Two numbers correcting the behavior of rational maps: the dynamical degree and the D-ratio.

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Notation

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K will denote a number field.

X will denote a non-singular projective algebraic variety over K.

$\varphi : X \longrightarrow X'$ *will denote a morphism from X to X'.*

$\varphi : X \dashrightarrow X'$ *will denote a rational map defined maybe only on an open set* $U \subset X$.

$\varphi = (\varphi_0 : \varphi_1 : \cdots : \varphi_n) : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ *will denote a self-map on the n-dimensional projective space* $\mathbb{P}^n$, *where* $\varphi_0, \varphi_1, \ldots, \varphi_n$ *are homogenous polynomials of degree d without common factors.*

D will denote a divisor on X and h_D *a height function relative to D.*

Divisors and linear equivalence

Definition

A divisor D is an element of the free abelian group generated by the subvarieties $V \subset X$ of codimension one. We can think of a divisor as a finite sum

$$D = \sum_i a_i V_i,$$

where a_i are in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ and V_i are subvarieties of codimension one in X.

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Definition

We say that two divisors D and D' are linearly equivalent ($D \sim D'$) if their difference D-D'=(f) is the principal divisor associated to a rational function $f \in K(X)$. We denote by $Pic(X) = Div / \sim$ the group of divisors module linear equivalences.

Intersection Theory

We have an intersection theory for subvarieties that extend to a multilinear form on divisors in X.

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For a surface X and C,C' two distinct subvarieties of dimension one, we can think of (C,C') as the numbers of points of the intersection counted with multiplicities.

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For a surface X and C,C' two distinct subvarieties of dimension one, we can think of (C,C') as the numbers of points of the intersection counted with multiplicities.

Remark

An important type of divisors will be the ones with positive intersection with all subvarieties. D is said to be ample if for all subvarieties V,

$$(\underbrace{D, D, \ldots, D}_{\dim(V)}, V) = (D^{\dim(V)}, V) > 0.$$

Numerical equivalence

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We say that two divisors D and D' are numerically equivalent divisors ($D \equiv D'$) if (D,C)=(D',C) for every irreducible curve C. We denote by $N(X) = \mathrm{Div} / \equiv$ the group of divisors module numerical equivalence.

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Remark

For complex projective non-singular algebraic varieties, the group of numerically equivalent classes $\mathrm{N}(X)$ *is the quotient of the Neron-Severi group* $\mathrm{NS}(X)$ *by its torsion subgroup, therefore* $\mathrm{NS}_{\mathbb{R}}(X) \cong \mathrm{N}(X)_{\mathbb{R}}$. *Also, any two linearly equivalent divisors are numerically equivalent.*

Induced maps on divisors

Remark

For a morphism $\varphi : X \longrightarrow X'$ *we have induced maps*

$$\varphi^* : \mathrm{NS}_{\mathbb{Q}}(X') \longrightarrow \mathrm{NS}_{\mathbb{Q}}(X), \qquad \varphi_* : \mathrm{NS}_{\mathbb{Q}}(X) \longrightarrow \mathrm{NS}_{\mathbb{Q}}(X').$$

In particular when we have a self-map $\varphi : X \longrightarrow X$, *we get:*

$$\varphi^* : \mathrm{NS}_{\mathbb{Q}}(X) \longrightarrow \mathrm{NS}_{\mathbb{Q}}(X), \qquad \varphi_* : \mathrm{NS}_{\mathbb{Q}}(X) \longrightarrow \mathrm{NS}_{\mathbb{Q}}(X).$$

For D *in* $\mathrm{Div}(X)$ *and* C *an algebraic curve* $C \subset X$ *we have a projection formula*

$$(\varphi^* D, C) = (D, \varphi_* C).$$

Degree of maps on the n-dimensional projective space

Definition

Suppose that $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ is a self-map on the n-dimensional projective space $\mathbb{P}^n$. The degree of the map φ can be defined in any of the following equivalent ways:

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2. The spectral radius $\rho(\varphi^*)$ of the linear map

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induced on the Neron-Severi group $\mathrm{NS}_{\mathbb{R}}(\mathbb{P}^n)$.

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induced on the Neron-Severi group $NS_{\mathbb{R}}(\mathbb{P}^n)$.
3. The intersection number $(\varphi^* H, H^{n-1})$, where H is an ample divisor.

Weil height on the projective space

Remark

When we have a map $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ *on the projective space, the structure of the Neron-Severi group is quite simple* $\mathrm{NS}_{\mathbb{R}}(\mathbb{P}^n) = \mathbb{R}$ *and the map* $\varphi^* : \mathrm{NS}_{\mathbb{R}}(\mathbb{P}^n) \longrightarrow \mathrm{NS}_{\mathbb{R}}(\mathbb{P}^n)$ *is just multiplication by* $\deg(\varphi)$*. That is the reason why we can define degree for maps on* $\mathbb{P}^n$ *in several equivalent ways.*

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Definition

The Weil logarithmic height h(P) of a point $\mathrm{P}=(\mathrm{x}_0 : \mathrm{x}_1 : \cdots : \mathrm{x}_\mathrm{n})$ in $\mathbb{P}^\mathrm{n}$ is defined by the sum over all places v of K:

$$\mathrm{h}(\mathrm{P}) = \sum_{\mathrm{v}} \log \sup(|\mathrm{x}_0|_\mathrm{v}, |\mathrm{x}_1|_\mathrm{v}, \ldots, |\mathrm{x}_\mathrm{n}|_\mathrm{v}).$$

Height for maps on the projective space

Remark

For $\deg(\varphi) > 1$, *the sequence defined by*

$$\left\{\frac{h(\varphi^n(P))}{\deg(\varphi)^n}\right\}$$

is a Cauchy sequence.

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$$\hat{h}_\varphi = \lim_n \frac{h(\varphi^n(P))}{\deg(\varphi)^n},$$

is a height function associated to the map φ. *It will satisfy the functional equation:*

$$\hat{h}_\varphi(\varphi(P)) = \deg(\varphi)h(P).$$

Degree of a self-map on a projective variety

Definition

Suppose that $\varphi : X \longrightarrow X$ is a self-map on the projective algebraic variety X. The degree $\deg(\varphi)$ of the map φ can be defined as the spectral radius or maximal eigenvalue $\rho(\varphi^*)$ of the linear map induced on the Neron-Severi group $\varphi^* : NS_{\mathbb{R}}(X) \longrightarrow NS_{\mathbb{R}}(X)$.

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Remark

The degree of a morphism behave properly in the sense that

$$\deg(\varphi^n) = \deg(\varphi)^n.$$

Canonical height for selfmaps on projective varieties

Theorem

(Canonical height) Let X be a smooth projective algebraic variety defined over a number field K. Suppose that $\varphi : X \longrightarrow X$ is a self-map of X in such a way that for some $\alpha > 1$ and for some divisor $D \in \mathrm{Div}(X)$, we have $\varphi^* D \sim \alpha D$. We have the following:

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(a) *For all $P \in X(\bar{K})$, the following limits exists*

$$\hat{h}_{D,\varphi}(P) = \lim_{n\to\infty} \alpha^{-n} h_D(\varphi^n(P)).$$

(b) $\hat{h}_\varphi(\varphi(P)) = \alpha \hat{h}_\varphi(P)$ *for all $P \in X(\bar{K})$.*

(c) *For D ample, $\hat{h}_\varphi(P) = 0$ if and only if P is pre-periodic for the map φ.*

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(c) *For D ample, $\hat{h}_\varphi(P) = 0$ if and only if P is pre-periodic for the map φ.*

The condition $\varphi^* D \sim \alpha D$ is saying that D is an eigenvector associated to the eigenvalue $\alpha > 1$.

Duplication on Elliptic curves

Let K be a number field. Consider the elliptic curve E with affine model $E : y^2 = G(x) = x^3 + px + q$, where $p, q \in K$. The map $[2] : E(K) \longrightarrow E(K)$ represents multiplication by 2 or adding a point to itself in the group law of the points in E(K).

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$$[2](x, y) = \left(\frac{G'(x)^2 - 8xG(x)}{4G(x)}, \frac{G'(x)^3 - 12xG(x)G'(x) + 8G(x)^2}{8G(x)y}\right).$$

The pre-periodic points of the map are the torsion points in $E(K)$. An ample symmetric line bundle $\mathcal{L} = \mathcal{L}^{-1}$ on E satisfies the equation $[2]^*\mathcal{L} \cong \mathcal{L}^4$. In this example, $\mathcal{L}$ is an eigenvector for the map $[2]^*$ associated to the eigenvalue $\lambda = 4$.

Dynamics on a family of K3 surfaces

Consider the family $S_{a,b}$ of K3 surfaces studied by Wehler and later by Silverman. The family $S_{a,b} \subset \mathbb{P}^2 \times \mathbb{P}^2$ is determined by the two equations

$$\sum_{i,j=1}^{3} a_{i,j} x_i y_j = 0, \qquad \sum_{i,j,k,l=1}^{3} b_{i,j,k,l} x_i x_k y_j y_l = 0,$$

with coefficients in a number field K.

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with coefficients in a number field K. When we fix each set of variables x or y, we get double coverings $p_1, p_2 : S_{a,b} \longrightarrow \mathbb{P}^2$ of $\mathbb{P}^2_K$ that determine morphisms $\sigma_1, \sigma_2 : S_{a,b} \longrightarrow S_{a,b}$.

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$$E^+ = (2+\sqrt{3})L_1 - L_2, \qquad E^- = -L_1 + (2+\sqrt{3})L_2,$$

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$$E^+ = (2 + \sqrt{3})L_1 - L_2, \qquad E^- = -L_1 + (2 + \sqrt{3})L_2,$$

the geometry of the family $S_{a,b}$ can be used to prove that

$$(\sigma_1 \circ \sigma_2)^{\pm 1 *}(E^\pm) = (7 + 4\sqrt{3})E^\pm.$$

Degree of rational maps on the projective spaces

Definition

Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a dominant rational map. The degree $\deg(\varphi)$ can be defined in the same way we did for morphisms on $\mathbb{P}^n$ as the degree of the polynomials defining $\varphi = (\varphi_0 : \varphi_1 : \cdots : \varphi_n)$.

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Example

Consider the map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $\varphi(x : y : z) = (x^2 : yx : zx)$. The map φ is defined in the open set $U = \{x \neq 0\}$. The second iterate gives

$$\varphi^2(x : y : z) = (x^4 : yx^3 : zx^3) = (x : y : z).$$

This shows that in general

$$\deg(\varphi^n) \neq \deg(\varphi)^n.$$

Definition of algebraically stable in codimension one

Definition

Consider a blowing up $\pi : \tilde{X} \dashrightarrow X$ of the indeterminacy I_φ of the rational map $\varphi : X \dashrightarrow X$. As explained in Example II.2.17.3 of Hartshorne, we can extend the rational map φ to a morphism $\tilde{\varphi} : \tilde{X} \longrightarrow X$. We define $\varphi^* : NS(X) \longrightarrow NS(X)$ by

$$\varphi^* = \pi_* \circ \tilde{\varphi}^*.$$

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Definition

(Fornaess and Sibony) A rational dominant map $\varphi : X \dashrightarrow X$ is said to be algebraically stable in codimension one if $(\varphi^n)^* = (\varphi^*)^n$. Morphisms are always algebraically stable maps by the functoriality $(\varphi \circ \varphi')^* = \varphi'^* \circ \varphi^*$.

Definition of first dynamical degree of rational maps

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Remark

For an algebraically stable map, $\delta_\varphi = \rho(\varphi)$ is an algebraic integer.

The case of rational maps on the projective space

Remark

For a dominant rational map $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$*, the structure of the Neron-Severi group* ($\mathrm{NS}_{\mathbb{R}}(\mathbb{P}^n) = \mathbb{R}$) *allows us to define the first dynamical also as*

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$$\delta_\varphi = \lim_{n\to\infty} \sup(\deg(\varphi^n))^{1/n}.$$

Example

Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be given by $\varphi(x : y : z) = (xy : xz : z^2)$ then $\varphi^2(x : y : z) = (x^2yz : xyz^2 : z^4) = (x^2y : xyz : z^3)$. So we have, $\deg(\varphi) = 2$, $\deg(\varphi^2) = 3$ and we can see in general that $\deg(\varphi^n) = F_{n+1}$ is given by the Fibonacci sequence and therefore

$$\deg(\varphi^n) \approx ((1+\sqrt{5})/2)^n \Rightarrow \delta_\varphi = (1+\sqrt{5})/2.$$

Examples of dynamical degrees

Example

Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be given by $\varphi(x : y : z) = (x^2 : xz : y^2)$. The indeterminacy locus is set-theoretically $(0 : 0 : 1)$ and the formula for the iterates are

$$\varphi^{2n}(x : y : z) = (x^{2^n} : y^{2^n} : z^{2^n}),$$

$$\varphi^{2n\text{-}1}(x : y : z) = (x^{2^n} : x^{2^{n-1}} z^{2^{n-1}} : y^{2^n}),$$

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Remark

The dynamical degree of all our examples are algebraic integers.

An algebraically stable rational map that is not a morphism

Example

Let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be given in projective coordinates by $\varphi(x : y : z) = (x^3 + yz^2 : xz^2 + y^2z : z^3)$. The indeterminacy locus is $(0 : 1 : 0)$. The iterates are

$$\begin{aligned}
\varphi(x : y : z) &= (x^3 + yz^2 : xz^2 + y^2z : z^3), \\
\varphi^2(x : y : z) &= (x^9 + zP(x, y, z) : z^3Q(x, y, z) : z^9), \\
\varphi^n(x : y : z) &= (x^{3^n} + zP_n(x, y, z) : z^{3^{n-1}}Q_n(x, y, z), z^{3^n}),
\end{aligned}$$

and therefore $\deg(\varphi^n) = 3^n$ and $\delta_\varphi = \deg(\varphi) = 3$.

Extension of Canonical height to numerical equivalence

Theorem

(Kawaguchi, Silverman) Assume that $\varphi : X \longrightarrow X$ *is a morphism defined over the number field* K *and suppose that* $D \in \mathrm{Div}_{\mathbb{R}}(X)$ *is a divisor that satisfy the numerical equivalence:*

$$\varphi^* D \equiv \beta D \quad \text{for some real number} \quad \beta > \sqrt{\delta_\varphi},$$

Then we have the following:

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Then we have the following:

(a) *For all* $P \in X(\bar{K})$*, the following limits exists:*

$$\hat{h}_{D,\varphi}(P) = \lim_{n\to\infty} \beta^{-n} h_D(\varphi^n(P)).$$

(b) *For all* $P \in X(\bar{K})$*,* $\hat{h}_{D,\varphi}(\varphi(P)) = \beta \hat{h}_{D,\varphi}(P)$*.*

(c) *For* D *ample,* $\hat{h}_{D,\varphi}(P) = 0$ *if and only if* P *is pre-periodic for* φ*.*

Some technical details: Ample and Nef divisors

Proposition

(Kawaguchi, Silverman) The δ_φ can be defined as the limit (no need the lim sup*)*

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The proof uses the introduction of different cones in $NS_{\mathbb{R}}(X)$: the cone Amp(X) of ample $\mathbb{R}$-divisors and the cone Nef(X) of numerically effective (**nef**) $\mathbb{R}$-divisors.

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Definition

(Nakai-Moishezon criterium) A divisor D is said to be ample if for all irreducible subvarieties V of dimension $k > 0$, $(D^k, V) > 0$.
(Kleiman's theorem) A divisor is said to be **nef** if for all irreducible subvarieties V of dimension $k > 0$, $(D^k, V) \geq 0$.

Morphisms with positive entropy admit polarization

One good reason to study **nef** divisors is provided by the following

Remark

Assume that $\varphi : X \longrightarrow X$ is a morphism defined over the number field K. There is always a **nef** *divisor D, such that*

$$\varphi^* D \equiv \delta_\varphi D.$$

For a morphism of positive entropy ($\delta_\varphi > 1$), we will have a canonical height satisfying:

(a) $\hat{h}_{D,\varphi}(P) = \lim_{n\to\infty} \delta_\varphi^{-n} h_D(\varphi^n(P))$.

(b) *For all* $P \in X(\bar{K})$, $\hat{h}_{D,\varphi}(\varphi(P)) = \delta_\varphi \hat{h}_{D,\varphi}(P)$.

Effective and pseudo-effective divisors

Definition

A divisor $D \in \mathrm{Div}_{\mathbb{R}}$ is effective ($D \succeq 0$) if it can be written as a sum $\sum c_i D_i$, where D_i are effective and $c_i \in \mathbb{R}$ are positive. The cone of $\mathbb{R}$-effective divisors is denoted by $\mathrm{Eff}(X)$. The closure $\overline{\mathrm{Eff}(X)}$ of $\mathrm{Eff}(X)$ represent the cone of pseudo-effective divisors.

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Example

The divisors $E^+ = (2+\sqrt{3})L_1 - L_2$ and $E^- = -L_1 + (2+\sqrt{3})L_2$ on Wheler family of K3 surfaces are in $\overline{\mathrm{Eff}(S)} \setminus \mathrm{Eff}(S)$.

Several norms for operators on the Neron-Severi group

We can define several norms for endomorphisms of the finitely dimension $\mathbb{R}$-vector space $NS_{\mathbb{R}}(X)$:

$$\|A\|' = \sup_{v \neq 0 \in Nef} \frac{|Av|}{|v|}, \qquad \|A\|'' = \sup_{v \neq 0 \in \overline{Eff}} \frac{|Av|}{|v|},$$

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Lemma

Let $\|.\| = \|.\|_\infty$ *denotes the usual* sup *norm of the matrix* A_{φ^*} *representing the map* φ^* *on a fixed basis of* $\mathrm{NS}(X)$*, for some positive constants* C*,* C_1 *and* C_2 *we can prove::*

$$\|(\varphi^{n+m})^*\|' \leq C\|(\varphi^n)^*\|'\|(\varphi^m)^*\|'',$$

$$\|(\varphi^n)^*\|' \leq C_1\|(\varphi^*)^n\|',$$

$$\|(\varphi^{n+m})^*\| \leq C_2\|(\varphi^n)^*\|\|(\varphi^m)^*\|.$$

Presentation of A family of four-folds with three involutions

A situation similar to Wheler family is as follows. Consider the family $\{X^{A,B}\}_{A,B}$ defined in $\mathbb{P}^2_K \times \mathbb{P}^2_K \times \mathbb{P}^2_K$ by equations

$$L(x,y,z) = \sum_{i,j,k=0}^{2} a_{i,j,k} x_i y_j z_k = 0,$$

$$Q(x,y,z) = \sum_{i,j,k,l,m,n=0}^{2} b_{i,j,k,l,m,n} x_i x_l y_j y_m z_k z_n = 0,$$

where $A = (a_{ijk})$, $B = (b_{i,j,k,l,m,n})$ and all indices are moving in the set $\{0,1,2\}$.

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where $A = (a_{ijk})$, $B = (b_{i,j,k,l,m,n})$ and all indices are moving in the set $\{0,1,2\}$.When we fix two of the variables we get the intersection on $\mathbb{P}^2$ of a quadric and a line, which is general, will give two points $P_i, P_i' \in X$ for $i = 1,2,3$ and will determine involutions $\sigma_1, \sigma_2, \sigma_3 : X \dashrightarrow X$. The involutions σ_i for $i = 1,2,3$, will not be in general morphisms but just rational maps defined on certain open sets $U_i \subset X$.

Pull-backs of Hyperplanes to X

To study the action of the σ_i^* on Pic(X) we denote by H, H' hyperplane sections representing the two fundamental classes in $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2)$,

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Also when we take points $P = (x_0 : x_1 : x_2, y_0 : y_1 : y_2, z_0 : z_1 : z_2)$ in X, we want to define the divisors D_x, D_y and D_z by:

$$D_x = \{P \in X : x_0 = 0\}, \qquad D_y = \{P \in X : y_0 = 0\},$$

$$D_z = \{P \in X : z_0 = 0\}.$$

Pull-backs of divisors in X

The pullbacks of H, H' by the different projections give back the divisors D_x, D_y, D_z,

$$p_{xy}^* H = p_3^* H = D_x, \quad p_{xy}^* H' = p_3^* H' = D_y,$$

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$$p_{yz}^* H = p_1^* H = D_y, \qquad p_{yz}^* H' = p_1^* H' = D_z.$$

Action of the σ_i^*.

The action of the σ_i^*'s on the divisors D_x, D_y, D_z is expressed by:

$$\begin{aligned}
\sigma_1^*(D_x) &= p_1^* p_{1*} D_x - D_x \sim 4D_y + 4D_z - D_x,\\
\sigma_1^*(D_y) &= \sigma_1^* p_1^* H = (p_1 \circ \sigma_1)^* H = D_y,\\
\sigma_1^*(D_z) &= \sigma_1^* p_1^* H' = (p_1 \circ \sigma_1)^* H' = D_z,\\
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\sigma_2^*(D_y) &= p_2^* p_{2*} D_y - D_y \sim 4D_x + 4D_z - D_y,\\
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\sigma_3^*(D_z) &= p_3^* p_{3*} D_z - D_z \sim 4D_x + 4D_y - D_z.
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Action of the involutions

Theorem

The involutions σ_1, σ_2 and σ_3 are algebraically stable maps.

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The matrices associated with action of these maps on $\mathrm{NS}_{\mathbb{R}}$ are:

$$\sigma_1^* = \begin{pmatrix} -1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \sigma_2^* = \begin{pmatrix} 1 & 0 & 0 \\ 4 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_3^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 4 & -1 \end{pmatrix}$$

Action of the composition of two involutions

The matrices associated to the composition of any two of the involutions are:

$$\sigma_{12}^* = \begin{pmatrix} -1 & -4 & 0 \\ 4 & 15 & 0 \\ 4 & 20 & 1 \end{pmatrix} \qquad \sigma_{21}^* = \begin{pmatrix} 15 & 4 & 0 \\ -4 & -1 & 0 \\ 20 & 4 & 1 \end{pmatrix}$$

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$$\sigma_{32}^* = \begin{pmatrix} 1 & 4 & 20 \\ 0 & -1 & -4 \\ 0 & 4 & 15 \end{pmatrix} \qquad \sigma_{23}^* = \begin{pmatrix} 1 & 20 & 4 \\ 0 & 15 & 4 \\ 0 & -4 & -1 \end{pmatrix}$$

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$$\sigma_{31}^* = \begin{pmatrix} -1 & 0 & -4 \\ 4 & 1 & 20 \\ 4 & 0 & 15 \end{pmatrix} \qquad \sigma_{13}^* = \begin{pmatrix} 15 & 0 & 4 \\ 20 & 1 & 4 \\ -4 & 0 & -1 \end{pmatrix}$$

Dynamical degree of rational maps on four-folds.

Theorem

Suppose that X is smooth and therefore the Picard number $p(X) = 3$, then the first dynamical degree $\delta_{\sigma_{ij}}$ of the algebraically stable map σ_{ij} is $\delta_{\sigma_{ij}} = \beta = 7 + 4\sqrt{3}$.

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Proof.

The divisors D_x, D_y, D_z represent three distinct classes in $NS(X)_{\mathbb{Q}}$. If the Picard number $p(X) = 3$, then we have that $NS(X)_{\mathbb{Q}}$ is the subspace spanned by D_x, D_y, D_z and the dynamial degree is:

$$\delta_{\sigma_{ij}} = \limsup_{n\to\infty} \rho((\sigma_{ij}^n)^*)^{1/n} =$$

$$\limsup_{n\to\infty} \rho((\sigma_{ij}^*)^n)^{1/n} = \limsup_{n\to\infty} (\beta^n)^{1/n} = \beta.$$

□

Questions on a family of four-folds.

Question

Can we describe the cones $\mathrm{Amp}(X_{A,B})$ *and* $\mathrm{Nef}(X_{A,B})$ *for smooth members of the family* $\{X^{A,B}\}_{A,B}$*?*

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Remark

(Silverman) In the similar case of $X = S_{a,b}$ *being a Wheler K3 surface we have*

$$\mathrm{Amp}(S_{a,b}) = \mathrm{Eff}(S_{a,b}),$$

$$\mathrm{Nef}(S_{a,b}) = \overline{\mathrm{Eff}(S_{a,b})},$$

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Remark

(Silverman) For example for Wheler family of K3 surfaces $S_{a,b}$ and the involutions $\sigma_1, \sigma_2 : S_{a,b} \longrightarrow S_{a,b}$ we have:

$$\|\sigma_i\|' = \|\sigma_i\|'' = \sqrt{17} + \frac{\sqrt{8\sqrt{3}}}{\sqrt{8+4\sqrt{3}}}.$$

Relation of heights of iterates on the n-projective space

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Proposition

(Silverman) Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map of degree d defined over $\bar{\mathbb{Q}}$. Let I_φ be the subset of common zeroes of the φ_i's.

(a) *There exists a constant C_1 such that $h(\varphi(P)) \leq dh(P) + C_1$ for all $P \in \mathbb{P}^n(\bar{\mathbb{Q}}) \setminus I_\varphi$.*

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(b) *Let $X \subset \mathbb{P}^n$ a subvariety with the property $X \cap I_\varphi = \emptyset$, then there exist a constant C_2 such that $h(\varphi(P)) \geq dh(P) - C_2$ for all $P \in X(\bar{\mathbb{Q}})$.*

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Remark

It is not true in general that $h(\varphi(P)) \geq dh(P) + C_2$ for all $P \in \mathbb{P}^n(\bar{\mathbb{Q}}) \setminus I_\varphi$, for example take $\varphi(x : y : z) = (x^2 : y^2 : xz)$.

The D-ration of a rational map

Motivation

For a morphism $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ *and a hyperplane* H*, the coefficient of* H *in* $\varphi^* H$ *is just* $\deg(\varphi)$ *and therefore we have that*

$$\frac{1}{\deg(\varphi)} \varphi^* H - H \succeq 0.$$

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$$\frac{1}{\deg(\varphi)}\varphi^* H - H \succeq 0.$$

For a rational map $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *whose indeterminacy locus* I_φ *is contained in a hyperplane* H*, we take a resolution of the indeterminacy* $\pi : \tilde{X} \dashrightarrow \mathbb{P}^n$ *and a morphism* $\tilde{\varphi} : \tilde{X} \longrightarrow \mathbb{P}^n$ *extending* φ *and consider the numbers* δ *such that the divisor*

$$\frac{1}{\delta}\tilde{\varphi}^* H - \pi^* H$$

is an effective divisor.

D-ratio: the definition

Definition

(Joey) The D-ratio is the constant $r(\varphi)$ such that:

$$\frac{\deg(\varphi)}{r(\varphi)} = \sup_{\delta}\{\delta : \frac{1}{\delta}\tilde{\varphi}^* H - \pi^* H \quad \text{is} \quad \mathbb{A}^n - \text{effective}\}.$$

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With the help of the D-ratio we can have part (b) of the proposition relating the height of a point and its image in a thicker set:

Theorem

(Joey) There exist a constant C_2', depending only on φ, such that $\frac{r(\varphi)}{\deg(\varphi)} h(\varphi(P)) + C_2' > h(P)$ for all $P \in \mathbb{P}^n(\bar{K}) \setminus H$.

Valuative criterion

Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map whose indeterminacy locus I_φ is contained in a hyperplane H. Let $\pi : \tilde{X} \dashrightarrow \mathbb{P}^n$ be a resolution of the indeterminacy and $\tilde{\varphi} : \tilde{X} \longrightarrow \mathbb{P}^n$ a resolving morphism. Suppose that $\{H_V = \pi^\star H, E_1, \ldots, E_r\}$ is a basis for $\mathrm{Pic}(\tilde{X})$, where $\pi^\star H$ represents the proper transform of H and the E_i are coming from the monomial transformations needed to obtain π. We have:

$$\pi^* H = a_0 H_V + \sum a_i E_i \quad \text{and} \quad \tilde{\varphi}^* H = b_0 H_V + \sum b_i E_i.$$

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$$\pi^* H = a_0 H_V + \sum a_i E_i \quad \text{and} \quad \tilde{\varphi}^* H = b_0 H_V + \sum b_i E_i.$$

Remark

If $b_i \neq 0$ for all $a_i \neq 0$ we can take the D-ratio as

$$r(\varphi) = \deg(\varphi) \max_i \left(\frac{a_i}{b_i}\right).$$

Examples of D-ratios

Example

Let $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map given in projective coordinates by $\psi(x, y, z) = (x^3 + yz^2, xz^2 + y^2z, z^3)$ and consider a hyperplane H containing the point of indeterminacy $(0, 1, 0)$ of the map ψ. In order to resolve the indeterminacy of ψ, we need to do three blow-ups along points. We obtain the pull-backs of H as:

$$\pi_\psi^* H = H_V + E_1 + 2E_2 + 3E_3, \qquad \tilde{\psi}^* H = 3H_V + 2E_1 + 4E_2 + 6E_3.$$

Thus $r(\psi) = 3/2$.

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$$\pi_\psi^* H = H_V + E_1 + 2E_2, \qquad \tilde{\psi}^* H = 2H_V + E_1 + 2E_2.$$

Thus $r(\psi) = 2$.

D-ratios and pre-periodic points

Proposition

(Joey) Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map defined over the number field K. Suppose that the indeterminacy locus $I(\varphi)$ is contained in a hyperplane $H \subset \mathbb{P}^n$ and $u = \frac{r(\varphi)}{\deg(\varphi)} < 1$, then the set of pre-periodic points for the map φ, is a set of bounded height and therefore there are at most finitely many pre-periodic points for φ in K.

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Example

As observed before, $\psi(x, y, z) = (x^3 + yz^2, xz^2 + y^2z, z^3)$ has $r(\psi) = 3/2 < \deg(\varphi) = 3$ and therefore it has only finitely many pre-periodic points on any number field.

D-ratio: technical details

Definition

(Joey) The cone of $\mathbb{A}^n$-effective divisors is the cone

$$\mathbb{R}^{\geq 0}H_V + \mathbb{R}^{\geq 0}E_1 + \mathbb{R}^{\geq 0}E_2 \cdots + \mathbb{R}^{\geq 0}E_r,$$

for a fixed basis $\{H_V, E_1, \ldots, E_r\}$ of $\mathrm{Pic}(\tilde{X})$.

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for a fixed basis $\{H_V, E_1, \ldots, E_r\}$ of $\mathrm{Pic}(\tilde{X})$.

Remark

A divisor that is $\mathbb{A}^n$-effective is in particular effective and therefore the associated height h_D is ≥ 0, wherever is defined.

Questions

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For maps on $\mathbb{P}^2$, can we compute the coefficients a_i and b_i and therefore the D-ratio using intersection theory?

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Question

In the particular case of the family $\{X^{A,B}\}_{A,B}$. Can we use the results to study the orbits of points under the iteration of the three involutions σ_1, σ_2 and σ_3?