1 Inverse of a matrix

Suppose that the we are given a matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix}$$

How do we determine whether or not the matrix is singular? In case is invertible (non-singular), how can we find the inverse?

Proposition 1. *(dimension two) A 2×2 square matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if and only if $ad - bc \neq 0$ and the inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Example 2. Determine if the matrix is invertible and find the inverse if possible for $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$. We compute $ad - bc = 1(5) - 2(2) = 1 \neq 0$. Since the result is not equal to zero, the matrix is non-singular or invertible and we can compute its inverse as $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$. Our answer can be checked when we do the two product of matrices:

$$\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \cdot \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Definition 3. A matrix E is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation.

Remark 4. Performing row operations on a matrix is the same as multiplying by elementary matrix: If the elementary matrix E results from performing a certain row operation on the identity matrix I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

Remark 5. Every elementary matrix is invertible, and the inverse is also an elementary matrix. The list of inverse operations are:

Operation	Inverse
Interchanging two rows	Interchanging the same two rows
Multiplying a row by $c \neq 0$	Multiplying the same row by $1/c$
Adding c times a row i to a row j	Adding $-c$ times the same row i to row j

Proposition 6. *Let A be a square matrix of $n \times n$, then the following statements are equivalent:*

(1) The matrix A is invertible.

(2) $Ax = 0$ has only the trivial solution.

(3) The reduced row echelon form of A is the identity I_n.

(4) A is expressible as a product of elementary matrices.

Algorithm 7. To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}. We can see that from the equations:

$$E_k \dots E_2 E_1 A = I_n \Rightarrow E_k \dots E_2 E_1 I_n = A^{-1}.$$

Example 8. Find (if possible) the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right). \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right).$$

Theorem 9. *If the matrix A is an invertible matrix of order n, system of equations $Ax = b$ has exactly one solution, namely, $x = A^{-1}b$.*

Example 10. Use the information you have to solve the system of equations:

$$\begin{cases} x + 2y + 3z = -1 \\ 2x + 5y + 3z = 2 \\ x \qquad + 8z = -6 \end{cases}$$

The solution of the system $Ax = b$ is given by $x = A^{-1}b$. Therefore in this case

$$x = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 18 \\ 3 \\ -3 \end{pmatrix}$$