MANIN-MUMFORD AND LATTÉS MAPS

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Abstract. The present paper is an introduction to the dynamical Manin-Mumford conjecture and an application of a theorem of Ghioca and Tucker to obtain counterexamples on certain family of Lattés maps.

1. Introduction

The language of algebraic varieties is developed to study problems involving the solution set of a system of polynomial equations. An algebraic variety is a topological space behaving locally like the zero set of a system of polynomials in the affine space $\mathbb{A}^n$. A projective variety is an algebraic variety that can be embedded in the projective space $\mathbb{P}^n$ for some n. The topology considered on an algebraic variety is the Zariski topology that has subvarieties of X as irreducible closed sets. The dimension of X is the maximum of all n such that we find a chain of subvarieties $Y_0 \subsetneq Y_1 \subsetneq ... \subsetneq Y_n = X$.
Many number theoretic questions are naturally expressed as diophantine problems. This means that the algebraic variety X is defined by equations in a number field K. In the case of a projective variety it means that X is defined by homogenous polynomials with coefficients in K.
We are interested in studying finite maps $X \to X$ from a projective algebraic variety to itself and defined over a number field K. More specifically we want to understand the subvarieties of X that have a finite forward orbit under the action of the map. For example for the map $x \to x^2$ on $\mathbb{P}^1_K$, the points 0, ∞ and the roots of unity in $\bar{K}$ have a finite forward orbit. In this case, the set made as the union of $\{0, \infty\}$ and the roots of unity, is dense for the Zariski topology in $\mathbb{P}^1$!
Suppose that $Y \subsetneq X$ is a subvariety of X the main question we would like to analyze is:

Question 1.1. *Can a subvariety $Y \subsetneq X$ be such that Y is not preperiodic but contains a dense set of preperiodic points ?*

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We will restrict ourselves to a special type of maps on X which will be called polarized maps and will be introduce in the next section. This question has a negative answer for varieties X of dimension one because a subvariety of dimension zero is just a point. On the other hand the question is open for $X = \mathbb{P}^2$. To give this question a positive answer for polarized maps in dimension two we will use projective varieties that are easily presented as product of two varieties. More specifically we will use the product of two varieties of dimension one called "elliptic curves": $X = E \times E$ and the product of two projective lines: $X = \mathbb{P}^1 \times \mathbb{P}^1$. The subvariety Y used in both cases is the diagonal subvariety and the maps to be defined in the next section are denoted by

$$([\omega], [\omega']) : E \times E \to E \times E \quad \textit{and} \quad (f_\omega, g'_\omega) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1.$$

The ω, ω' can be interpreted as elements in an order R in an imaginary quadratic extension $L = \mathbb{Q}(\sqrt{-D})$ of $\mathbb{Q}$ and we say that E admits "complex multiplication" by ω, ω'. The maps f_ω, g'_ω are in fact strongly related to $[\omega], [\omega']$. There is a natural projection $\pi : E \to \mathbb{P}^1$ and the maps $f_\omega, g_{\omega'}$ are called Lattés map associated to the maps $[\omega], [\omega'] : E \to E$. This means that we have commutative diagrams

$$\begin{array}{ccc} E & \xrightarrow{[\omega]} & E \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{P}^1 & \xrightarrow{f_\omega} & \mathbb{P}^1 \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{[\omega']} & E \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{P}^1 & \xrightarrow{g_{\omega'}} & \mathbb{P}^1 \end{array}$$

The main results presented are, a theorem of Ghioca and Tucker and an application of this theorem to certain family of Lattés maps:

Theorem 1.2. *(Ghioca-Tucker) Let E be an elliptic curve with complex multiplication defined over a number field K. Let R be an order in an imaginary quadratic extension of $\mathbb{Q}$ such that there is an isomorphism $\iota : R \longrightarrow \mathrm{End}(E)$ written as $\iota(\omega) = [\omega]$. Suppose that $\omega, \omega' \in R$ are such that $|\omega| = |\omega'| > 1$ and ω/ω' is not a root of unity. Then question 1.1 has a positive answer for the diagonal subvariety Δ in $E \times E$ under the action of $([\omega], [\omega'])$.*

Theorem 1.3. *Let $\varphi = (f, g) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, where $f = f_\omega$ and $g = g_{\bar\omega}$ are Lattés maps associated respectively to multiplication by $\omega, \bar\omega, (|\omega| > 1)$ on an elliptic curve E with complex multiplication by an order R in $L = \mathbb{Q}(\sqrt{-D})$. Then question 1.1 has a positive answer for the diagonal subvariety $\Delta' \subset \mathbb{P}^1 \times \mathbb{P}^1$ under the action of (f, g), unless we are in one of the following cases:*

(i) *$\omega = \pm a + ai$ with $0 \neq a \in \mathbb{Z}$ and $D = 1$,*

(ii) $\omega = a$ *with* $0 \neq a \in \mathbb{Z}$,
(iii) $\omega = b\sqrt{-D}$ *with* $0 \neq b \in \mathbb{Z}$,
(iv) $\omega = \frac{\pm 3b + b\sqrt{-3}}{2}$ *with* $0 \neq b \in \mathbb{Z}$ *and* $D = 3$,
(v) $\omega = \frac{\pm b + b\sqrt{-3}}{2}$ *with* $0 \neq b \in \mathbb{Z}$ *and* $D = 3$.

The main ingredients of the proofs are the group properties of the points on elliptic curves and the properties of roots of unity in quadratic fields.

2. The dynamical Manin-Mumford conjecture

2.1. **Arithmetic dynamics, elliptic curves and Lattés maps.** Let K be a number field. We want to consider projective algebraic varieties X defined over K.

Definition 2.1. *An arithmetic dynamical system $\varphi : X \to X$ over K is a map from the algebraic variety X to itself also defined over K.*

Is not easy to come up with non-trivial examples of self-maps, because in most cases they are consequence of deeper geometric properties. Consider for instance a plane curve with equation $E : zy^2 = G(x, z)$ where $G \in K[x, z]$ is a homogeneous and irreducible polynomial of degree 3. These algebraic objects are called elliptic curves and their arithmetic properties have been intensively studied by many authors ([Sil86], [Sil94]). The distinctive property of elliptic curves defined over K is that for any extension L/K, points on $E(L)$ have a group structure, that is, we can define a K-morphism $+ : E \times E \to E$, a point $P_0 = (0, 1, 0) \in E(K)$ and an involution $[-1](x, y, z) = (x, -y, z)$ making $(E(L), +, P_0)$ into a group. In particular we can define K-maps

$$[n] : E \to E,$$

for all $n > 1$.
To illustrate the complexity of these self-maps we take the elliptic curve $E : zy^2 = z^3 G(x/z, 1)$ with $G \in K[x]$ of degree 3, without repeated roots and $P = (x : y : 1) \in E$ with $y \neq 0$. Under this conditions, multiplication by [2] is given by formulaes:

$$[2](x, y) = \left(\frac{G'(x)^2 - 8xG(x)}{4G(x)}, \frac{G'(x)^3 - 12xG(x)G'(x) + 8G^2(x)}{8G(x)y} \right).$$

One thing to notice is that the x-component of the map depends only on x, so this map will descend to a map on $\mathbb{P}^1$. This fact is not specific to multiplication by 2. In general, maps $[n] : E \to E$ for $n \in \mathbb{Z}$ will give rise to maps $\varphi_n : \mathbb{P}^1 \to \mathbb{P}^1$ when we mod out by the involution $[-1] : E \to E$. In some cases we have maps $[\omega] : E \to E$, where

$\omega \notin \mathbb{Z}$ can be interprete as an element in the ring of integers of an imaginary quadratic field, and the theory of elliptic functions produces maps $\varphi_\omega : \mathbb{P}^1 \to \mathbb{P}^1$ associated to them. If $\pi : E \to \mathbb{P}^1$ represents the $(2:1)$-projection associated to the hyperelliptic involution on E we have the diagram

$$\begin{tikzcd} E \arrow[r, "{[\omega]}"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "\varphi_\omega"] & \mathbb{P}^1 \end{tikzcd}$$

Definition 2.2. *An elliptic curve E is said to have complex multiplication if $\mathbb{Z} \subsetneq \mathrm{End}(E) =$ Order in an imaginary quadratic number field.*

For example the elliptic curve E_1 with affine equation $y^2 = x^3 + x$ admits the automorphism $[i](x, y) = (-x, iy)$ and $\mathrm{End}(E) = \mathbb{Z} + \mathbb{Z}i$.

Definition 2.3. *The maps φ_ω are called Lattés maps associated to the maps $[\omega] : E \to E$ where in general ω can be seen as an element in an order in an imaginary quadratic field.*

2.2. **Preperiodic subvarieties and preperiodic points.** In this section we present the idea of subvarieties with a finite forward orbit.

Definition 2.4. *Let $\varphi : X \to X$ be an arithmetic dynamical system defined over K. A subvariety Y of X is said to be preperiodic if there exist natural numbers m and $k > 0$ such that $\varphi^{m+k}(Y) = \varphi^m(Y)$. The set $\mathrm{Prep}_\varphi(X)$ is defined to be the set of all preperiodic points of $X(\bar{K})$ under the action of φ.*

We explain the concepts of preperiodic points and preperiodic subvarieties in a special type of variety that generalizes elliptic curves: Abelian varieties. Abelian varieties A are the higher dimensional analogous of elliptic curves. They are connected complete algebraic groups and similarly to elliptic curves we can define self-maps $[n] : A \to A$ for $n > 1$. The preperiodic points for any map $[n] : A \to A$ are just the torsion points $A(\bar{K})_{tors}$. On the other hand we can find examples of preperiodic subvarieties as translated $P + B$, where P is torsion and B is Abelian subvariety.
The torsion points for Abelian varieties (and Abelian subvarieties) are always Zariski dense. We have the following theorem of Raynaud characterizing subvarieties $Y \subsetneq A$ with Zariski dense set of preperiodic points.

Theorem 2.5. *(Raynaud) Let A be an Abelian variety. If $Y \subsetneq A$ is a subvariety of A and contains a Zariski dense subset of preperiodic points then $Y = P+B$, where P is torsion and B is Abelian subvariety.*

Proof. This is the main result in [Ray83]. In the case that Y does not contain a translated of an Abelian subvariety, the finiteness of the torsion points in Y is established. In this case the idea of the proof is as follows: choose p prime such that the p-primary torsion contained in $X+a$ is finite and bounded independent of $a \in A$. The other part of the torsion, prime to p, can be proved to be finite by p-adic methods. □

As a consequence of this result, subvarieties $Y \subsetneq A$ containing a Zariski dense set of preperiodic points will be also preperiodic.

2.3. **Pic, ampleness and polarization.** More than the group properties of Abelian varieties, many results associated to dynamics are consequence of certain equation in the Picard group of the Variety. Any algebraic variety X, as ringed space, has (functorially) associated an Abelian group $\mathrm{Pic}(X)$. The group $\mathrm{Pic}(X)$ is the group of locally free sheafs of rank one or invertible sheafs on X, with tensor product as group operation. Elements of $\mathrm{Pic}(X)$ are also called line bundles on X and any algebraic dynamical system $\varphi : X \to X$ determines by functoriality a group homomorphism $\varphi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$.
On Abelian varieties there exist this special type of line bundles $\mathcal{L} \in \mathrm{Pic}(A)$, called symmetric line bundles, that satisfy $[-1]^*\mathcal{L} = \mathcal{L}$. As a consequence of Mumford formula (cor. A.7.2.5 in [HS00]), symmetric line bundles on A satisfy a relation of the form $[n]^*\mathcal{L} = \mathcal{L}^{[n^2]}$ for all n, that is, they represent eigenvectors associated to eigenvalues n^2 for all n and in particular for $n > 1$. A line bundle $\mathcal{L}$ satisfying this equation is what constitute a polarization for the system $(A, [n])$ for $n > 1$.
Now let's see the notion of ampleness, treated for example in A.3.1 and A.3.2 of [HS00]. A system of sections $s_i \in \Gamma(X, \mathcal{L}), i = 0..r$ determines a rational map $\phi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^r$. If for some natural $m > 0$, the map $\phi_{\mathcal{L}^m} : X \dashrightarrow \mathbb{P}^r$ becomes an isomorphic embedding of X onto its image we say that $\mathcal{L}$ is ample. It is not hard to find ample and symmetric line bundles on an Abelian variety A, for example on an elliptic curve E we have the line bundle $\mathcal{O}(P+[-1]P)$ associated to the divisor $P+[-1]P$. In this sense, the dynamics $(A, [n])$ on Abelian varieties always admit a polarization by an ample line bundle.

Definition 2.6. *An arithmetic dynamical system $\varphi : X \longrightarrow X$ is said to have a polarization if there exists an ample line bundle $\mathcal{L}$ on X such that $\varphi^*\mathcal{L} = \mathcal{L}^{\otimes\alpha}$ for some $\alpha > 1$. A polarized dynamical system will be denoted by $(X, \varphi, \mathcal{L}, \alpha)$.*

Another example of polarized dynamical system is given by maps $\varphi = (p_0 : ... : p_n) : \mathbb{P}^n \to \mathbb{P}^n$ for homogeneous polynomials $p_i \in K[x_0, ..., x_n]$ of the same degree $d > 1$ and the ample line bundle $\mathcal{L} = \mathcal{O}(1)$ associated to hyperplanes. In this case the polarization is given by the equation $\varphi^*\mathcal{O}(1) = \mathcal{O}(d) = \mathcal{O}(1)^d$. To describe the preperiodic points and preperiodic subvarieties in this situation is in general complicated. The simplest case $\varphi_m(x_0, ..., x_n) = (x_0^m, ..., x_n^m), m \neq \pm 1$ represents the multiplicative group analogous of Abelian varieties and we have that preperiodic points are points with coordinates x_i equal to zero or to a root of unity.
From now on we always work with polarized dynamical systems. Motivated by the way preperiodic points are related to preperiodic subvarieties on Abelian varieties, we present the Manin-Mumford conjecture for polarized dynamical systems.

Conjecture 2.7. *(Arithmetic Dynamical Manin-Mumford) Suppose that $(X, \varphi, \mathcal{L}, \alpha)$ is a polarized dynamical system defined over K. A subvariety Y of X is preperiodic if and only if $Y \cap \mathrm{Prep}_\varphi(X)(\bar{K})$ is Zariski dense in $Y(\bar{K})$.*

When we have a preperiodic subvariety Y, our system $(X, \varphi, \mathcal{L}, \alpha)$ can be restricted to have a dynamical system $(\varphi^m(Y), \varphi^k, \mathcal{L})$ and we will get a Zariski dense set of preperiodic (even periodic) points by theorem 5.1 in [Fak03]. So what the conjecture is actually saying that being preperiodic as subvariety is the only way to have a Zariski dense of preperiodic points.

2.4. **Counterexamples to Manin-Mumford.** We begin this section by studying the product of two polarized dynamical systems.

Proposition 2.8. *Suppose that we have two polarized dynamical systems $(X, \varphi, \mathcal{L}, \alpha)$ and $(X', \varphi', \mathcal{L}', \alpha)$, then we have a polarized dynamical system also on the product variety, that is $(X \times X', \varphi \times \varphi', p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}', \alpha)$.*

Proof. We compute the action of the product of maps on $p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}'$,

$$\begin{aligned}
(\varphi \times \varphi')^*(p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}') &= (\varphi \times \varphi')^*(p_1^*\mathcal{L}) \otimes (\varphi \times \varphi')^*(p_2^*\mathcal{L}') \\
&= (\varphi \times \varphi' \circ p_1)^*\mathcal{L} \otimes (\varphi \times \varphi' \circ p_2)^*\mathcal{L}' \\
&= (p_1 \circ \varphi)^*(\mathcal{L}) \otimes (p_2 \circ \varphi')^*(\mathcal{L}') \\
&= p_1^*\varphi^*\mathcal{L} \otimes p_2^*\varphi'^*\mathcal{L}' \\
&= p_1^*\mathcal{L}^\alpha \otimes p_2^*\mathcal{L}'^\alpha \\
&= (p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}')^\alpha
\end{aligned}$$

So we have a polarization on the new system. ☐

The following theorem in [GT09] provides a family of counterexamples to Conjecture 2.7 for the diagonal subvariety Δ on the product $E \times E$, where E is an elliptic curve with complex multiplication.

Theorem 2.9. *(Ghioca-Tucker) Let E be an elliptic curve with complex multiplication defined over a number field K. Let R be an order in an imaginary quadratic extension of $\mathbb{Q}$ such that there is an isomorphism $\iota : R \longrightarrow \mathrm{End}(E)$ written as $\iota(\omega) = [\omega]$. Suppose that $\omega, \omega' \in R$ are such that $|\omega| = |\omega'| > 1$ and ω/ω' is not a root of unity. Then Conjecture 2.7 fails for the diagonal subvariety Δ in $E \times E$ under the action of $([\omega], [\omega'])$.*

Proof. Suppose that $([\omega]^{n+k}, [\omega']^{n+k})(\Delta) = ([\omega]^n, [\omega']^n)(\Delta)$ for some $n, k > 0$. Consider a non-torsion point $P \in E$, then there exist $Q \in E$ also non-torsion such that $([\omega]^{n+k}, [\omega']^{n+k})(P, P) = ([\omega]^n, [\omega']^n)(Q, Q)$. But then $[\omega]^{n+k}(P) = [\omega]^n(Q)$ and $[\omega']^{n+k}(P) = [\omega']^n(Q)$ or equivalently $[\omega]^n([\omega]^k(P) - Q) = 0$ and $[\omega']^n([\omega']^k(P) - Q) = 0$. These last two equations are saying that there are torsion points P_1, P_2 such that $[\omega]^k(P) - Q = P_1$ and $[\omega']^k(P) - Q = P_2$ and therefore

$$[\omega]^k(P) - [\omega']^k(P) = ([\omega]^k - [\omega']^k)(P)$$

will also be a torsion point, and that cannot be for P non-torsion unless $[\omega]^k - [\omega']^k = 0$ or ω/ω' is a root of unity. On the other hand the system $(E, ([\omega], [\omega']))$ is polarized after proposition 2.8 whenever $\deg([\omega]) = |\omega| = |\omega'| = \deg([\omega']) > 1$ and Δ contains infinitely many preperiodic points, namely all points (P, P) with $P \in E$ torsion point. This is proving that Conjecture 2.7 fails for the diagonal subvariety Δ in $E \times E$ under the action of $([\omega], [\omega'])$. □

Remark 2.10. Previous theorem gives rise to many counterexamples. For instant, take any ω such that $|\omega| > 1$ and $\omega/\overline{\omega}$ is not root of unity, and let $\omega' = \overline{\omega}$. See later lemma 2.12.

Corollary 2.11. *Let $\varphi = (f, g) : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, where $f = f_\omega$ and $g = g_{\omega'}$ are Lattés maps associated respectively to multiplication by ω, ω' on an elliptic curve E with complex multiplication by R as in previous theorem. Suppose also that $\omega, \omega' \in R$ are such that $|\omega| = |\omega'| > 1$ and ω/ω' is not a root of unity. Then, conjecture 2.7 fails for the diagonal subvariety $\Delta' \subset \mathbb{P}^1 \times \mathbb{P}^1$ under the action of (f, g).*

Proof. Suppose that the elliptic curve E admits multiplication by ω, ω' and this gives rise to Lattés maps $f_\omega, g_{\omega'}$. We have a commutative diagram

$$\begin{array}{ccc} \Delta \subset E\times E & \xrightarrow{(\omega,\omega')} & E\times E \\ {\scriptstyle (\pi,\pi)}\big\downarrow & & {\scriptstyle (\pi,\pi)}\big\downarrow \\ \Delta' \subset \mathbb{P}^1\times\mathbb{P}^1 & \xrightarrow{(f_\omega, g_{\omega'})} & \mathbb{P}^1\times\mathbb{P}^1 \end{array}$$

and a line bundle $\mathcal{L}_0 = p_1^*\mathcal{O}(1)\otimes p_2^*\mathcal{O}(1)$ on $\mathbb{P}^1\times\mathbb{P}^1$. If we consider the polarized systems

$$S' = (\mathbb{P}^1\times\mathbb{P}^1, f_\omega\times g_{\omega'}, \mathcal{L}_0, |\omega|) \quad S = (E\times E, \omega\times\omega', (\pi,\pi)^*\mathcal{L}_0, |\omega|),$$

a subvariety $Y \subsetneq E\times E$ is a preperiodic for S if and only if the projection $Y' = (\pi,\pi)(Y)$ is preperiodic for S'. $\square$

Lemma 2.12. *Suppose that $\omega, \bar\omega \in R$, where R is an order in an imaginary quadratic extension $L = \mathbb{Q}(\sqrt{-D})$ with $D\in\mathbb{Z}$ squarefree. Then the fact that $\omega/\bar\omega$ is a root of unity is equivalent to one of the following cases:*

(i) $\omega = \pm a + ai$ *with* $0\neq a\in\mathbb{Z}$ *and* $D=1$

(ii) $\omega = a$ *with* $0\neq a\in\mathbb{Z}$,

(iii) $\omega = b\sqrt{-D}$ *with* $0\neq b\in\mathbb{Z}$,

(iv) $\omega = \dfrac{\pm 3b + b\sqrt{-3}}{2}$ *with* $0\neq b\in\mathbb{Z}$ *and* $D=3$,

(v) $\omega = \dfrac{\pm b + b\sqrt{-3}}{2}$ *with* $0\neq b\in\mathbb{Z}$ *and* $D=3$.

Proof. Elements in R are integers in L and therefore written as

$$\omega = a + b\sqrt{-D} \quad or \quad \omega = a/2 + b\sqrt{-D}/2 \quad 2\mid a-b$$

depending on $-D\equiv 2,3 \mod (4)$ or $-D\equiv 1 \mod (4)$. In any case if ω/ω' is a root of unity then

$$\Re(\omega/\bar\omega) = \frac{a^2-b^2D}{a^2+b^2D} = \cos(2\pi/n)$$

for some natural $n>0$.

The fact that $\exp(2\pi i/n)$ generates the n-th cyclotomic field means that, for $n>2$, the degree of the field extensions

$$[\mathbb{Q}(\exp(2\pi i/n):\mathbb{Q}] = \phi(n) \qquad [\mathbb{Q}(\cos(2\pi/n)):\mathbb{Q}] = \phi(n)/2.$$

As a consequence, $\cos(2\pi/n)$ is only rational if $n=\{1,2,3,4,6\}$, in which cases $\cos(2\pi/n)$ takes the values $\{1,-1,-1/2,0,1/2\}$ respectively. So, we have the following possibilities for ω, D:

i) $\dfrac{a^2-b^2D}{a^2+b^2D}=0$ which forces a^2-b^2D-0 and therefore we get

$$D=1, \quad a=\mp b, \quad \omega=\pm a+ai.$$

ii) $\dfrac{a^2-b^2D}{a^2+b^2D}=1$ which forces $2b^2D=0$ and therefore we get

$$b=0, \quad \omega=a.$$

iii) $\dfrac{a^2-b^2D}{a^2+b^2D}=-1$ which forces $2a^2=0$ and therefore we get

$$a=0, \quad \omega=b\sqrt{-D}.$$

iv) $\dfrac{a^2-b^2D}{a^2+b^2D}=\frac{1}{2}$ which forces $a^2=3Db^2$ and therefore we get

$$D=3, \quad a=\pm 3b, \quad \omega=\frac{\pm 3b+b\sqrt{-3}}{2}.$$

v) $\dfrac{a^2-b^2D}{a^2+b^2D}=\frac{-1}{2}$ which forces $3a^2=Db^2$ and therefore we get

$$D=3, \quad a=\pm b \quad \omega=\frac{\pm b+b\sqrt{-3}}{2}.$$

Which proves one direction. In the other direction the cases will give $\Re(\omega/\bar{\omega})=1,-1,0,1/2,-1/2$ which mean that $\omega/\bar{\omega}$ is in fact a root of unity. □

Theorem 2.13. *Let $\varphi=(f,g):\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1\times\mathbb{P}^1$, where $f=f_\omega$ and $g=g_{\bar{\omega}}$ are Lattés maps associated respectively to multiplication by $\omega,\bar{\omega},(|\omega|>1)$ on an elliptic curve E with complex multiplication by an order R in $L=\mathbb{Q}(\sqrt{-D})$. Then question 1.1 has a positive answer for the diagonal subvariety $\Delta'\subset\mathbb{P}^1\times\mathbb{P}^1$ under the action of (f,g), unless we are in one of the following cases:*

(i) $\omega=\pm a+ai$ *with* $0\neq a\in\mathbb{Z}$ *and* $D=1$,

(ii) $\omega=a$ *with* $0\neq a\in\mathbb{Z}$,

(iii) $\omega=b\sqrt{-D}$ *with* $0\neq b\in\mathbb{Z}$,

(iv) $\omega=\dfrac{\pm 3b+b\sqrt{-3}}{2}$ *with* $0\neq b\in\mathbb{Z}$ *and* $D=3$,

(v) $\omega=\dfrac{\pm b+b\sqrt{-3}}{2}$ *with* $0\neq b\in\mathbb{Z}$ *and* $D=3$.

Proof. This is an application of lemma 2.12 and corollary 2.11. □

2.5. **Further inquiries.** The following question is related to the question stated in the introduction of the paper. It only pays attention to the dimension of the varieties X and Y.

Question 2.14. *Consider the pair of natural numbers $(n, m), n > m$. Can we have a polarized dynamics $(X, \varphi, \mathcal{L}, .)$, where X has dimension n and $Y \subsetneq X$ is a subvariety of dimension m which is not a preperiodic subvariety of X but contains a Zariski dense set of preperiodic points?*

The discussion on this paper provides a possitive answer for the pair $(2, 1)$. We can use analogous techniques for pairs $(n, 1)$. Consider the diagonal subvariety Δ inside the n-th fold $E \times ... \times E$, where E is an elliptic curve with complex multiplication by an order R. The subvariety Δ will not be preperiodic for the action of the polarized map $[\omega_1] \times ... \times [\omega_n]$ where $\omega_i \in R$ are of the same norm with at least one quotient ω_i/ω_j that is not a root of unity. On the other hand Δ contains infinitely many points of the form $(x, x, ..., x)$ where x is torsion.

Question 2.15. *Can we find an example for $(n, 2)$ for some $n \geq 3$?*

Question 2.16. *What is the answer for the projective plane $X = \mathbb{P}^2$?*

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