverman. The family $S_{a,b}$ is determined by equations:

$$\sum_{i,j=1}^{3} a_{i,j} x_i y_j = \sum_{i,j,k,l=1}^{3} b_{i,j,k,l} x_i x_k y_j y_l = 0.$$

The projections p_x and p_y represent double coverings of $\mathbb{P}^2_K$ and determine rational maps σ_x and σ_y in each of the members of the family. Suppose that σ_x, σ_y are morphisms and we have denoted the pull-back divisors by $D^x_n = p^*_x\{x_n = 0\}$ and $D^y_m = p^*_y\{y_m = 0\}$ respectively. We can determine the eigenvalues of $(\sigma_y \circ \sigma_x)^{\pm 1*}$ in the subspace of generated by D^x_n and D^y_m. Indeed for $\varphi = \sigma_y \circ \sigma_x$ and $\beta = 2 + \sqrt{3}$, the real divisors

$$E^+ = E^+_{mn} = \beta D^n_x - D^m_y, \qquad E^- = E^-_{mn} = -D^n_x + \beta D^m_y$$

will satisfy the two identities

$$\varphi^* E^+ \sim \beta^2 E^+ \text{ and } (\varphi^{-1})^* E^- \sim \beta^2 E^-.$$

The family S_c of $K3$ surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Consider a smooth projective variety $S = S_c$ defined by a $(2, 2, 2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. For $(x, y, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, the surface S can be viewed as zero locus of the polynomial

$$F(x, y, z) = \sum_{i_l + j_l = 2,\ l = 0,1,2} c_{i_1,i_2,i_3,j_1,j_2,j_3} x_0^{i_1} x_1^{j_1} y_0^{i_2} y_1^{j_2} z_0^{i_3} z_1^{j_3},$$

where the coefficients $c_{i_1,i_2,i_3,j_1,j_2,j_3}$ belong to a field K. We define rational maps $\sigma_1 = \sigma_{2,3} \colon S \longrightarrow S$, $\sigma_2 = \sigma_{1,3} \colon S \longrightarrow S$ and $\sigma_3 = \sigma_{1,2} \colon S \longrightarrow S$. For generic members of the family, the maps σ_1, σ_2 and σ_3 are well defined automorphisms of the surface S. We call this case, the generic case, and work from now on with generic surfaces of the family S. Let $\{t_0\}$ be a point in $\mathbb{P}^1_K$ and $p^i \colon S \longrightarrow \mathbb{P}^1$ the projection onto the i-th component. Let D_i, for $i = 1, 2, 3$, be the ample divisor $D_i = (p^i)^*[t_0]$ in S

Eigenvalues and eigenvectors for $\sigma^*_{i,j,k} \colon S_c \longrightarrow S_c$

For generic surfaces in S_c, the Picard number is three and we will work with the basis $\mathcal{D} = \{D_1, D_2, D_3\}$ of $\mathrm{Car}(S)_{\mathbb{R}}$. We put $\beta = \frac{3+\sqrt{5}}{2}$, $a = \frac{-3+\sqrt{5}}{2}$ and $b = \frac{-1+\sqrt{5}}{2}$, and name the divisors

$$E_1 = [1, a, b]_{\mathcal{D}} \qquad E_5 = [b, 1, a]_{\mathcal{D}} \qquad E_3 = [a, b, 1]_{\mathcal{D}}$$

$$E_4 = [1, b, a]_{\mathcal{D}} \qquad E_2 = [a, 1, b]_{\mathcal{D}} \qquad E_6 = [b, a, 1]_{\mathcal{D}}$$

The eigenvectors associated to the different eigenvalues λ for the maps $\sigma^*_{i,j,k} = (\sigma_i \circ \sigma_j \circ \sigma_k)^* = \sigma^*_k \circ \sigma^*_j \circ \sigma^*_i$ can be computed as:

Morphism	$\lambda = \beta^3$	$\lambda = \beta^{-3}$	$\lambda = -1$
$\sigma^*_{3,2,1} = \sigma^*_1 \circ \sigma^*_2 \circ \sigma^*_3$	$E_3 = [a, b, 1]$	$E_4 = [1, b, a]$	$[1, -3, 1]$
$\sigma^*_{1,3,2} = \sigma^*_2 \circ \sigma^*_3 \circ \sigma^*_1$	$E_1 = [1, a, b]$	$E_2 = [a, 1, b]$	$[1, 1, -3]$
$\sigma^*_{2,1.3} = \sigma^*_3 \circ \sigma^*_1 \circ \sigma^*_2$	$E_5 = [b, 1, a]$	$E_6 = [b, a, 1]$	$[-3, 1, 1]$
$\sigma^*_{3,1,2} = \sigma^*_2 \circ \sigma^*_1 \circ \sigma^*_3$	$E_6 = [b, a, 1]$	$E_5 = [b, 1, a]$	$[-3, 1, 1]$
$\sigma^*_{2,3,1} = \sigma^*_1 \circ \sigma^*_3 \circ \sigma^*_2$	$E_2 = [a, 1, b]$	$E_1 = [1, a, b]$	$[1, 1, -3]$
$\sigma^*_{1,2,3} = \sigma^*_3 \circ \sigma^*_2 \circ \sigma^*_1$	$E_4 = [1, b, a]$	$E_3 = [a, b, 1]$	$[1, -3, 1]$

Pairs of inverse maps in the family S_c

We have three pairs of polarized dynamical systems on S given by a map $\varphi\colon S \longrightarrow S$ and its inverse $\varphi^{-1}\colon S \longrightarrow S$, namely the pairs:

$$\tau_1 = [(S, \sigma_{3,2,1}, E_3, \beta^3)\,,\,(S, \sigma_{1,2,3}, E_4, \beta^3)],$$
$$\tau_2 = [(S, \sigma_{1,3,2}, E_1, \beta^3)\,,\,(S, \sigma_{2,3,1}, E_2, \beta^3)],$$
$$\tau_3 = [(S, \sigma_{2,1,3}, E_5, \beta^3)\,,\,(S, \sigma_{3,1,2}, E_6, \beta^3)].$$

We can check that the divisors $E_3 + E_4$, $E_1 + E_2$ and $E_5 + E_6$ are ample real divisors divisors in $\mathrm{Car}(S)_{\mathbb{R}}$. Authors like Wang, Baragar and Billard worked out the action of the maps σ_i^* on $\mathrm{Car}(S)_{\mathbb{R}}$, as well as the eigenvalues and eigenvectors. Inspired by the work of Silverman in the family $S_{a,b}$, the pair of maps τ_2 was work out in full details by Billard.

Hyperbolic polarizations and pairs of inverse maps

Definition

A polarized dynamics (X, φ, E, α) defined over K on X will be called hyperbolic polarized if there exist a real divisor $0 \neq E' \in \mathrm{Car}(X)_{\mathbb{R}}$ such that $\varphi^* E' \sim \frac{1}{\alpha} E'$. Furthermore, if $E + E'$ is ample, we will say that the algebraic dynamical system (X, φ, E, α) is ample hyperbolic polarized.

Remark

Let X be a normal projective surface defined over a field K and (X, φ, E, α) a polarized dynamical system associated to an automorphism $\varphi \in Aut(X)$ over K. The system (X, φ, E, α) is hyperbolic polarized with divisor E' if and only if we can find a pair of polarized dynamical systems (X, φ, E, α) and $(X, \varphi^{-1}, E', \alpha)$ associated to the map $\varphi \colon X \longrightarrow X$ and its inverse $\varphi^{-1} \colon X \longrightarrow X$. Again in this situation we will say that (X, φ, E, α) is ample hyperbolic polarized if $E + E'$ is ample.

Properties of hyperbolic polarizations

Proposition

Let (X, φ, E, α) be polarized system on the smooth surface X over the field K. Suppose that (X, φ, E, α) is ample hyperbolic polarized. Then, for every nonzero effective divisor $0 \neq D \in \mathrm{Eff}(X)_{\mathbb{R}}$, we have $(E, D) > 0$.

Proposition

Let X be a smooth projective surface and $\varphi \in \mathrm{Aut}(X)$ such that we have a hyperbolic polarization given by two polarized dynamical systems (X, φ, E, α) and $(X, \varphi^{-1}, E', \alpha)$, with $E + E'$ is ample. Then, for every real nonzero effective divisor D, the intersection numbers (E, D) and (E', D) are both positive. In particular, the divisors E, E' can not be $\mathbb{R}$-linearly equivalent to effective divisors.

Polarized dynamics conditions the geometry of X

Theorem

(Fakhrudin for projective, Zhang for compact Käler manifolds) Let (X, φ, E, α) be a polarized dynamical system on X with E ample. Then the Kodaira dimension $\kappa(X)$ of X is $\kappa(X) \leq 0$.

As a corollary of our intersection results, we get:

Corollary

Let (X, φ, E, α) be ample hyperbolic polarized system on the smooth surface X over the field K associate to an étale map $\varphi \colon X \longrightarrow X$ such that $\deg(\varphi) \neq \alpha$. Let us denote by K_X the canonical divisor of X. Then, for any real number m, the divisor mK_X is either zero or not effective on X. In particular, the Kodaira dimension $\kappa(X) \leq 0$.

Examples of hyperbolic polarizations

Example

For the family $S_{a,b}$ of Wehler K3 surfaces, we have ample hyperbolic polarizations associated to the pair of polarized dynamical systems $(S_{a,b}, \varphi^+, E^+, \beta^2)$ and $(S_{a,b}, \varphi^-, E^-, \beta^2)$, for $\beta = 2 + \sqrt{3}$. We can check that $E^+ + E^- = (1+\sqrt{3})(D_1 + D_2)$ is ample.

Example

Let S_c be the family of K3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. For $\beta = \frac{3+\sqrt{5}}{2}$, the pairs τ_1, τ_2 and τ_3 of inverse maps, provide ample hyperbolic polarizations in S_c:

$$\tau_1 = [(S, \sigma_{3,2,1}, E_3, \beta^3), (S, \sigma_{1,2,3}, E_4, \beta^3)],$$

$$\tau_2 = [(S, \sigma_{1,3,2}, E_1, \beta^3), (S, \sigma_{2,3,1}, E_2, \beta^3)],$$

$$\tau_3 = [(S, \sigma_{2,1,3}, E_5, \beta^3), (S, \sigma_{3,1,2}, E_6, \beta^3)].$$

Adelic metrized divisors and the Dirichlet property

We are going to work with a polarized dynamical system (X, φ, E, α) defined over a number field K. We denote by $\mathcal{O}_K$ its ring of integers.

Consider analytic spaces $\pi_v \colon X_v^{an} \longrightarrow X$ associated to X at each place v of K and metrics on the analytifications $\mathcal{L}_v = \pi_v^* \mathcal{O}(E)$ for each place v of K. For a real divisor E polarizing a dynamical system $\varphi \colon X \longrightarrow X$, there exist a canonical way to put a metric $\|.\|_{\varphi,v}$, associated to the map φ, on the analytification $\mathcal{L}_v = \pi_v^* \mathcal{L}$ of the line bundle $\mathcal{L} = \mathcal{O}(E)$.

Definition

A metric on a line bundle $\mathcal{L}$ is a collection of metrics $\|.\| = (\|.\|_v)_v$, where $\|.\|_v$ is a metric on $\mathcal{L}_v = \pi_v^* \mathcal{L}$ for every place v of K. A metric will be called quasi-algebraic or adelic if it is induced (algebraic) by the same model for almost finite places v of K.

Algebraic metrics: metrics induced by a model

Let v be a finite place. For proper varieties we have a reduction map $red_v \colon X_v^{an} \longrightarrow X_v = X \otimes_K K_v$, from the analytic space X_v^{an} onto the special fibre.

Definition

Suppose that K^0 is a discrete valuation ring and put $S = \mathrm{Spec}(K^0)$. Let $(\tilde{X}, \tilde{\mathcal{L}}, e)$ be a proper flat model of $(X, \mathcal{L}^e)$ over $S = \mathrm{Spec}(K^0)$. Let s be a local section of $\mathcal{L}^{an}$ defined at a point $P \in X^{an}$. Let $\tilde{U} \subset \tilde{X}$ be a trivializing open neighborhood of the reduction $red(P)$ and σ a generator of $\tilde{\mathcal{L}}|\tilde{U}$. Let $U = \tilde{U} \cap X$ and $\lambda \in \mathcal{O}_{U^{an}}$ such that $s^{\otimes e} = \lambda\sigma$ on U^{an}. Then, the metric $\|.\|_{\tilde{X},\tilde{\mathcal{L}},e}$ induced by the proper model $(\tilde{X}, \tilde{\mathcal{L}}, e)$ on $\mathcal{L}^{an}$ is given by

$$\|s(P)\|_{\tilde{X},\tilde{\mathcal{L}},e} = |\lambda(P)|^{1/e}.$$

The metrics obtained in this way are independent of the choice of $\tilde{U}$ and σ and are called **algebraic metrics** on $\mathcal{L}$.

Canonical metric and compactified divisors

The metrized divisor $\widehat{(f)}$ associated to $f \in \mathrm{Rat}(X)^*_{\mathbb{R}}$ has the absolute value $|.|_v$ as associated metric on $\mathcal{O}((f))^{an}_v$. There is a canonical way to associated an adelic metric $\|.\|_\varphi = (\|.\|_{\varphi,v})_v$ to a polarized dynamical system.

Definition

Let (X, φ, E, α) be a polarized dynamical system on X defined over K and let $\|.\|$ be a metric on E. We have $\varphi^* E = \alpha E + (f)$ for some $f \in \mathrm{Rat}(X)^*_{\mathbb{R}}$ and, for every place v, a continuous function $\lambda_{v,(E,\|.\|)} \colon X^{an}_v \longrightarrow K_v$ such that

$$\varphi^* \|.\|_v = |f|_v \|.\|^{\alpha}_v \lambda_{v,(E,\|.\|)}.$$

The function $\lambda_{v,(E,\|.\|_\varphi)} \equiv 1$, for **almost all places**. The canonical metric associated to the map φ is the unique metric $\|.\|_\varphi$ on E satisfying that $\lambda_{v,(E,\|.\|_\varphi)} \equiv 1$ **for all places** v. In this sense we have

$$\varphi^*(E, \|.\|_\varphi) = \alpha(E, \|.\|_\varphi) + \widehat{(f)}.$$

Compactified divisors and the Dirichlet property

Definition

The metric $(\|.\|_{\varphi,v})$ is called the canonical metric on E associated to the map φ. The adelic metrized divisor $\bar{E} = (E, \|.\|_{\varphi,v})$ is called the canonical compactification of E.

In the work of Moriwaki and Chen, canonical compactifications are studied in relation to a higher dimensional analogue of Dirichlet unit's theorem. Let us extend our notion of effective to metrized divisors $\bar{D}$.

Definition

The arithmetic $\mathbb{R}$-divisor $\bar{D}$ is effective ($\bar{D} \succeq 0$) if D is effective and the canonical section s_D satisfies $\|s_D\|_{v,\sup} \leq 1$ for all places $v \in \mathcal{M}_K$. We say that the adelic metrized $\mathbb{R}$-divisor $\bar{D}$ satisfies the Dirichlet property if there exist an $\mathbb{R}$-rational function $f \in \mathrm{Rat}(X)^*_{\mathbb{R}}$ such that $\bar{D} + \widehat{(f)} \succeq 0$.

The classical Dirichlet property on Spec(K)

Proposition

(Dirichlet's unit theorem in dimension zero) Let $X = \text{Spec}(K)$ and let $\bar{D} = (0, (\xi_\sigma)_\sigma)$ be a real metrized divisor on X, the following two are equivalent:

(1) The degree $\widehat{\deg}(\bar{D}) = \sum_\sigma \varepsilon_\sigma \xi_\sigma \geq 0$.

(2) $\bar{D}$ has the Dirichlet property.

Corollary

(Classical Dirichlet's unit theorem) Let Φ a complete set of places at infinity of K without conjugate pairs. Let $\xi = (\xi_\sigma) \in \mathbb{R}^{|\Phi|}$ such that $\sum_{\sigma \in \Phi} \varepsilon_\sigma \xi_\sigma = 0$, where $\varepsilon_\sigma = 1$ for every real embedding $\sigma : K \longrightarrow \mathbb{C}$ and $\varepsilon_\sigma = 2$ otherwise. Then there exist units $u_1, u_2, \ldots, u_s \in \mathcal{O}_K^*$ and $a_1, a_2, \ldots, a_s$ such that $\xi_\sigma = a_1 \log|u_1|_\sigma + \cdots + a_s \log|u_s|_\sigma$ for all $\sigma \in \Phi$.

Another two cases when we have the Dirichlet property

Definition

Let X be a projective normal variety over K and $\bar{D}$ an adelic metrized $\mathbb{R}$-metrized divisor on X. The divisor $\bar{D}$ is pseudo-effective if $\bar{D} + \bar{A}$ is big for every big divisor $\bar{A}$.

If $\bar{D}$ has the Dirichlet property, then it is pseudo-effective. The converse, on the other hand, does not always holds.

Example

(Moriwaki, 2011) If the arithmetic divisor $\bar{D}$ is pseudo-effective and $D_{\mathbb{Q}}$ is numerically trivial, the Dirichlet property holds for $\bar{D}$.

Example

(Burgos Gil, Moriwaki, Phillippon, Sombra, 2012) If X is a toric variety and $\bar{D}$ is an toric metrized $\mathbb{R}$-divisor, the Dirichlet property holds for $\bar{D}$.

Polarizations and the Dirichlet property

Canonically compactified divisors gives examples and non-examples of metrized divisors with the Dirichlet property.

Example

(Chen/Moriwaki negative answer in Abelian varieties) Let A be an abelian variety and E a symmetric ample divisor such that $[2]^*E = 4E + div(f)$. For any place σ at infinity, the set $Prep([2])_\sigma$ of points with finite forward orbit, is dense in $A(\mathbb{C})_\sigma$. Therefore $\bar{E}$ does not have the Drichlet property.

Example

(Chen/Moriwaki positive answer for polynomial maps) Whenever a polarized system (X, φ, E, α) gives equality $\varphi^*E = \alpha D$ with E effective, the Dirichlet property holds for $\bar{E}$. For example take a surjective polynomial map $\varphi\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$ of degree $\deg(\varphi) = \alpha > 1$ and the hyperplane $H = \{P = (x_0, \ldots, x_n) \in \mathbb{P}^n \,|\, x_0 = 0\}$. We have $\varphi^*H = \alpha H$ and the Dirichlet property holds for $\bar{H}$.

Hyperbolic polarizations and the Dirichlet property

Ample hyperbolic polarizations will give examples of compactified divisors **without** the Dirichlet property:

Proposition

Let (X, φ, E, α) be an ample hyperbolic polarized system in the smooth surface X, with $\deg(\varphi) \neq \alpha^2$. Then, the canonical compactification $\bar{E}$ do not satisfy the Dirichlet property. Also, for a pair of polarized systems (X, φ, E, α), $(X, \varphi^{-1}, E', \alpha)$ with $E + E'$ ample, the canonically compactified divisors $\bar{E}$ and $\bar{E}'$ do not satisfy the Dirichlet property.

Corollary

In the family $S_{a,b}$, the compactified divisors $\bar{E}^+, \bar{E}^-$ do not have the Dirichlet property. In the family S_c, the compactified divisors $\bar{E}_i$ do not have the Dirichlet property.

Rational points on surfaces with hyperbolic polarizations

Let X be a smooth projective surface over a field K and (X, φ, E, α) a polarized dynamical system on X. Let us assume furthermore that (X, φ, E, α) is ample hyperbolic polarized.

Definition

Let us define the orbit of a point $P \in X(K)$ under the action of $\varphi\colon X \longrightarrow X$ as

$$\mathcal{A}(P) = \{\varphi^n(P) \mid n \in \mathbb{N}\}.$$

Proposition

If the orbit $\mathcal{A}(P)$ is infinite, then $\mathcal{A}(P)$ is Zariski dense in $X(K)$.

The above result was proved by Silverman for the family $S_{a,b}$ and by Billard for the family S_c under the action of the system τ_2.

Polarized dynamical systems and canonical heights

Consider a polarized dynamics (X, φ, E, α) over a number field K. Following Silverman (1993), we can define a canonical height function $\hat{h}_\varphi \colon X(\bar{K}) \longrightarrow \mathbb{R}$ satisfying the following properties:

1. $\hat{h}_\varphi$ is a Weil height function associated to E.
2. $\hat{h}_\varphi(\varphi(P)) = \alpha \hat{h}_\varphi(P)$.

Remark

If the orbit $\mathcal{A}(P)$ of a point P is finite, the canonical height $\hat{h}_\varphi(P) = 0$. Under some conditions we can prove the converse of this result.

Assumption

Suppose that a Weil height $h' = h_{E'}$ for the divisor E' is bounded on the orbit $\mathcal{A}(P)$. This is, there exist $M = M(P) > 0$ such that for all $n \in \mathbb{N}$, we have

$$|h'(\varphi^n(P))| < M.$$

Canonical height functions and rational points

Proposition

Under our assumption, the ample hyperbolic polarized system (X, φ, E, α) over the number field K satisfies:

$$\text{The orbit } \mathcal{A}(P) \text{ of a point } P \text{ is finite } \iff \hat{h}_\varphi(P) = 0.$$

Proposition

Under the assumption, the canonical height $\hat{h}_\varphi$ associated to our ample hyperbolic polarized system (X, φ, E, α) satisfies $\hat{h}_\varphi \geq 0$.

One way to satisfy our Assumption is to have an ample hyperbolic polarization (X, φ, E, α) associated to an automorphism $\varphi \colon X \longrightarrow X$ and choose the canonical height $\hat{h}' = \hat{h}_{\varphi^{-1}}$ associated $(X, \varphi^{-1}, E', \alpha)$.

Rational points and heights for pairs of inverse maps

The following result was obtained by Silverman on $S_{a,b}$ and by Billard on the family S_c:

Corollary

Let K be a number field and suppose and X a smooth surface defined over K. Suppose that (X, φ, E, α) is an ample hyperbolic polarization over K associated to an automorphism $\varphi \colon X \longrightarrow X$ on X. Let us denote by $\hat{h} = \hat{h}_{\varphi}$ and $\hat{h}' = \hat{h}_{\varphi^{-1}}$, the canonical heights associated to the dynamical systems (X, φ, E, α), and $(X, \varphi^{-1}, E', \alpha)$ respectively. For a rational point $P \in X(K)$, the following are equivalent:

(1) $\mathcal{A}(P)$ *is finite.*

(2) $(\hat{h} + \hat{h}')(P) = 0$.

(3) $\hat{h}(P) = 0$.

(4) $\hat{h}'(P) = 0$.

Arithmetic results

Suppose that (X, φ, E, α) is an ample hyperbolic polarization over K associated to an automorphism $\varphi\colon X \longrightarrow X$ on X.

Corollary

The set $\{P \in X(\bar{K}) \,|\, \mathcal{A}(P) \text{ is finite}\}$ is a set of bounded height. In particular, the set $\{P \in X(K) \,|\, \mathcal{A}(P) \text{ is finite}\}$ is a finite set.

Definition

For an orbit $C = \mathcal{A}(P)$, let us define the height of the orbit C as the number $h(\mathcal{C}) = \sqrt{\hat{h}(P)\hat{h}'(P)}$. The number $h(\mathcal{C})$ will measure the arithmetic complexity of the orbit $\mathcal{C}$.

Proposition

(Silverman 93, Billard 97) If $h(\mathcal{C}) > 0$ and B is sufficiently big, we have

$$\#\{Q \in \mathcal{C} \,|\, \hat{h}(Q) + \hat{h}'(Q) < B\} = \kappa(\mathcal{C}) \log_{\alpha}(B/h(\mathcal{C})) + O(1).$$

Open questions

So far, our examples of hyperbolic polarizations, involve automorphisms $\varphi\colon X \longrightarrow X$ and pairs of polarized systems (X, φ, E, α), $(X, \varphi^{-1}, E', \alpha)$.

Question

Can we find a hyperbolic polarization (X, φ, E, α), where $\varphi\colon X \longrightarrow X$ is not an automorphism? In case of a positive answer, can we find one such polarization satisfying our assumption on the height $h_{E'}$?

Question

Can we get results for the counting of rational points in orbits, when $\varphi\colon X \longrightarrow X$ is not an automorphism and we have an ample hyperbolic polarization (X, φ, E, α) in X?

Question

Can we find a hyperbolic polarization (X, φ, E, α) on a geometrically ruled surface $\pi\colon X \longrightarrow C$? Is the map $\varphi \in \mathrm{Aut}(X)$?

To the organizers for the great conference and school, thanks!