Taken from Lecture notes for Analysis I by Prof. Mitchell Faulk

1 Lecture 8: Cauchy sequences

Definition 1. Let x_n be a sequence in a metric space X. Say that x_n is a **Cauchy** sequence if the following is true. For each $\epsilon > 0$, there is a positive integer N such that if $m \geqslant N$ and $n \geqslant N$, then $d(x_n, x_m) < \epsilon$.

Heuristically, a Cauchy sequence is one in which the terms in the sequence become close to *one another*. This differs slightly from the notion of a convergent sequence where the terms become close to a specific point in the metric space.

Theorem 2. *Let X be a metric space.*

(i) Every convergent sequence is a Cauchy sequence.

(ii) If X is compact and if p_n is a Cauchy sequence of X, then p_n converges to a point of X.

Proof. For (i), suppose p_n converges to p. Let $\epsilon > 0$ be given. There is an $N > 0$ such that if $n \geqslant N$, then $d(p_n, p) < \epsilon/2$. Then for $m \geqslant N$ and $n \geqslant N$, the triangle inequality implies that

$$d(p_n, p_m) \leqslant d(p_n, p) + d(p, p_m) < \epsilon/2 + \epsilon/2 = \epsilon.$$

For (ii), let p_n be a Cauchy sequence in the compact space X. For each positive integer N, let E_N be the subset of X consisting of the points $p_N, p_{N+1}, p_{N+2}, \ldots$. The closure $\bar{E}_N$ is a closed subset of a comapact space, and hence $\bar{E}_N$ is compact itself. Also we have the nesting property that $E_N \supset E_{N+1}$, which implies also that $\bar{E}_N \supset \bar{E}_{N+1}$. By compactness, the intersection $\cap_n \bar{E}_n$ is nonempty. Let p be a point in this intersection.

Let $\epsilon > 0$ be given. Because p_n is Cauchy, there is an $N > 0$ such that if $m, n \geqslant N$, then $d(p_n, p_m) < \epsilon/2$. Because p belongs to $\bar{E}_N$, there is an $n_0 \geqslant N$ such that $d(p, p_{n_0}) < \epsilon/2$. Then if $m \geqslant N$, we have

$$d(p, p_m) \leqslant d(p, p_{n_0}) + d(p_{n_0}, p_m) < \epsilon/2 + \epsilon/2 = \epsilon.$$

We conclude that p_m converges to p. □

Corollary 3. *Every Cauchy sequence in $\mathbb{R}^k$ converges.*

Proof. Let x_n be a Cauchy sequence of $\mathbb{R}^k$. Let $E_N \subset \mathbb{R}^k$ be the set of points $x_N, x_{N+1}, x_{N+2}, \ldots$. Because x_n is Cauchy, there is an $N_0 > 0$ such that if $p, q \in E_{N_0}$, then $d(p, q) < 1$, which implies E_{N_0} is bounded. The subset E of $\mathbb{R}^k$ determined by x_n is then also bounded because it consists of E_{N_0} together with the finite set of points $\{x_1, \ldots, x_{N_0-1}\}$. The closure $\bar{E}$ is then closed and bounded, and hence compact by Heine-Borel. Then the result follows from the previous result part (ii). □

Definition 4. A metric space (X, d) in which each Cauchy sequence converges is called **complete**.

Example 5. The previous corollary shows that $\mathbb{R}^n$ is complete. The preceding result also shows than any compact metric space is complete.

Example 6. The set of rational numbers $\mathbb{Q}$ together with the usual metric is not complete, because there exist Cauchy sequences in $\mathbb{Q}$ which do not converge to any point of $\mathbb{Q}$. (For example, consider a sequence of rational numbers converging to an irrational number.)

Practice Questions:

(1) Prove or disprove: The sum of two Cauchy sequences is again a Cauchy sequence.
(2) Consider the sequence of Fibonacci $F_0 = 1$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, for $n > 1$. Show that the sequence of rational numbers F_{n+1}/F_n has a limit. Compute the limit.