# 1 Matrices

As we have seen earlier, a matrix is a rectangular array of numbers, in general in $m$ rows and $n$ columns. The size of a matrix is given by the pair $(m, n)$, we say that $A$ is a $m \times n$-matrix when it has $m$ rows and $n$ columns.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

As we saw also before, matrices are a useful way to express systems of linear equations. For example:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

represents the system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

with augmented matrix:

$$\left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{array}\right)$$

**Remark 1.** It is important to understand that the entries (numbers) inside the matrix can be any real numbers, not need to be integers or even fractions. We could have for example:

$$A = \begin{pmatrix} \frac{2}{3} & -\pi & -2 \\ -.33 & \frac{1}{2} & 1 \end{pmatrix} \qquad \text{or} \qquad B = \begin{pmatrix} 3 & -2 & e \\ -\frac{8}{5} & -7 & e^2 \\ \sqrt{2} & \sqrt{3} & 0 \end{pmatrix}.$$

A matrix $A$ with $n$ rows and $n$ columns is called a square matrix of order $n$, and the entries $a_{11}, a_{22}, ..., a_{nn}$ are said to be on the main diagonal of $A$. A square matrix of order $n$ looks like

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$$

For example $A = \begin{pmatrix} 3 & -2 \\ 8 & -7 \end{pmatrix}$ is a square matrix of order 2 or a $2 \times 2$-matrix with diagonal elements $a_{1,1} = 3$ and $a_{2,2} = -7$. The identity matrix is the matrix of order $n$

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}_{n \times n} .$$

**Definition 2.** (Equality - addition -subtraction) Two matrices are defined to be equal if they have the same size and their corresponding entries are equal. If $A$ and $B$ are matrices of the same size, then the sum $A + B$ is the matrix obtained by adding the entries of $B$ to the corresponding entries of $A$, and the difference $A - B$ is the matrix obtained by subtracting the entries of $B$ from the corresponding entries of $A$. Matrices of different sizes cannot be added or subtracted.

**Definition 3.** (Product by scalar) If $A$ is a matrix and $c$ is a real number, the product $cA$ is the matrix obtained by multiplying each entry of the matrix $A$ by $c$. The matrix $cA$ is said to be a scalar multiple of $A$.

**Definition 4.** (Matrix multiplication) If $A$ is an $m \times r$-matrix and $B$ is an $r \times n$-matrix, then the product $AB$ is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row $i$ and column $j$ of $AB$, single out row $i$ from the matrix $A$ and column $j$ from the matrix $B$. Multiply the corresponding entries from the row and column together, and then add up the resulting products. In other words the entry $c_{ij}$ in $AB$ is the scalar product of the vectors formed by the $i$-th row of $A$ and the $j$-column of $B$.

**Example 5.** Let $A = \begin{pmatrix} 3 & -2 & 1 \\ 8 & -7 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 6 \\ 2 & 3 \\ 1 & 1 \end{pmatrix}$. The matrix $A$ is an $2 \times 3$-matrix and $B$ is a $3 \times 2$-matrix, hence, we are allowed to do $A \cdot B$ and the result should be a $2 \times 2$-matrix:

$$A \cdot B = \begin{pmatrix} 3 & -2 & 1 \\ 8 & -7 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 6 \\ 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

Actually we did the scalar product of the vectors

$(3,-2,1)\cdot(1,2,1) = 3(1)-2(2)+1(1) = 0 \qquad (3,-2,1)\cdot(6,3,1) = 3(6)-2(3)+1(1) = 13$

$(8,-7,-3)\cdot(1,2,1) = 8(1)-7(2)-3(1) = 9 \quad (8,-7,-3)\cdot(6,3,1) = 8(6)-7(3)-3(1) = 24$

And the answer is $\begin{pmatrix} 0 & 13 \\ -9 & 24 \end{pmatrix}$.

Matrix multiplication was already used in writing our system of linear equations. We express the system using a matrix $A_{m\times n}$ and a vector of unknowns $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ as a $n \times 1$-matrix standing-up and did the operation $A \cdot \mathbf{x}$.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}_{m\times n} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix}_{m\times 1}$$

The system can be expressed in short by the equation $A\mathbf{x} = \mathbf{b}$, where both $\mathbf{x}$ and $\mathbf{b}$ are vectors in $n$-dimensional space $\mathbb{R}^n$ and we choose to express them as standing-up $n \times 1$-vectors to be able to do the product and have an equality.

Properties of the matrix multiplication or product of matrices:

(1) $A \cdot (B + C) = A\dot{B} + A \cdot C$.

(2) $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.

(3) For any square matrix $A_{n\times n}$ of order $n$ we have $A \cdot I_n = I_n \cdot A = A$.

**Remark 6.** In general, given matrices $A$ and $B$, is not always true that $A \cdot B = B \cdot A$. For example:

$$\begin{pmatrix} 3 & -2 \\ 6 & -7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & -2 \\ 6 & -7 \end{pmatrix} = \begin{pmatrix} 9 & -9 \\ 9 & -9 \end{pmatrix}$$

The product of matrices **is not a commutative operation**. Another example, when we take $A = \begin{pmatrix} 3 & -2 & 1 \\ 8 & -7 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 6 \\ 2 & 3 \\ 1 & 1 \end{pmatrix}$. The matrix $A$ is an $2 \times 3$-matrix and

$B$ is a $3 \times 2$-matrix, hence, we are allowed to do $A \cdot B$ and the result should be a $2 \times 2$-matrix:

$$A \cdot B = \begin{pmatrix} 3 & -2 & 1 \\ 8 & -7 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 6 \\ 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 13 \\ -9 & 24 \end{pmatrix}.$$

On the other hand $B_{3\times 2} \cdot A_{2\times 3}$ should be a $3 \times 3$-matrix

$$B \cdot A = \begin{pmatrix} 1 & 6 \\ 2 & 3 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & -2 & 1 \\ 8 & -7 & -3 \end{pmatrix} = \begin{pmatrix} 51 & -44 & -17 \\ 30 & -25 & -7 \\ 11 & -9 & -2 \end{pmatrix}.$$

**Remark 7.** The product of two non-zero matrices could multiply to give the zero matrix. For example:

$$\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

**Definition 8.** If $A$ is any $m \times n$ matrix, then the transpose of $A$, denoted by $A^T$ , is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of $A^T$; that is, the first column of $A^T$ is the first row of $A$,the second column of $A^T$ is the second row of $A$, and so forth.

**Example 9.** For $A = \begin{pmatrix} 3 & -2 & 1 \\ 8 & -7 & -3 \end{pmatrix}$, the tranpose $A^T = \begin{pmatrix} 3 & 8 \\ -2 & -7 \\ 1 & -3 \end{pmatrix}$

**Definition 10.** If $A$ is a square matrix, then the trace of $A$, denoted by $\mathrm{tr}(A)$, is defined to be the sum of the entries on the main diagonal of $A$. The trace of $A$ is undefined if $A$ is not a square matrix.

**Example 11.** For $A = \begin{pmatrix} 3 & -2 & 1 \\ 8 & -7 & -3 \end{pmatrix}$ we cannot do the trace, because it is not a square matrix. On the other hand, for $B = \begin{pmatrix} 3 & -2 \\ 8 & -7 \end{pmatrix}$, we can compute the trace as $\mathrm{tr}(B) = 3 - 7 = -4$.

**Definition 12.** If $A_{n\times n}$ is a square matrix, and if a matrix $B_{n\times n}$ of the same size can be found such that $A \cdot B = B \cdot A = I_n$ , then $A$ is said to be invertible (or nonsingular) and $B$ is called an inverse of $A$. If no such matrix $B$ can be found, then $A$ is said to be singular. The inverse of $A$ is denoted $A^{-1}$.

**Example 13.** For $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$ we can choose $B = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$ and check

$$\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \cdot \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence $A$ is invertible and $A^{-1} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$.