# Canonical metrics and local canonical heights associated to several maps

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## Definition

Let $X$ be a projective algebraic variety, together with absolute value function $|.|$ defined on the structural sheaf $\mathcal{O}_X$. A metric $\|.\|$ on a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ is a collection of metrics $\|.\|_x$ on the fibres $\mathcal{L}_x$, varying continuously on $x$, in such a way that for any open set $U \subset X$ and $s \in \Gamma(U, \mathcal{L})$, the continuous function $\|s\|_U : U \to \mathbb{R}_+$ satisfies $\|fs\| = |f|\|s\|$ for $f \in \mathcal{O}_X(U)$.

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## Example

Consider a projective variety $X$ defined over over a number field $K$ and $\mathcal{L} \in \mathrm{Pic}(X)$. For a place $\sigma : K \hookrightarrow \mathbb{C}$ at infinity, we can take the norm $|x|_\sigma = |\sigma(x)|$ in $K$ and put hermitian metrics on $\mathcal{L}_\sigma = \mathcal{L} \otimes_\sigma \mathbb{C}$. The datum $(\mathcal{L}, \|.\|_\sigma)$, where $\sigma$ is moving in places at infinity is called a hermitian line bundle.

Ultrametric situation: Let $K$ be a complete ultrametric field which is the field of fractions of a complete discrete valuation ring $\mathcal{O}_K$. Let $X$ be a projective variety over $K$ and $\mathcal{L}$ a line bundle on $X$.

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## Definition

(Algebraic metrics) A model $(\tilde{X}, \tilde{\mathcal{L}})$ of some power $(X, \mathcal{L}^e)$ defines a metric on $\mathcal{L}$ in the following way:
Let $\varepsilon_{\mathcal{U}} : \mathcal{O}_{\tilde{X}}(\mathcal{U}) \xrightarrow{\sim} \tilde{\mathcal{L}}|\mathcal{U}$ be a local frame for $\tilde{\mathcal{L}}|\mathcal{U}$, then every non-zero section $s_{\mathcal{U}} : \mathcal{U} \to \tilde{\mathcal{L}}|\mathcal{U}$ will be written as $s_{\mathcal{U}} = \varepsilon_{\mathcal{U}} f_{\mathcal{U}}$ for some $f_{\mathcal{U}} \in \mathcal{O}_{\tilde{X}}(\mathcal{U})$. When we restrict ourself to $U = \mathcal{U} \times \mathrm{Spec}(K)$, we get $\varepsilon_U : \mathcal{O}_X(U) \xrightarrow{\sim} \mathcal{L}^e|U$ and for any section $s_U$ of $\mathcal{L}|U$ we could write $s_U^e = \varepsilon_U f_U$. We declare $\|s_U\|_{\tilde{\mathcal{L}}} = |f_U|^{1/e}$ or equivalently $\|\varepsilon_U\|_{\tilde{\mathcal{L}}} = 1$ on $U$.

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## Definition

A metric $\|.\|$ on $\mathcal{L}$ is called bounded and continuous if there is a model $(\tilde{X}, \tilde{\mathcal{L}})$ such that $\log \frac{\|.\|}{\|.\|_{\tilde{\mathcal{L}}}}$ is bounded an continuous on $X(K)$. The space of bounded and continuous metrics on $\mathcal{L}$ will be denoted by $\mathrm{BdM}(X, \mathcal{L})$.

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### Definition

We say that the system $(X, \{\varphi_1, \ldots, \varphi_k\}, \mathcal{L}, q)$ is a polarized dynamical system of $k$ maps if there exist an ample line bundle $\mathcal{L}$ on $\mathrm{Pic}(X) \otimes \mathbb{R}$ such that $\bigotimes_{i=1}^{k} \varphi_i^* \mathcal{L} \xrightarrow{\sim} \mathcal{L}^q$ for some real number $q > k$.

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A polarization in other words is saying that the operator

$$\Phi_q^* = (\prod_{i=1}^{k} \varphi_i^*)^{1/q} : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X)$$

has a fixed point $\mathcal{L}$ for some real number $q > k$. When $\mathcal{L} \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ and $q = d \in \mathbb{Q}$ we will say that system is polarized by a $\mathbb{Q}$-bundle.

## Lemma

*Let $(X, \{\varphi_1, ..., \varphi_k\}, \mathcal{L}, d)$ be a system polarized by a $\mathbb{Q}$-bundle over a algebraically closed valuation field. Suppose that each of the $\varphi_i : X \longrightarrow X$ is surjective and we have fixed an isomorphism $\Phi : \mathcal{L}^d \simeq \bigotimes_{i=1}^k \varphi_i^* \mathcal{L}$. Let us assume that we have a bounded and continuous metric $\|.\|_0$ in $\mathcal{L}$. Then, there exist a bounded and continuous metric $\|.\|_{\{\varphi_1,...,\varphi_k\}}$ on $\mathcal{L}$ satisfying the equation*

$$\|.\|^d_{\{\varphi_1,...,\varphi_k\}} = \Phi^*(\varphi_1^*\|.\|_{\{\varphi_1,...,\varphi_k\}} \cdots \varphi_k^*\|.\|_{\{\varphi_1,...,\varphi_k\}})$$

*The metric $\|.\|_{\{\varphi_1,...,\varphi_k\}}$ is called the canonical metric associated to the system $(X, \{\varphi_1, ..., \varphi_k\}, \mathcal{L}, d)$.*

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The metric $\|.\|_{\{\varphi_1,\ldots,\varphi_k\}}$ is then a fixed point of the operator

$$\Phi_d^* = (\prod_{i=1}^k \varphi_i^*)^{1/d} : \mathrm{BdM}(X, \mathcal{L}) \longrightarrow \mathrm{BdM}(X, \mathcal{L}).$$

## Proof.

For the proof of the Lemma we construct recursively the metrics $\|.\|_n^d = \Phi^*(\varphi_1^*\|.\|_{n-1} \dots \varphi_k^*\|.\|_{n-1})$ and consider the bounded and continuous function $h(x) = \log \frac{\|.\|_1}{\|.\|_0}$ to obtain the identity

$$\log \|.\|_n = \sum_{j=0}^{n-1} (\sum_{i=1}^{k} \frac{1}{d} \Phi^* \varphi_i^*)^j h + \log \|.\|_0$$

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Because $\|(\sum_{i=1}^{k} \frac{1}{d} \Phi^* \varphi_i^*)^j h\|_{sup} \leq (\frac{k}{d})^j \|h\|_{sup}$, we will get that the series $\sum_{j=0}^{n-1} (\sum_{i=1}^{k} \frac{1}{d} \Phi^* \varphi_i^*)^j h$ converges absolutely and uniformly to a bounded and continuous function $h_{\varphi_1 \ldots \varphi_k}$ and the metric $\|.\|_n$ converges uniformly to the continuous and bounded metric

$$\|.\|_{\{\varphi_1, \ldots, \varphi_k\}} = \|.\|_0 \exp(h_{\varphi_1 \ldots \varphi_k}).$$

□

Global Situation: Let $X$ be a projective algebraic variety defined over a number field $K$:

## Definition

A bounded and continuous adelic metric $\|.\|$ on a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ is a collection of bounded and continuous metrics $\|.\| = (\|.\|_v)_{v \in M_K}$, where $v$ runs over the places of $K$; in such a way that for all but finitely many $v$, the metric is induced by the same $\mathcal{O}_K$-model.

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## Example

Let $X$ is a projective variety and $\mathcal{L}$ an ample line bundle on $X$. Some power of $\mathcal{L}$ will be very ample and therefore we can build a model $(\tilde{X}, \tilde{\mathcal{L}})$ of some power $(X, \mathcal{L}^e)$ over $\mathrm{Spec}(\mathcal{O}_K)$, with $\tilde{\mathcal{L}}$ hermitian line bundle. This induces an adelic metric $\|.\|$ on $\mathcal{L}$.

Let $(X, \{\varphi_1, ..., \varphi_k\}, \mathcal{L}, d)$ be a polarized dynamical system of $k$ surjective morphisms over a number field $K$ and let us fix an isomorphism $\Phi : \mathcal{L}^d \simeq \bigotimes_{i=1}^k \varphi_i^* \mathcal{L}$. Suppose that $X$ is a projective variety and $\mathcal{L}$ is an ample line bundle. We can build a model $(\tilde{X}, \tilde{\mathcal{L}})$ of some power $(X, \mathcal{L}^e)$ over $\mathrm{Spec}(\mathcal{O}_K)$, with $\tilde{\mathcal{L}}$ hermitian line bundle. This induces an adelic metric $\|.\|_0$ on $\mathcal{L}$.

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## Remark

There is an open $U \subset \mathrm{Spec}(\mathcal{O}_K)$ such that the maps extend to $\varphi_i : \tilde{X}_U \longrightarrow \tilde{X}_U$ and $\Phi_U : \mathcal{L}_U^d \simeq \bigotimes_{i=1}^k \varphi_i^* \mathcal{L}_U$ in $\mathrm{Pic}(\tilde{X}_U)$. It follows that for $v \in U$

$$\|\cdot\|_{0,v}^d = \Phi^*(\varphi_1^* \|\cdot\|_{0,v} \dots \varphi_k^* \|\cdot\|_{0,v})$$

We can build a sequence of models $(\tilde{X}_n, \tilde{\mathcal{L}}_n)$ inducing adelic metrics $\|.\|_n$ that converge to a limit adelic metric $\|.\|_n \longrightarrow \|.\|_{\{\varphi_1, \dots, \varphi_k\}}$, which we call the canonical metric.

Let us define the normalization $\tilde{\varphi}_i : \tilde{X}^i \longrightarrow \tilde{X}$ of the composition of morphisms $\varphi_i : \tilde{X}_U \longrightarrow \tilde{X}_U \hookrightarrow \tilde{X}$. Let $\tilde{X}_1$ be the Zariski closure of

$$\tilde{X}_U \xrightarrow{\Delta} \tilde{X}_U \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \tilde{X}_U \hookrightarrow \tilde{X}^1 \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \tilde{X}^k$$

where $\Delta$ is the diagonal map. Let $p_i : \tilde{X}^1 \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \tilde{X}^k \longrightarrow \tilde{X}^i$ denote the projection onto the $i$-th factor.

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## Remark

We can build a new model $(\tilde{X}_1, \tilde{\mathcal{L}}_1)$ of $(X, \mathcal{L}^e)$ when we put

$$\tilde{\mathcal{L}}_1 = [((\tilde{\varphi}_1 \circ p_1)^* \tilde{\mathcal{L}} \otimes \cdots \otimes (\tilde{\varphi}_k \circ p_k)^* \tilde{\mathcal{L}}))]^{1/d}.$$

This new model induces an adelic metric $\|.\|_1$ on $\mathcal{L}$ with the property $\|.\|_{v,1}^d = \Phi^*(\varphi_1^*\|.\|_{v,0} \dots \varphi_k^*\|.\|_{v,0})$ for all places $v$ of $K$. In this way starting from a model $(\tilde{X}_n, \tilde{\mathcal{L}}_n)$ we can build a new model $(\tilde{X}_{n+1}, \tilde{\mathcal{L}}_{n+1})$ and for each place $v$ of $K$ we will have a sequence of metrics $\|.\|_{v,n}$ converging to the metric $\|.\|_{v,\{\varphi_1,\dots,\varphi_k\}}$.

The work of Kawaguchi showed the existence of canonical height and local canonical heights associated to systems of several maps.

**Theorem**

*Let $(X, \{\varphi_1, \ldots, \varphi_k\}, \mathcal{L}, q)$ be a polarized dynamical system of $k$ morphisms over a number field $K$. Then there exist unique a real valued function*

$$\hat{h}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}} : X(\bar{K}) \longrightarrow \mathbb{R}$$

*with the following properties*

(1) $\hat{h}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}$ *is a Weil height associated to* $\mathcal{L}$,

(2) $\sum_{i=1}^{k} \hat{h}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}(\varphi_i(x)) = q\hat{h}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}(x)$ *for all* $x \in X(\bar{K})$.

The function $\hat{h}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}$ is called the canonical height function associated to $(X, \{\varphi_1, \ldots, \varphi_k\}, \mathcal{L}, q)$.

Local heights: Suppose that $X$ is a normal projective variety defined over a number field $K$ and $U \subset X$ is a non-empty open set. Let us denote by $M_K$ the set of absolute values on $K$ and by $M = M_{\bar{K}}$, the set of absolute values on $\bar{K}$ extending those of $K$.

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### Definition

A function $\lambda : U(\bar{K}) \times M \longrightarrow \mathbb{R}$ is called $M_K$-continuous if, for every $v \in M_K$, $\lambda_v : U(\bar{K}) \longrightarrow \mathbb{R}$, $x \mapsto \lambda(x, v)$ is continuous in the $v$-adic topology. A function $\gamma : M_K \longrightarrow \mathbb{R}$ is called $M_K$-constant if $\gamma(v) = 0$ for all but finitely many $v \in M_K$.
A function $\alpha : U(\bar{K}) \times M \longrightarrow \mathbb{R}$ is called $M_K$-bounded if there is a $M_K$-constant function $\gamma$ such that $|\alpha(x, v)| \leq \gamma(v)$ for all $(x, v) \in U(\bar{K}) \times M$.

## Definition

Let $D \in \mathrm{Div}(X) \otimes \mathbb{R}$. A function $\lambda_D : X \setminus \mathrm{Supp}(D)(\bar{K}) \times M \longrightarrow \mathbb{R}$ is a said to be a local height associated to $D$ if there is an affine covering $\{U_i\}$ of $X$, a Cartier divisor $\{(U_i, f_i)\}$ representing $D$ such that the function $\alpha(x, v) = \lambda_D(x, v) - v \circ f_i(x)$ is $M_K$-bounded and $M_K$-continuous for $x \in (U_i \setminus \mathrm{Supp}(D))(\bar{K})$ and $v \in M$.

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Let $\mathcal{L} \in \mathrm{Pic}(X)$ and $\|.\| = \{\|.\|_v, v \in M_K\}$ a bounded and continuous adelic metric on $\mathcal{L}$. Take a section $s \in \Gamma(X, \mathcal{L})$ and consider the function

$$\alpha(x, v) = -\log \|s(x)\|_v - v \circ f_i(x) = -\log \|\varepsilon_{U_i}(x)\|_v$$

for a local frame $\varepsilon_{U_i} : \mathcal{O}_X(U_i) \xrightarrow{\sim} \mathcal{L}|U_i$. By the definition of the adelic metric induced by a model, we will have $\|\varepsilon_{U_i}(x)\| = 1$ for all $x \in U_i$ and almost all $v \in M_K$, therefore $\alpha$ is a $M_K$-bounded and $M_K$-continuous and $\lambda_D(x, v) = -\log \|s(x)\|_v$ is a local height associated to $D = \mathrm{div}(s)$.

## Theorem

*Let $X$ be a normal projective variety and consider the polarized system $(X, \{\varphi_1, ..., \varphi_k\}, \mathcal{L}, q)$ of $k$ morphisms over a number field $K$. Suppose that $E$ is a divisor on $X$ associated to the line bundle $\mathcal{L}$ such that $\varphi_i(X)$ is not contained in $\mathrm{Supp}(E)$ and $\varphi_1^* E + \cdots + \varphi_k^* E = qE + \mathrm{div}(f)$ for some rational function $f \in \bar{K}(X)^* \otimes \mathbb{R}$. There exist a unique function $\hat{\lambda}_{\mathcal{L},\{\varphi_1,...,\varphi_k\}}$ satisfying:*

(1) *$\hat{\lambda}_{\mathcal{L},\{\varphi_1,...,\varphi_k\}}$ is a Weil local height associated to $E$.*

(2) *for any $x \in X \setminus (\mathrm{Supp}(E) \cup \mathrm{Supp}(\varphi_1^* E) \cup \cdots \cup \mathrm{Supp}(\varphi_k^* E))$ and for all $v \in M$,*

$$\sum_{i=1}^{k} \hat{\lambda}_{\mathcal{L},\{\varphi_1,...,\varphi_k\}}(\varphi_i(x), v) = q\hat{\lambda}_{\mathcal{L},\{\varphi_1,...,\varphi_k\}}(x, v) + v(f(x)).$$

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## Theorem

*Let $X$ be a normal projective variety over the number field $K$ and the polarization $(X, \{\varphi_1, ..., \varphi_k\}, \mathcal{L}, d)$ by a $\mathbb{Q}$-bundle $\mathcal{L}$. Suppose that $E$ is a divisor on $X$ associated to the line bundle $\mathcal{L}$ such that $\varphi_i(X)$ is not contained in $\mathrm{Supp}(E)$ and, for some rational function $f \in \bar{K}(X)^*$, $\varphi_1^* E + \cdots + \varphi_k^* E = dE + \mathrm{div}(f)$ Then, for $s \in \Gamma(X, \mathcal{L})$, $v \in M_K$ and $x \in X \setminus \mathrm{Supp}(s)$, the function:*

$$\hat{\lambda}_{\mathcal{L},\{\varphi_1,\dots,\varphi_k\}}(x, v) = -\log \|s(x)\|_{v,\{\varphi_1,\dots,\varphi_k\}}$$

*will be the local canonical height function associated to the polarized dynamical system $(X, \{\varphi_1, ..., \varphi_k\}, \mathcal{L}, d)$.*

Proof: For a section $s \in \Gamma(X, \mathcal{L})$, $v \in M_K$ and $x \in X \setminus \mathrm{Supp}(s)$, consider the function

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The function so defined is a local height associated to $E$ because the canonical metric is the uniform limit of bounded and continuous adelic metrics. The functional equation satisfied by the canonical metric forces

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$$\begin{aligned}
\sum_{i=1}^{k} \hat{\lambda}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}(\varphi_i(x), v) &= \sum_{i=1}^{k} -\log \|s(\varphi_i(x))\|_{v,\{\varphi_1,\ldots,\varphi_k\}} \\
&= -\log \Phi^{-1*} \|s(x)\|^{d}_{v,\{\varphi_1,\ldots,\varphi_k\}} \\
&= -\log |f(x)|_v \|s(x)\|^{d}_{v,\{\varphi_1,\ldots,\varphi_k\}} \\
&= -d \log \|s(x)\|_{v,\{\varphi_1,\ldots,\varphi_k\}} + v(f(x)) \\
&= d\hat{\lambda}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}(x) + v(f(x))
\end{aligned}$$

We want to consider metrics on real line bundles:

### Definition

An element $\mathcal{L} \in \mathrm{Pic}(X) \otimes \mathbb{R}$ can be written as a formal product

$$\mathcal{L} = \bigotimes_{1 \leq j \leq t} \mathcal{L}_j^{r_j} = \mathcal{L}_1^{\otimes r_1} \otimes \cdots \otimes \mathcal{L}_t^{\otimes r_t},$$

where $\mathcal{L}_j \in \mathrm{Pic}(X)$ for $j = 1, \ldots, t$ and the $r_1, \ldots, r_t \in \mathbb{R}$ are real numbers.

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## Definition

A section $s \in \Gamma(U, \mathcal{L})$ over an open set $U \subset X$, can be written as the formal product $s = s_1^{r_1} \otimes \cdots \otimes s_t^{r_t}$, where $s_j \in \Gamma(U, \mathcal{L}_j)$. In particular for some open cover $\{U^j\}_{j \in J}$ of $X$, we have local frames

$$\varepsilon_{U^j} = \varepsilon_{U_1^j}^{r_1} \otimes \cdots \otimes \varepsilon_{U_t^j}^{r_t} : \mathcal{O}_X(U^j) \xrightarrow{\sim} \mathcal{L}|U^j.$$

## Definition

To put a metric on the fibre $\mathcal{L}_x$ is to be able to measure the length of a non-zero section $s(x) = \bigotimes_{j=1}^{t} s_j(x)^{r_j} \in \mathcal{L}_x$, that is, to put a metric $\|.\|_j$ on each $\mathcal{L}_j(x)$ for $j = 1 \dots t$ and declare

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## Definition

Given a map $\varphi : X' \longrightarrow X$ and a real line bundle $\mathcal{L}$, we can define the pullback by: $\mathcal{L} = \mathcal{L}_1^{\otimes r_1} \otimes \cdots \otimes \mathcal{L}_t^{\otimes r_t} \Rightarrow \varphi^*\mathcal{L} = \bigotimes_j (\varphi^*\mathcal{L}_j)^{\otimes r_j}$
In the same way if $\|.\|$ is a metric on $\mathcal{L}$, we define the pullback metric by $\varphi^*\|s'(x')\| = \prod_j \|s'_j(\varphi(x'))\|_j^{r_j}$, for $U'$ an open set, $x' \in U' \subset X'$ and $s' \in \Gamma(U', \mathcal{L}')$.

## Definition

(Definition of the complete and algebraically closed field $\mathbb{C}_v$) Let $K$ be a number field and $v$ a finite place of $K$. First complete $K$ for the absolute value given by $v$, then take its algebraic closure; this field admits a unique absolute value extending $v$ and we take its completion for that absolute value to obtain $\mathbb{C}_v$.

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Let $X$ be a projective variety over the complete ultrametric field $\mathbb{C}_v$ and $\mathcal{L}$ an element of $\mathrm{Pic}(X) \otimes \mathbb{R}$. Given a model $(\tilde{X}, \tilde{\mathcal{L}})$, where $\tilde{X}$ is a $\mathcal{O}_{\mathbb{C}_v}$-scheme with generic fibre $X$ and $\tilde{\mathcal{L}}$ is a real line bundle on $\tilde{X}$ such that $\tilde{\mathcal{L}} \otimes \mathbb{C}_v = \mathcal{L}^e$ for some power $e > 0$.

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## Definition

We define a metric $\|.\|_{\tilde{\mathcal{L}}}$ induced by the model $(\tilde{X}, \tilde{\mathcal{L}})$ in a similar way as we did with $\mathbb{Q}$-bundle declaring $\|\varepsilon_U\|_{\tilde{\mathcal{L}}} = 1$ on $U$ for a local frame $\varepsilon_U : \mathcal{O}_{\tilde{X}}(U) \xrightarrow{\sim} \tilde{\mathcal{L}}|U$.

## Lemma

*Let $K$ be a number field and $v$ a finite place of $K$. Let $(X, \{\varphi_1, ..., \varphi_k\}, \mathcal{L}, q)$ be a polarized dynamical system over algebraically closed valuation field $\mathbb{C}_v$. Assume that each of the $\varphi_i : X \longrightarrow X$ is surjective and we have fixed an isomorphism $\Phi : \mathcal{L}^q \simeq \bigotimes_{i=1}^k \varphi_i^* \mathcal{L}$. Suppose that we have a bounded and continuous metric $\|.\|_0$ in $\mathcal{L}$. Then the metrics $\|.\|_n$ on $\mathcal{L}$ defined inductively by $\|.\|_n^q = \Phi^*(\varphi_1^* \|.\|_{n-1} \dots \varphi_k^* \|.\|_{n-1})$, converge uniformly to a continuous and bounded metric on $\mathcal{L}$, which we denote by $\|.\|_{\{\varphi_1,...,\varphi_k\}}$. The metric $\|.\|_{\{\varphi_1,...,\varphi_k\}}$ is the unique metric with the property*

$$\|.\|_{\{\varphi_1,...,\varphi_k\}}^q = \Phi^*(\varphi_1^* \|.\|_{\{\varphi_1,...,\varphi_k\}} \cdots \varphi_k^* \|.\|_{\{\varphi_1,...,\varphi_k\}})$$

*and will be called the canonical metric associated to $\mathcal{L}$ and the maps $\varphi_1, \dots, \varphi_k$.*

## Definition

Let $X$ be defined over a number field $K$. An bounded and continuous adelic metric $\|.\|$ for a line bundle $\mathcal{L} \in \mathrm{Pic}(X) \otimes \mathbb{R}$ is a collection of continuous and bounded metrics $\|.\| = (\|.\|_v)_{v \in M_K}$, where $v$ runs over the places of $K$; in such a way that for all but finitely many $v$, the metric is induced by the same $\mathcal{O}_K$-model.

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## Remark

Given a polarized dynamical system $(X, \{\varphi_1, \ldots, \varphi_k\}, \mathcal{L}, q)$ over a number field $K$, the idea will be, to build a sequence of models $(\tilde{X}_n, \tilde{\mathcal{L}}_n)$ inducing metrics $\|.\|_n$ satisfying the functional equation

$$\|.\|_n^q = \Phi^*(\varphi_1^*\|.\|_{n-1} \ldots \varphi_k^*\|.\|_{n-1}),$$

so we can apply the previous lemma and obtain an adelic metric which is the canonical metric at every place.

The models $(\tilde{X}_n, \tilde{\mathcal{L}}_n)$ inducing the adelic metrics $\|.\|_n$ on $\mathcal{L}$ can be defined recursively as follows: Suppose that a model $(\tilde{X}_{n-1}\tilde{\mathcal{L}}_{n-1})$ is already being defined. The map $\varphi_i : X \longrightarrow X$ can be extended to an open set $\varphi_i : \tilde{X}_{n-1}(U) \longrightarrow \tilde{X}_{n-1}(U)$.

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$$\tilde{X}_{n-1}(U) \xrightarrow{\Delta} \tilde{X}_{n-1}(U) \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \tilde{X}_{n-1}(U) \hookrightarrow \tilde{X}^1_{n-1} \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \tilde{X}^k_{n-1}$$

The models $(\tilde{X}_n, \tilde{\mathcal{L}}_n)$ inducing the adelic metrics $\|.\|_n$ on $\mathcal{L}$ can be defined recursively as follows: Suppose that a model $(\tilde{X}_{n-1}\tilde{\mathcal{L}}_{n-1})$ is already being defined. The map $\varphi_i : X \longrightarrow X$ can be extended to an open set $\varphi_i : \tilde{X}_{n-1}(U) \longrightarrow \tilde{X}_{n-1}(U)$. Let us consider the normalization $\tilde{\varphi}_i : \tilde{X}^i_{n-1} \longrightarrow \tilde{X}_{n-1}$ of the composition of morphisms $\varphi_i : \tilde{X}_{n-1}(U) \longrightarrow \tilde{X}_{n-1}(U) \hookrightarrow \tilde{X}_{n-1}$ and take $\tilde{X}_n$ to be the Zariski closure of

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Let $p_i : \tilde{X}^1_{n-1} \times_{\mathcal{O}_K} \cdots \times_{\mathcal{O}_K} \tilde{X}^k_{n-1} \longrightarrow \tilde{X}^i_{n-1}$ denote the projection onto the $i$-th factor and define:
$\tilde{\mathcal{L}}_n = [((\tilde{\varphi}_1 \circ p_1)^* \tilde{\mathcal{L}}_{n-1} \otimes \cdots \otimes (\tilde{\varphi}_k \circ p_k)^* \tilde{\mathcal{L}}_{n-1}))]^{1/d}$.

The models $(\tilde{X}_n, \tilde{\mathcal{L}}_n)$ inducing the adelic metrics $\|.\|_n$ on $\mathcal{L}$ can be defined recursively as follows: Suppose that a model $(\tilde{X}_{n-1}\tilde{\mathcal{L}}_{n-1})$ is already being defined. The map $\varphi_i : X \longrightarrow X$ can be extended to an open set $\varphi_i : \tilde{X}_{n-1}(U) \longrightarrow \tilde{X}_{n-1}(U)$. Let us consider the normalization $\tilde{\varphi}_i : \tilde{X}^i_{n-1} \longrightarrow \tilde{X}_{n-1}$ of the composition of morphisms $\varphi_i : \tilde{X}_{n-1}(U) \longrightarrow \tilde{X}_{n-1}(U) \hookrightarrow \tilde{X}_{n-1}$ and take $\tilde{X}_n$ to be the Zariski closure of

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## Remark

The induced metrics so defined satisfy the functional equation:

$$\|.\|^q_n = \Phi^*(\varphi_1^* \|.\|_{n-1} \ldots \varphi_k^* \|.\|_{n-1})$$

## Theorem

*(Local canonical height associated to polarizations by real line bundles) Let $X$ be a normal projective variety and consider the polarized system $(X, \{\varphi_1, \ldots, \varphi_k\}, \mathcal{L}, q)$ of $k$ morphisms over a number field $K$, where $\mathcal{L}$ is a real line bundle. Suppose that $E$ is a real divisor divisor on $X$ associated to the line $\mathcal{L}$ such that $\varphi_i(X)$ is not contained in $\mathrm{Supp}(E)$ and $\varphi_1^* E + \cdots + \varphi_k^* E = qE + \mathrm{div}(f)$ for some rational function $f \in \bar{K}(X)^* \otimes \mathbb{R}$. The function defined by:*

$$\hat{\lambda}_{\mathcal{L}, \{\varphi_1, \ldots, \varphi_k\}}(x, v) = -\log \|s(x)\|_{v, \{\varphi_1, \ldots, \varphi_k\}},$$

*for a section $s \in \Gamma(X, \mathcal{L})$, $v \in M_K$ and $x \in X \setminus \mathrm{Supp}(s)$, is the local canonical associated to $(X, \{\varphi_1, \ldots, \varphi_k\}, \mathcal{L}, q)$.*

## Theorem

*The function* $\hat{\lambda}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}(x,v) = -\log \|s(x)\|_{v,\{\varphi_1,\ldots,\varphi_k\}}$ *satisfies:*

(1) $\hat{\lambda}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}$ *is a Weil local height associated to* $E$*.*

(2) *for any* $x \in X \setminus (\mathrm{Supp}(E) \cup \mathrm{Supp}(\varphi_1^* E) \cup \cdots \cup \mathrm{Supp}(\varphi_k^* E))$ *and for all* $v \in M$*,*

$$\sum_{i=1}^{k} \hat{\lambda}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}(\varphi_i(x),v) = q\hat{\lambda}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}(x,v) + v(f(x)).$$

(3) *For* $L$ *an extension of* $K$*,* $w$ *and extension of* $v$ *and* $x \in X(L)$*, we have the decomposition:*

$$\hat{h}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}(x) = \frac{1}{[L:K]} \sum_{w \in M_L} [L_w : K_v] \hat{\lambda}_{\mathcal{L},\{\varphi_1,\ldots,\varphi_k\}}(x,w).$$

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