1 Introduction

Systems of linear equations and their solutions constitute one of the major topics that we will study in this course. A system of three linear equations is an array of simultaneous linear equations:

$$\begin{cases} 3x + y - z = 1 \\ x - 2y + z = \frac{5}{2} \\ -x + 2y + z = 5 \end{cases}$$

In general a system of linear equations looks like an array of m-linear equations in n-unknowns variables $x_1, \ldots, x_n$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Definition 1. A solution of a linear system in n unknowns $x_1, \ldots, x_n$ is a sequence of n numbers $s_1, \ldots, s_n$ for which the substitution

$$x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$$

makes each equation a true statement.

Definition 2. Two systems of equation are said to be equivalent when they have the same solution set.

Definition 3. We say that a linear system is consistent if it has at least one solution and inconsistent if it has no solutions.

Example 4. The system of equations

$$\begin{cases} 3x + y - z = 1 \\ x - 2y + z = \frac{5}{2} \\ -x + 2y + z = 5 \end{cases}$$

admits solution $x = \dfrac{33}{28}$, $z = \dfrac{15}{4}$, $y = \dfrac{17}{14}$. It is therefore consistent. On the other hand the (very similar) system

$$\begin{cases} 3x + \ \ y - z = 1 \\ \ \ x - 2y + z = \frac{5}{2} \\ -x + 2y - z = 5 \end{cases}$$

is inconsistent. When we add the last two equations term by term, we get $0 = \dfrac{15}{2}$. This contradiction is saying that the last two equations cannot be full-filled simultaneously.

Remark 5. Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Example 6. The system

$$\begin{cases} 3x + \ \ y - z = 1 \\ \ \ x - 2y + z = -5 \\ -x + 2y - z = 5 \end{cases}$$

has, for example, infinitely many solutions. It is consistent, yes!!! and more than that it has infinitely many triples (s_1, s_2, s_3) that constitute solutions. When we add the last two equations we get $0 = 0$. We can therefore eliminate the last equation and reduce the question to solve:

$$\begin{cases} 3x + \ \ y - z = 1 \\ \ \ x - 2y + z = -5 \end{cases}$$

Multiplying by 2 the first equation and adding the result to the second equation we get:

$$\begin{cases} 3x + y - z = 1 \\ 7x \qquad - z = -3. \end{cases}$$

We obtain from here the solutions $x = \dfrac{z-3}{7}$, $y = -\dfrac{-4z-16}{7}$. To describe the whole set of solutions we use the parametric equations:

$$x = \frac{t-3}{7},\ y = -\frac{-4t-16}{7},\ z = t.$$

We can obtain specific numerical solutions from these equations by substituting numerical values for the parameter t. For instance $t = 0$ will give the particular solution $x = \dfrac{-3}{7}, y = \dfrac{-16}{7}, z = 0$.

One way to simplify the notation in systems, is to use matrix notation. For example, the system

$$\begin{cases} 3x + \ \ y - z = 1 \\ \ \ x - 2y + z = \frac{5}{2} \\ -x + 2y + z = 5 \end{cases}$$

can be represented using matrices as:

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{2} \\ 5 \end{pmatrix}$$

We also denote, the so-called augmented matrix of the system, containing all information to find the solutions. The augmented matrix of the system is:

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 1 & -2 & 1 & \frac{5}{2} \\ -1 & 2 & 1 & 5 \end{array}\right)$$

In general a linear system of equations can be written in matrix form as:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

with augmented matrix:

$$\left(\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{array}\right)$$

In the course of solving our systems in previous examples, we use some operations on the equations. Basically, we were multiplying a row by a number and adding the result to another row. These operations constitute the basic operations to reduce a system (or a matrix) to a more elementary form.

Definition 7. Elementary row operations on the matrix of a system are the following:

(1) Multiply a row through by a nonzero constant. ($R_i \mapsto aR_i$ for some $a \neq 0$)

(2) Interchange two rows. ($R_i \leftrightarrow R_j$)

(3) Add a constant times one row to another. ($R_i \mapsto R_i + aR_j$)

Theorem 8. *Elementary row operations determine equivalent systems of linear equations. This means, if we perform elementary row operations on a system, we obtain a system with the same solution set.*

Remark 9. The process of Gaussian elimination is an algorithm that uses elementary row operations to reduce the augmented matrix of a linear system to a form where solutions can be obtained by simple inspection. The simplified forms for the matrix are called **Row echelon form** and **Reduced row echelon form**.

Example 10. The following matrices are in **reduced row echelon** form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{5}{2} \\ 0 & 0 & 1 & 5 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The following matrices, on the other hand, are in **echelon form, but not reduced**:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & \frac{5}{2} \\ 0 & 0 & 1 & 5 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Definition 11. A matrix is in **reduced row echelon form**, when it has the following properties:

(a) If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a leading 1.

(b) If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

(c) In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

(d) Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in **row echelon form**.

Algorithm 12. Roughly speaking, the steps of the Gauss-Jordan elimination (also called row reduction) are:

(1) Interchange the top row with another row, if necessary, to bring a nonzero entry to the top.

(2) If the entry that is now at the top of the column found in Step 1 is a, multiply the first row by $\frac{1}{a}$ in order to introduce a leading 1.

(3) Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

(4) Now cover the top row in the matrix and begin again with Step 1 applied to the sub-matrix that remains. Continue in this way until the entire matrix is in row echelon form.

(5) To find the reduced row echelon form we need the following additional step. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leading 1's.

When we do only steps from 1 to 4, we are doing only Gauss elimination.

Example of row reduction or Gauss-Jordan elimination:

$$\left(\begin{array}{ccccc|c} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right) \quad R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{ccccc|c} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right) \quad R_1 \mapsto \frac{1}{2}R_1$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right) \quad R_2 \mapsto R_2 - 2R_1$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right) \quad R_2 \mapsto -\frac{1}{2}R_2$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{array}\right) \quad R_3 \mapsto R_3 - 5R_2$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad R_3 \mapsto 2R_3$$

The matrix

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

is already in echelon form. To further continue and find the reduced echelon form we do:

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad R_2 \mapsto R_2 + \frac{7}{2}R_3$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad R_1 \mapsto R_1 - 6R_3$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad R_1 \mapsto R_1 + 5R_2$$

The system of equations corresponding to the last matrix is:

$$\begin{cases} x_1 + 2x_2 \quad + 3x_4 \quad = 7 \\ \qquad\quad x_3 \qquad\quad = 1 \\ \qquad\qquad\qquad x_5 = 2 \end{cases}$$

where we can easily see the presence of infinitely many solutions using two parameters t, s for x_2 and x_4 respectively. The general solution of the system is hence:

$$S = \{x_1 = 7 - 2t - 3s,\ x_2 = t,\ x_3 = 1,\ x_4 = s, x_5 = 2\}.$$

1.1 Homogeneous linear systems of equations

A homogeneous system of linear equations looks like an array of m-linear equations in n-unknowns variables $x_1, \ldots, x_n$ where all constant terms are equal zero

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \qquad\qquad\qquad\qquad\vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0 \\ \qquad\qquad\qquad\qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

Remark 13. Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, ..., x_n = 0$ as a solution. This solution is called the trivial solution; if there are other solutions, they are called nontrivial solutions. There are only two possibilities for a homogeneous system:

(1) The system has only the trivial solution.

(2) The system has infinitely many solutions in addition to the trivial solution.

Example 14. An homogeneous system with infinitely many solutions:

$$\begin{cases} x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 = 0 \\ \qquad\qquad 5x_3 + 10x_4 \qquad = 0 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 = 0 \end{cases}$$

$$\left(\begin{array}{ccccc|c} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & 0 \\ 0 & 0 & 5 & 10 & 0 & 0 \\ 2 & 6 & 0 & 8 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The matrix on the right side is written in reduced row echelon form. The associated system of equations is:

$$\begin{cases} x_1 + 3x_2 \quad + 4x_4 + 2x_5 = 0 \\ \qquad\qquad x_3 + 2x_4 \qquad = 0 \end{cases}$$

We have infinitely many solutions using parameters r, s and t for the variables x_2, x_4 and x_5 respectively. The general solution of the system is hence:

$$S = \{x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s, x_5 = t\}.$$

Example 15. A homogeneous system with unique trivial solution:

$$\begin{cases} 2x + 3y + 2z = 0 \\ \qquad 6y - 5z = 0 \\ \qquad\qquad z = 0 \end{cases}$$

The matrix associated is:

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 0 \\ 0 & 6 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3/2 & 1 & 0 \\ 0 & 6 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 3/2 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Theorem 16. *We have the following properties:*
(1) If a homogeneous linear system has n unknowns and the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.
(2) A homogeneous linear system with more unknowns than equations has infinitely many solutions.
(3) An homogeneous system with the same amount of equations as unknowns has only trivial solution if and only if, the reduced echelon form of the augmented matrix has the identity matrix on the left side:

$$\left(\begin{array}{cccc|c} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \dots & 1 & 0. \end{array}\right)$$