Taken from Lecture notes for Analysis I by Prof. Mitchell Faulk

1 Lecture 16: Mean value theorem

Definition 1. Let f be a real function defined on a metric space X. Say that f has a **local maximum at** $p \in X$ if there is a $\delta > 0$ such that whenever $q \in X$ satisfies $d(p, q) < \delta$, then $f(q) \leqslant f(p)$. The notion of local minimum is defined similarly.

Theorem 2. *Let f be defined on $[a, b]$. Suppose f is differentiable at $x \in (a, b)$ and f has a local maximum at x. Then $f'(x) = 0$.*

Proof. Exercise. Hint: compute the left-hand and right-hand limits separately. □

Let $f : [a, b] \to \mathbb{R}$, and let $G \subset \mathbb{R}^2$ be the graph of f given by

$$G = \{(x, f(x)) : x \in [a, b]\}.$$

Note that the secant line connecting the points $(a, f(a))$ and $(b, f(b))$ has slope given by

$$\frac{f(b) - f(a)}{b - a}.$$

The mean value theorem asserts that if f is differentiable on $[a, b]$, then this slope is equal to the slope of some tangent line.

Theorem 3 (Mean Value Theorem)**.** *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let $h(t)$ be the function defined for $t \in [a, b]$ by

$$h(t) = t(f(b) - f(a)) - (b - a)f(t).$$

Then h is continuous on $[a, b]$ and differentiable on (a, b). We also have that

$$h(b) = b(f(b) - f(a)) - (b - a)f(b) = a \cdot f(b) - b \cdot f(a)$$

and

$$h(a) = a(f(b) - f(a)) - (b - a)f(a) = a \cdot f(b) - b \cdot f(a),$$

which means that $h(a) = h(b)$. If h is constant, then we are done because we can let c be any point. Otherwise, there is a point $t \in (a, b)$ such that $h(t) \neq h(a)$. Without loss of generality we may assume that $h(t) > h(a)$. Because h is continuous, there is a point $x \in (a, b)$ where h achieves its maximum. The previous result then shows that $h'(x) = 0$, which is what we require. □

There is a generalized version of the previous result too.

Theorem 4 (Generalized Mean Value Theorem)**.** *Let f, g be continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $x \in (a, b)$ such that*

$$g'(x)[f(b) - f(a)] = f'(x)[g(b) - g(a)].$$

Proof. Replicate the proof above but with

$$h(x)[f(b) - f(a)]g(t) - [g(b) - g(a)]f(t)$$

instead. □

Definition 5. Let f be defined on $[a, b]$. We say that

(a) f is **monotonically increasing** if whenever $x \leqslant y$, then $f(x) \leqslant f(y)$

(b) f is **monotonically decreasing** if whenever $x \leqslant y$, then $f(x) \geqslant f(y)$.

Corollary 6. *Suppose f is differentiable on (a, b).*

(a) If $f'(x) \geqslant 0$ for each $x \in (a, b)$, then f is increasing.

(b) If $f'(x) \leqslant 0$ for each $x \in (a, b)$, then f is decreasing.

(c) If $f'(x) = 0$ for each $x \in (a, b)$, then f is constant.

Proof. The proofs follow from the equation

$$f'(c)(x - y) = f(x) - f(y)$$

which holds for each $x < y$ and for some c satisfying $x < c < y$. □

1.1 L'Hospital's rule

Theorem 7 (L'Hospital's Rule: Version 1)**.** *Let f and g be differentiable on (a, b). Suppose that*

(a) $g'(x) \neq 0$ for each $x \in (a, b)$

(b) $\lim_{x \to a} \frac{f'(x)}{g'(x)} = A$

(c) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = A.$$

Proof. We proceed in steps.
Assertion 1. If q satisfies $A < q$, then there is a point $c \in (a, b)$ such that whenever $a < x < c$ we have

$$\frac{f(x)}{g(x)} < q.$$

Proof of Assertion 1. Let r be a number satisfying $A < r < q$. Because $f'(x)/g'(x) \to A$, there is a point $c \in (a, b)$ such that whenever $a < x < c$ we have

$$\frac{f'(x)}{g'(x)} < r.$$

If x, y satisfy $a < x < y < c$, then the generalized mean value theorem shows that there is a point $t \in (x, y)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r.$$

We then take the limit of this inequality as $x \to a$ and use hypothesis (c) to find that

$$\frac{f(y)}{g(y)} \leqslant r < q$$

for each $a < y < c$. This completes the proof of Assertion 1.
Assertion 2. If p satisfies $p < A$, then there is a point $d \in (a, b)$ such that whenever $a < x < d$ we have

$$p < \frac{f(x)}{g(x)}.$$

Proof of Assertion 2. The proof is similar to that of Assertion 1.

The proof now follows from Assertions 1 and 2 together. □

The following version is also useful. Note that hypothesis (c) is replaced by a slightly different hypothesis (c') involving only the limit of $g(x)$ as $x \to a$.

Theorem 8 (L'Hospital's Rule: Version 2)**.** *Let f and g be differentiable on (a, b). Suppose that*

(a) $g'(x) \neq 0$ *for each* $x \in (a, b)$

(b) $\lim_{x\to a} \frac{f'(x)}{g'(x)} = A$

(c') $\lim_{x\to a} g(x) = \infty$.

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = A.$$

Proof. See Rudin. □

1.2 Taylor's Theorem

Definition 9. If f has a derivative f' on an interval and if f' is itself differentiable, then we will denote the derivative of f' by f''. If we can continue this process, then we obtain functions

$$f, f', f'', f^{(3)}, \ldots, f^{(n)}$$

where $f^{(n)}$ is the nth derivative of f.

Theorem 10. *Let f be defined on $[a, b]$. Suppose that $f^{(n-1)}$ is continuous on $[a, b]$ and $f^{(n)}(t)$ exists for each $t \in (a, b)$. Let α and β be distinct points of $[a, b]$ and define the polynomial*

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then there is a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n.$$

Remark 11. For the case $n = 1$, the polynomial $P(t)$ is just constant $P(t) = f(\alpha)$, and the statement is the mean value theorem.

Proof. Let M be the number defined by

$$f(\beta) = P(\beta) + M(\beta-\alpha)^n.$$

Note that our goal is to find an x between α and β such that $f^{(n)}(x) = Mn!$. Define a function g by the rule

$$g(t) = f(t) - P(t) - M(t-\alpha)^n.$$

Note that the nth derivative of g satisfies

$$g^{(n)}(t) = f^{(n)}(t) - 0 - Mn!.$$

It follows that the proof will be complete if we can find a point x between α and β such that $g^{(n)}(x) = 0$.

Specializing to the point $t = \alpha$, we note that because $f^{(k)}(\alpha) = P^{(k)}(\alpha)$ for each $k = 1, \ldots, n-1$, we have that

$$g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0.$$

At the point $t = \beta$, the choice of M implies directly that $g(\beta) = 0$. The mean value theorem asserts that there is a point x_1 between α and β such that $g'(x_1) = 0$. For the same reason, there is a point x_2 between α and x_1 such that $g''(x_2) = 0$. Iteratively, we obtain a point x_n between α and x_{n-1} such that $g^{(n)}(x_n) = 0$. $\square$