LOCAL CANONICAL HEIGHTS ASSOCIATED TO SEVERAL MORPHISMS

JORGE PINEIRO

Abstract. We use canonical metrics on Q-bundles to have an explicit formula for the local canonical height associated to several maps.

1. introduction

The theory of height functions lies at the center of the solution of many important problems in Diophantine Geometry. With the introduction of a hermitian structure on line bundles on arithmetic surfaces [1],[11], a geometric content was given to the computation of heights as intersection numbers. Later on, a special type of heights were introduced by Call and Silverman [2] to study dynamics: canonical heights associated to self-maps. These canonical heights were obtained as limits of Weil heights, making use of the Weil height machine [7] and required the map to be "polarized" (c.f. section 2.3) by a line bundle. The theory of adelic metrized line bundles developed by Shou-Wu in [12], allow us to interpret canonical heights of morphisms polarized by $\mathbb{Q}$-bundles as (limits of) intersection numbers. In [8], Kawaguchi generalizes the work of Shou-Wu on adelic metrics and adelic intersection to dynamical systems of several maps, dealing with, heights of points, heights of subvarieties, as well as the decomposition, in the normal case, of canonical heights into sum of local canonical heights in the spirit of [9].

Let K be a number field with set of places M_K. We consider a projective variety X defined over K and a system of k maps $\varphi_i : X \longrightarrow X$ also defined over K. We study the notion of canonical metrics $\{\|.\|_v\}_{v\in M_K}$ for a system of several maps $\{\varphi_1, \ldots, \varphi_k\}$ on a line bundle $\mathcal{L}$, satisfying a polarization property $\bigotimes_{i=1}^k \varphi_i^*\mathcal{L} \cong \mathcal{L}^d$, for some rational number $d > k$. With the use of canonical metrics we can obtain a closed expression

$$\hat{\lambda}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}(P, v) = -\log \|s(P)\|_{v,\{\varphi_1,\ldots,\varphi_k\}},$$

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for the local canonical height $\hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}$, whose existence we know from section 4 of [8]. For a clear survey in Diophantine Geometry containing an approach to canonical metrics using analytic spaces we refer to [5].

2. Metrics on line bundles

It was an invention of Arakelov [1] to add places at infinity to the non-archimedean places of a number field and consider hermitian metrics on line bundles on the associated complex varieties.

2.1. **Metrics on line bundles.** Let X be a projective algebraic variety defined over a field K, together with an absolute value $|.|$ defined over all residue fields $K(x)$ for $x \in X$. A metric $\|.\|$ on a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ is a collection of metrics $\|.\|_x$ on the fibres $\mathcal{L}_x$, varying continuously on x, in such a way that for any open set $U \subset X$ and $s \in \Gamma(U, \mathcal{L})$, the continuous function $\|s\|_U : U \to \mathbb{R}_+$ satisfies $\|fs\| = |f|\|s\|$ for $f \in \mathcal{O}_X(U)$.

Example 2.1. For X a projective variety defined over over a number field K and $\sigma : K \hookrightarrow \mathbb{C}$ a place at infinity, we can take the norm $|x|_\sigma = |\sigma(x)|$ in K and put hermitian continuous metrics on line bundles $\mathcal{L}$ over $\mathbb{C}_\sigma$. The datum $(\mathcal{L}, \|.\|_\sigma)$, where σ is moving in places at infinity is called a hermitian line bundle.

Let K be a complete ultrametric field which is the field of fractions of a complete discrete valuation ring K_0. Let X be a projective variety over K and $\mathcal{L}$ a line bundle on X. Given a K_0-scheme $\mathcal{X}$ and a line bundle $\tilde{\mathcal{L}}$ on $\mathcal{X}$, we say that $(\mathcal{X}, \tilde{\mathcal{L}})$ is a model of $(X, \mathcal{L}^e)$ for some power $e > 0$, if the generic fibre $\mathcal{X} \times_{\mathrm{Spec}(K_0)} \mathrm{Spec}(K) \cong X$ and $\tilde{\mathcal{L}}|X \cong \tilde{\mathcal{L}} \otimes K \cong \mathcal{L}^e$. A model $(\mathcal{X}, \tilde{\mathcal{L}})$ defines a metric on $\mathcal{L}$ in the following way: Let $\varepsilon_{\mathcal{U}} : \mathcal{O}_{\mathcal{X}}(\mathcal{U}) \cong \tilde{\mathcal{L}}|\mathcal{U}$ be a local frame for $\tilde{L}|\mathcal{U}$, then every non-zero section $s_{\mathcal{U}} : \mathcal{U} \to \tilde{\mathcal{L}}|\mathcal{U}$ will be written as $s_{\mathcal{U}} = \varepsilon_{\mathcal{U}} f_{\mathcal{U}}$ for some $f \in \mathcal{O}_{\mathcal{X}}(\mathcal{U})$. When we restrict ourself to $U = \mathcal{U} \times \mathrm{Spec}\, K$, we get $\varepsilon_U : \mathcal{O}_X(U) \cong \mathcal{L}^e|U$ and for any section s_U of $\mathcal{L}|U$ we could write $s_U^e = \varepsilon_U f_U$. We declare $\|s_U\|_{\tilde{\mathcal{L}}} = |f_U|^{1/e}$ or equivalently $\|\varepsilon_U\|_{\tilde{\mathcal{L}}} = 1$ on U. A metric so defined is called algebraic.

Example 2.2. Let K be a number field and v a non-archimedean place. Let X be a projective variety defined over the ultrametric complete field K_v and $\mathcal{L} \in \mathrm{Pic}(X)$ a line bundle on X. A model $(\mathcal{X}, \tilde{\mathcal{L}})$ of $(X, \mathcal{L}^e)$ over the ring of integers of K_v will define a metric $\|.\|_{\tilde{\mathcal{L}},v}$ on $\mathcal{L}$. If we take $\tilde{\mathcal{L}} = (\tilde{\mathcal{L}}, \|.\|_\sigma)$ to be a hermitian line bundle, we will have metrics at all places $v \in M_K$.

Adelic metrics for line bundles $\mathcal{L} \in \mathrm{Pic}(X)$ were introduced by S. Zhang in [12] to add a little more flexibility in choosing our metrics at finitely many places:

Definition 2.3. *Let X be defined over a number field K. An adelic metric $\|.\|$ for a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ is a collection of metrics $\|.\| = (\|.\|_v)_{v \in M_K}$, where v runs over the places of K; in such a way that for all but finitely many v, the metric is induced by the same $\mathcal{O}_K$-model. We say that a sequence $\|.\|_n$ of adelic metrics on $\mathcal{L}$ converges to an adelic metric $\|.\|_\infty$ if for all but finitely many places $\log(\|.\|_{v,n}/\|.\|_{v,\infty}) = 0$ and $\log(\|.\|_{v,n}/\|.\|_{v,\infty}) \to 0$ uniformly on $X(K)$ for all $v \in M_K$.*

2.2. **Canonical metric.** Canonical metrics are special types of metrics associated to dynamical systems. A dynamical system of one map on a projective variety X is a self-map $\varphi : X \longrightarrow X$. A dynamical system is said to be polarized by an ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ if $\varphi^*\mathcal{L} \cong \mathcal{L}^d$ for some real number $d > 1$. We denote a polarized dynamical system by $(X, \varphi, \mathcal{L}, d)$. In the case that $d > 1$ is a natural number, in section 2 of [12] a sequence of models $(\mathcal{X}_n, \tilde{\mathcal{L}}_n)$ for $(X, \mathcal{L}^{e_n})$ is defined in such a way that the induced adelic metrics $\|.\|_{n,v}$ converge to an adelic metric $\|.\|_{\varphi,v}$. The metric $\|.\|_{\varphi,v}$ is called the canonical metric on $\mathcal{L}$. It is the only metric that makes the isomorphism $\varphi^*\mathcal{L} \cong \mathcal{L}^d$ into an isometry.

2.3. **Canonical height.** Let X a projective algebraic variety defined over a number field K. A polarized dynamical system $(X, \varphi, \mathcal{L}, d)$ over K, has associated a unique function $\hat{h}_\varphi : X(\bar{K}) \longrightarrow \mathbb{R}^+$, which satisfies the properties:

(1) $\hat{h}_\varphi(\varphi(P)) = d\hat{h}_\varphi(P)$ for all $P \in X(\bar{K})$,
(2) $\hat{h}_\varphi(P) \geq 0$ for all $P \in X(\bar{K})$,
(3) $\hat{h}_\varphi(P) = 0$ if and only if the forward orbit of P by φ is finite.
(4) There exist $C > 0$, such that $|h_{\mathcal{L}}(P) - \hat{h}_\varphi(P)| < C$ for all $P \in X(\bar{K})$.

The function $\hat{h}_\varphi$ is called the canonical height function associated to the system $(X, \varphi, \mathcal{L}, d)$. A proof of the existence and uniqueness was first established in [2]. A presentation of canonical heights in terms of adelic metrics and intersection numbers can be found in [12].

3. Canonical height for several morphisms

As a generalization of canonical heights associated to polarized dynamical systems $(X, \varphi, \mathcal{L}, d)$ of one map [2], [12], [7], the work of

Kawaguchi in [8] showed the existence of canonical height functions associated to polarized dynamical systems of several maps.

Definition 3.1. *Let K be a number field. Let X be a projective variety defined over K and $\varphi_i : X \longrightarrow X (i = 1, \dots, k)$ be morphisms over K. We say that the system $(X, \{\varphi_1, \dots, \varphi_k\}, \mathcal{L}, d)$ is a polarized dynamical system of k morphisms over K if there exist an ample $\mathbb{R}$-line bundle $\mathcal{L}$ on $\mathrm{Pic}(X) \otimes \mathbb{R}$ such that $\bigotimes_{i=1}^{k} \varphi_i^* \mathcal{L} \cong \mathcal{L}^d$ for some $d > k$.*

Let's denote by $C(P) = \{P\} \cup \{\varphi_{i_1} \circ \varphi_{i_2} \circ \cdots \circ \varphi_{i_l}(P) \mid l > 0, 1 \leq i_1 \leq i_2 \leq \cdots \leq i_l \leq k\}$ the forward orbit of a point $P \in K$ under the maps $\varphi_1, \dots, \varphi_k$. The following theorem is a combination of theorems 1.2.1 and 1.3.1 in [8]. It establishes the existence of a canonical height function for varieties with a family of morphisms.

Theorem 3.2. *Let $(X, \{\varphi_1, ..., \varphi_k\}, \mathcal{L}, d)$ be a polarized dynamical system of k morphisms over a number field K. Then there exist a real valued function*

$$\hat{h}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}} : X(\bar{K}) \longrightarrow \mathbb{R}$$

with the following properties

(1) $\sum_{i=1}^{k} \hat{h}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(\varphi_i(P)) = d\hat{h}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(P)$ *for all* $P \in X(\bar{K})$,
(2) $\hat{h}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(P) \geq 0$ *for all* $P \in X(\bar{K})$,
(3) $\hat{h}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(P) = 0$ *if and only if* $C(P)$ *is finite.*

The idea of the proof [8] is similar to the case of one morphism. The canonical height is obtained as limit of a sequence, whose convergence is proven using a contracting map type argument due to Tate. The properties of the canonical height are consequence of the Weil height machine developed for example in [7].

Example 3.3. The main illustration of height associated to several maps is the dynamics of two automorphisms on the family of K3 surfaces studied by Silverman in [10].

Consider the family of $K3$ surfaces $S_{a,b} \subset \mathbb{P}^2 \times \mathbb{P}^2$ determined by the two equations with coefficients in a number field K,

$$\sum_{i,j=1}^{3} a_{i,j} x_i y_j = 0 \qquad \sum_{i,j,k,l=1}^{3} b_{i,j,k,l} x_i x_k y_j y_l = 0.$$

The projections $p_1, p_2 : S_{a,b} \longrightarrow \mathbb{P}^2$ represents double coverings of $\mathbb{P}^2_K$ and determine morphisms $\sigma_1, \sigma_2 : S_{a,b} \longrightarrow S_{a,b}$ in a generic members of the family $S_{a,b}$. If we take $H \in \mathrm{Pic}(\mathbb{P}^2)$ a hyperplane section and $D_j = p_j^*(H) \in \mathrm{Pic}(S_{a,b})$ for $i = 1, 2$; the ample divisor $D = D_1 + D_2$

will give the polarization $\sigma_1^*D + \sigma_2^*D \sim 4D$ associated to σ_1, σ_2 and, by theorem 3.2, we will get a canonical height $\hat{h}_{D,\sigma_1,\sigma_2} : S(\bar{K}) \longrightarrow \mathbb{R}^+$.

3.1. **Local canonical heights associated to several morphisms.** One valuable tool to compute canonical heights will be, to express the height as the sum of local components. This motivates the definition of local canonical heights. In this section we are following [8] and ultimately Chapter 10 of [9]. Suppose that X is a normal projective variety defined over a number field K and $U \subset X$ is a non-empty open set. Let us denote by M_K the set of absolute values on K and by $M = M_{\bar{K}}$, the set of absolute values on $\bar{K}$ extending those of K. A function $\lambda : U(\bar{K}) \times M \longrightarrow \mathbb{R}$ is called M_K-continuous if, for every $v \in M_K$, $\lambda_v : U(\bar{K}) \longrightarrow \mathbb{R}$, $P \mapsto \lambda(P, v)$ is continuous in the v-adic topology. A function $\gamma : M_K \longrightarrow \mathbb{R}$ is called M_K-constant if $\gamma(v) = 0$ for all but finitely many $v \in M_K$. For $v' \in M$ extending $v \in M_K$, we set $\gamma(v') = \gamma(v)$. In this way γ is extended to a function on M, which is also said to be M_K-constant. A function $\alpha : U(\bar{K}) \times M \longrightarrow \mathbb{R}$ is called M_K-bounded if there is a M_K-constant function γ such that $|\alpha(P, v)| \leq \gamma(v)$ for all $(P, v) \in U(\bar{K}) \times M$. With these definitions in our hands we can introduce local height functions:

Definition 3.4. *Let $D \in \mathrm{Div}(X) \otimes \mathbb{R}$. A function $\lambda_D : X \backslash \mathrm{Supp}(D)(\bar{K}) \times M \longrightarrow \mathbb{R}$ is a said to be a local height associated to D if there is an affine covering $\{U_i\}$ of X, a Cartier divisor $\{(U_i, f_i)\}$ representing D such that the function $\alpha(P, v) = \lambda_D(P, v) - v \circ f_i(P)$ is M_K-bounded and M_K-continuous for $P \in (U_i \setminus \mathrm{Supp}(D))(\bar{K})$ and $v \in M$.*

Example 3.5. Let $\mathcal{L} \in \mathrm{Pic}(X)$ be a line bundle on X and $\|.\| = \{\|.\|_v, v \in M_K\}$ an adelic metric on $\mathcal{L}$. Take a section $s \in \Gamma(X, \mathcal{L})$ and consider the function

$$\alpha(P, v) = -\log \|s(P)\|_v - v \circ f_i(P) = -\log \|\varepsilon_{U_i}(P)\|_v$$

for a local frame $\varepsilon_{U_i} : \mathcal{O}_X(U_i) \cong \mathcal{L}|U_i$. By the definition of the adelic metric induced by a model, we will have $\|\varepsilon_{U_i}(P)\| = 1$ for all $P \in U_i$ and almost all $v \in M_K$, therefore α is a M_K-bounded and M_K-continuous and $\lambda_D(P, v) = -\log \|s(P)\|_v$ is a local height associated to $D = \mathrm{div}(s)$.

The following theorem is a combination of theorem 4.2.1 and 4.3.1 in [8]. It states the existence and properties of the local canonical height functions associated to systems of morphisms on varieties over number fields. The proof however will be constructive using the theory of canonical metrics on line bundles.

Theorem 3.6. *Let X be a normal projective variety and consider the polarized system $(X, \{\varphi_1, ..., \varphi_k\}, \mathcal{L}, d)$ of k morphisms over a number* *field K. Suppose that E is a divisor on X associated to the line bundle $\mathcal{L}$ such that $\varphi_i(X)$ is not contained in* $\mathrm{Supp}(E)$ *and $\varphi_1^*E+\cdots+\varphi_k^*E = dE + \mathrm{div}(f)$ for some rational function $f \in \bar{K}(X)^* \otimes \mathbb{R}$. Then there exist a unique function*

$$\hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}} : \{X \setminus \mathrm{Supp}(E)(\bar{K})\} \times M \longrightarrow \mathbb{R}$$

with the properties:

(1) *$\hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}$ is a Weil local height associated to E.*
(2) *for any $P \in X \setminus (\mathrm{Supp}(E) \cup \mathrm{Supp}(\varphi_1^*E) \cup \cdots \cup \mathrm{Supp}(\varphi_k^*E))$ and for all $v \in M$,*

$$\sum_{i=1}^{k} \hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(\varphi_i(P), v) = d\hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(P, v) + v(f(P)).$$

(3) *For L an extension of K, w and extension of v and $P \in X(L)$, we have the decomposition:*

$$\hat{h}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(P) = \frac{1}{[L:K]} \sum_{w \in M_L} [L_w : K_v] \hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(P, w).$$

We would like to prove the theorem by finding explicitly local canonical heights functions in terms of canonical metrics. For that reason we start with stating and proving a local lemma for real line bundles that is a generalization of theorem 2.2 in [12] in the same way Kawaguchi does it for places at infinity in Theorem 3.1.1 of [8].

Lemma 3.7. *Let $(X, \{\varphi_1, ..., \varphi_k\}, \mathcal{L}, d)$ be a polarized dynamical system over an algebraically closed valuation field K. Suppose that each of the $\varphi_i : X \longrightarrow X$ is surjective and we have fixed an isomorphism $\Phi : \mathcal{L}^d \simeq \bigotimes_{i=1}^k \varphi_i^* \mathcal{L}$. Suppose that we have a bounded and continuous metric $\|.\|_0$ in $\mathcal{L}$. Then the metrics $\|.\|_n$ on $\mathcal{L}$ defined inductively by*

$$\|.\|_n^d = \Phi^*(\varphi_1^*\|.\|_{n-1} \dots \varphi_k^*\|.\|_{n-1}) \tag{3.7.1}$$

converges uniformly to a metric on $\mathcal{L}$, which we denote by $\|.\|_{\{\varphi_1,\dots,\varphi_k\}}$. The metric $\|.\|_{\{\varphi_1,\dots,\varphi_k\}}$ is the unique metric with the property

$$\|.\|_{\{\varphi_1,\dots,\varphi_k\}}^d = \Phi^*(\varphi_1^*\|.\|_{\{\varphi_1,\dots,\varphi_k\}} \dots \varphi_k^*\|.\|_{\{\varphi_1,\dots,\varphi_k\}})$$

and will be called the canonical metric associated to $\mathcal{L}$ and the maps $\varphi_1, \dots, \varphi_k$.

Proof. The idea of the proof is taken from theorem (2.2) in [12]. Denote by h the bounded and continuous function $\log \frac{\|.\|_1}{\|.\|_0}$ on $X(K)$. Then

$$\begin{aligned}\log \|.\|_n &= (\sum_{i=1}^{k} \frac{1}{d}\Phi^*\varphi_i^*)^{n-1} \log \|.\|_1 \\ &= (\sum_{i=1}^{k} \frac{1}{d}\Phi^*\varphi_i^*)^{n-1}(h + \log \|.\|_0) \\ &= (\sum_{i=1}^{k} \frac{1}{d}\Phi^*\varphi_i^*)^{n-1}h + \sum_{i=1}^{k}(\frac{1}{d}\Phi^*\varphi_i^*)^{n-1} \log \|.\|_0 \\ &= (\sum_{i=1}^{k} \frac{1}{d}\Phi^*\varphi_i^*)^{n-1}h + \log \|.\|_{n-1}\end{aligned}$$

Using induction we get $\log \|.\|_n = \sum_{j=0}^{n-1}(\sum_{i=1}^{k} \frac{1}{d}\Phi^*\varphi_i^*)^j h + \log \|.\|_0$ and because $\|(\sum_{i=1}^{k} \frac{1}{d}\Phi^*\varphi_i^*)^j h\|_{sup} \leq (\frac{k}{d})^j \|h\|_{sup}$, we will get that the series $\sum_{j=0}^{n-1}(\sum_{i=1}^{k} \frac{1}{d}\Phi^*\varphi_i^*)^j h$ converges absolutely and uniformly to a bounded and continuous function $h_{\varphi_1 \ldots \varphi_k}$ and the metric $\|.\|_n$ converges uniformly to the continuous and bounded metric $\|.\|_{\{\varphi_1,\ldots,\varphi_k\}} = \|.\|_0 \exp(h_{\varphi_1 \ldots \varphi_k})$. The invariant property follows when we let the operator $(\sum_{i=1}^{k} \frac{1}{d}\Phi^*\varphi_i^*)$ act on $\log \|.\|_{\{\varphi_1,\ldots,\varphi_k\}}$. On the other hand if there are two different metrics $\|.\|_{\{\varphi_1,\ldots,\varphi_k\}}$ and $\|.\|'_{\{\varphi_1,\ldots,\varphi_k\}}$ satisfying the functional equation 3.7.1, the continuous and bounded function $g = \log(\|.\|_{\{\varphi_1,\ldots,\varphi_k\}}/\|.\|'_{\{\varphi_1,\ldots,\varphi_k\}})$ will satisfy the equation $(\frac{1}{d}\Phi^*\varphi_i^*)g = g$ and therefore it will be identically zero. □

Following section 2.2 of [8] we go back now to the global situation, where we have a polarized dynamical system $(X, \{\varphi_1, \ldots, \varphi_k\}, \mathcal{L}, d)$ of k surjective morphisms over a number field K and we have fixed an isomorphism $\Phi : \mathcal{L}^d \simeq \bigotimes_{i=1}^{k} \varphi_i^*\mathcal{L}$. Since X is a projective variety and $\mathcal{L}$ is ample we can build a model $(\mathcal{X}, \tilde{\mathcal{L}})$ of some power $(X, \mathcal{L}^e)$ over $\mathrm{Spec}(\mathcal{O}_K)$, with $\tilde{\mathcal{L}}$ hermitian line bundle. This induces an adelic metric $\|.\|_0$ on $\mathcal{L}$. There is an open $U \subset \mathrm{Spec}(\mathcal{O}_K)$ such that the maps extend to $\varphi_i : \tilde{X}_U \longrightarrow \tilde{X}_U$ and $\Phi_U : \mathcal{L}_U^d \simeq \bigotimes_{i=1}^{k} \varphi_i^*\mathcal{L}_U$ in $\mathrm{Pic}(\tilde{X}_U)$. It follows that for $v \in U$

$$\|.\|_{0,v}^d = \Phi^*(\varphi_1^*\|.\|_{0,v} \ldots \varphi_k^*\|.\|_{0,v})$$

Let's define the normalization $\tilde{\varphi}_i : \tilde{X}^i \longrightarrow \tilde{X}$ of the composition of morphisms $\varphi_i : \tilde{X}_U \longrightarrow \tilde{X}_U \hookrightarrow \tilde{X}$. Let $\tilde{X}_1$ be the Zariski closure of

$$\tilde{X}_U \xrightarrow{\Delta} \tilde{X}_U \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \tilde{X}_U \hookrightarrow \tilde{X}^1 \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \tilde{X}^k$$

where Δ is the diagonal map. Let $p_i : \tilde{X}^1 \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \tilde{X}^k \longrightarrow \tilde{X}^i$ denote the projection onto the i-th factor. We can build a new model $(\tilde{X}_1, \tilde{\mathcal{L}}_1)$ of $(X, \mathcal{L}^e)$ when we put

$$\tilde{\mathcal{L}}_1 = [((\tilde{\varphi}_1 \circ p_1)^* \tilde{\mathcal{L}} \otimes \cdots \otimes (\tilde{\varphi}_k \circ p_k)^* \tilde{\mathcal{L}}))]^{1/d}.$$

This new model induces an adelic metric $\|.\|_1$ on $\mathcal{L}$ with the property $\|.\|_{1,v}^d = \Phi^*(\varphi_1^*\|.\|_{0,v} \dots \varphi_k^*\|.\|_{0,v})$ for all places v of K. In this way starting from a model $(\tilde{X}_n, \tilde{\mathcal{L}}_n)$ we can build a new model $(\tilde{X}_{n+1}, \tilde{\mathcal{L}}_{n+1})$ and for each place v of K we will have a sequence of metrics $\|.\|_{n,v}$ satisfying the recurrence 3.7.1 and therefore a canonical metric $\|.\|_{v,\{\varphi_1,\dots,\varphi_k\}}$ associated $\mathcal{L}$ and the maps $\varphi_1, \dots, \varphi_k$ over each place v. We are ready now to prove theorem 3.6.

Proof. For a section $s \in \Gamma(X, \mathcal{L})$, $v \in M_K$ and $P \in X \setminus \mathrm{Supp}(s)$, let's define the function

$$\hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(P, v) = -\log \|s(P)\|_{v,\{\varphi_1,\dots,\varphi_k\}}.$$

The function so defined is a local height associated to E because the canonical metric is an adelic metric on $\mathcal{L}$. The functional equation satisfied by the canonical metric forces

$$\begin{aligned}
\sum_{i=1}^k \hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(\varphi_i(P), v) &= \sum_{i=1}^k -\log \|s(\varphi_i(P))\|_{v,\{\varphi_1,\dots,\varphi_k\}} \\
&= -\log \prod_{i=1}^k \varphi_i^* \|s(P)\|_{v,\{\varphi_1,\dots,\varphi_k\}} \\
&= -\log \Phi^{-1*} \|s(P)\|_{v,\{\varphi_1,\dots,\varphi_k\}}^d \\
&= -\log |f(P)|_v \|s(P)\|_{v,\{\varphi_1,\dots,\varphi_k\}}^d \\
&= -d \log \|s(P)\|_{v,\{\varphi_1,\dots,\varphi_k\}} + v(f(P)) \\
&= d\hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(P) + v(f(P))
\end{aligned}$$

The uniqueness of the canonical height follows from the properties (1) and (2) as discussed in the proof of theorem 4.2.1 of [8]. Suppose that $\hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}$ and $\hat{\lambda}'_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}$ were two canonical height functions with properties (1) and (2), then their difference $\delta(P, v)$ can be extended to a M_K-bounded and M_K-continuous function on $X(\bar{K})$ satisfying

$$|\delta(P, v)| \leq \frac{1}{d^l} |\sum_{\varphi \in \mathcal{F}_l} \delta(\varphi(P), v)| \leq \frac{k^l}{d^l} \to 0.$$

(3) The weighted sum $h_{\hat{\lambda}}(P)$ on the right hand side of (3) is a Weil height associated to E, so by 3.2 it will be sufficient to prove that it satisfy the functional equation $h_{\hat{\lambda}}(\varphi_1(P)) + \cdots + h_{\hat{\lambda}}(\varphi_k(P)) = dh_{\hat{\lambda}}(P)$ for all $P \in X(K)$. For $P \in X \setminus (\mathrm{Supp}(E) \cup \mathrm{Supp}(\varphi_1^* E) \cup \cdots \cup \mathrm{Supp}(\varphi_k^* E))$, the functional equation follows from (2) and the product formula. For a point $P \in \mathrm{Supp}(E) \cup \mathrm{Supp}(\varphi_1^* E) \cup \cdots \cup \mathrm{Supp}(\varphi_k^* E)$ choose a rational function t such that $P \notin \mathrm{Supp}(E') \cup \mathrm{Supp}(\varphi_1^* E') \cup \cdots \cup \mathrm{Supp}(\varphi_k^* E')$ for the new divisor $E' = E - \mathrm{div}(t)$. Now using the function $f' = \frac{ft^d}{\prod_{i=1}^k t \circ \varphi_i}$ we get again $\varphi_1^* E' + \cdots + \varphi_k^* E' = dE' + \mathrm{div}(f')$ and the formula still holds. □

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Department of Mathematics and Computer Science. Bronx Community College of CUNY. 2155 University Ave. Bronx, NY 10453

E-mail address: jorge.pineiro@bcc.cuny.edu