Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 9: Functions. One-to-one and onto functions. Section 2.3

1 Functions. One-to-one and onto functions

Definition 1. Let A and B be nonempty sets. A **function** f from A to B is an assignment of **exactly one element** of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f\colon A \to B$. The set A is the domain of f.

Definition 2. A function f is said to be **one-to-one**, or an injunction, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be **injective** if it is one-to-one.

Definition 3. A function f from A to B is called **onto**, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called **surjective** if it is onto.

Definition 4. The function f is a one-to-one correspondence, or a **bijection**, if it is both one-to-one and onto. We also say that such a function is **bijective**.

Example 5. Let A be a set. The identity function on A is the function $\iota_A\colon A \to A$, where $\iota_A(x) = x$ for all $x \in A$. In other words, the identity function ι_A is the function that assigns each element to itself. The function ι_A is one-to-one and onto, so it is a bijection.

Example 6. Consider $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. The function $f\colon A \to B$ defined as

$$f(1) = a, \quad f(2) = b \quad \text{and} \quad f(3) = c,$$

is injective but not surjective. The element d is not in the image or the range of f. On the hand, if we take $g\colon A \to B$ given by

$$g(1) = a, \quad g(2) = a \quad \text{and} \quad g(3) = c,$$

is neither injective, not surjective since now two different elements in the domain hit the same element in the range $g(1) = g(2) = a$. To obtain a bijective function, we need to change to co-domain B, for example, the function $h\colon \{1, 2, 3\} \to \{a, b, c\}$ defined by

$$h(1) = a, \quad h(2) = b \quad \text{and} \quad h(3) = c,$$

is both injective and surjective.

Example 7. Consider the rule $f(x) = x^2$ in three different functions

$$f_1 \colon \mathbb{R} \to \mathbb{R}, \qquad f_2 \colon \mathbb{R} \to [0, \infty) \quad \text{and} \quad f_3 \colon [0, \infty) \to [0, \infty).$$

The function f_1 is neither injective nor surjective, since $f_1(1) = f_1(-1) = 1$ and at the same time negative numbers are not in the range of f_1. The function f_2 is still not injective because $f_2(-1) = f_2(1) = 1$ but now it is surjective, every element in $[0, \infty)$ is in the range of f_2. The function f_3 is both injective and surjective.

Definition 8. Let f be a function from the set A to the set B. The **graph** Γ_f of the function f is the set of ordered pairs

$$\{(a, b) \mid b = f(a)\}.$$

The graph of a function $f \colon A \to B$ is a subset of $A \times B$.

1.1 Some important functions

Definition 9. The **floor function** assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The **ceiling function** assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Remark 10. Suppose that x, y are real numbers, we have

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq x + y.$$

Observe that it is always true that $\lfloor x \rfloor \leq x$ and $\lfloor y \rfloor \leq y$. Hence $\lfloor x \rfloor + \lfloor y \rfloor$ is an integer and

$$\lfloor x \rfloor + \lfloor y \rfloor \leq x + y.$$

By the definition of the floor function, it must be the case that

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq x + y.$$