GEOMETRY OF FOUR-FOLDS WITH THREE NON-COMMUTING INVOLUTIONS

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Abstract. In this paper we adapt some techniques developed for K3 surfaces, to study the geometry of a family of projective varieties in $\mathbb{P}^2_K \times \mathbb{P}^2_K \times \mathbb{P}^2_K$ defined as the intersection of a form of degree $(2,2,2)$ and a form of degree $(1,1,1)$. Members of the family will be equipped with dominant rational self-maps and we will study the actions of those maps on divisors and compute the first dynamical degrees of the composition of any pair.

1. Introduction

The study of a family of K3 surfaces presented as intersection of a $(2,2)$-form and a $(1,1)$-form in $\mathbb{P}^2 \times \mathbb{P}^2$ goes back to Joachin Wehler [13]. In the family considered by Wehler, generic members were equipped with a pair of non-commuting involutions σ_1 and σ_2 generating a group of automorphisms isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. The work of Silverman in [10] provide us with canonical height functions associated to the involutions σ_1 and σ_2 in generic members of Wehler's family. Motivated by this work of Silverman, we study dynamics on a family of varieties $\{X^{A,B}\}_{A,B}$ in $\mathbb{P}^2_K \times \mathbb{P}^2_K \times \mathbb{P}^2_K$ defined as the intersection of a form of degree $(2,2,2)$ and a form of degree $(1,1,1)$. Individual members of the family $\{X^{A,B}\}_{A,B}$ come equipped with $(2:1)$-projections $p_1, p_2, p_3 : X^{A,B} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ that generate involutions $\sigma_1, \sigma_2, \sigma_3$ on $X^{A,B}$. In this situation however the maps σ_i for $i = 1,2,3$ are not morphisms of the whole $X = X^{A,B}$, but only rational dominant maps. Still it is possible to induce maps $\sigma_i^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X)$ and $\tilde{\sigma}_i^* : NS(X)_{\mathbb{Q}} \longrightarrow NS(X)_{\mathbb{Q}}$, on divisors modulo linear and numerical equivalence. The computations with divisors in the case of three involutions are going to be similar to the computations on K3 surfaces of type $(2,2,2)$ in $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ studied by several authors, like Wang [12] and Baragar [2], [3] and [4].

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The work with rational maps leads to consider different degrees or measures of the entropy of the map. The following degree associated to the dynamics was initially studied by Arnold in [1], and particularly for dominant rational maps by Silverman in [11].

Definition 1.1. *Let X be an algebraic variety and $\varphi : X \dashrightarrow X$ a dominant rational map. The first dynamical degree of φ is*

$$\delta_\varphi = \limsup_{n\to\infty} \rho(\widetilde{\varphi^{n*}})^{1/n},$$

where $\rho(\widetilde{\varphi^{n*}})$ represents the spectral radius or maximal eigenvalue of the map $\widetilde{\varphi^{n*}} : NS(X)_{\mathbb{Q}} \longrightarrow NS(X)_{\mathbb{Q}}$.

It is also possible to extend the notion of polarization, with respect to one rational map or, more general, in the sense of Kawaguchi [9], associated to several rational maps:

Definition 1.2. *Let X be a projective variety and $\varphi_i : X \dashrightarrow X$ for $i = 1, \dots, k$ dominant rational maps. We say that the system $(X, \{\varphi_1, \dots, \varphi_k\}, \mathcal{L}, d)$ is a polarized dynamical system of k maps if there exist an ample line bundle $\mathcal{L} \in \mathrm{Pic}(X) \otimes \mathbb{R}$ such that $\bigotimes_{i=1}^k \varphi_i^* \mathcal{L} \cong \mathcal{L}^d$ for some $d > k$.*

Let X be a element of the family $\{X^{A,B}\}_{A,B}$. By studying the actions of the maps σ_1^*, σ_2^* and σ_3^* on $\mathrm{Pic}(X) \otimes \mathbb{R}$ we will exhibit a polarization for the system of three maps $\{\sigma_1, \sigma_2, \sigma_3\}$. Also, under the condition that the Picard number is the least possible value $p(X) = 3$, the first dynamical degree of any of the maps $\sigma_{ij} = \sigma_i \circ \sigma_j$ will be computed. The computations will produce the same dynamical degree as the maps on K3 surfaces [11], that is, the algebraic integer $\beta = 7 + 4\sqrt{3}$. The key point for studying the actions of the involutions on divisors will be lemma 2.3, that explains the action of push-forwards on pullbacks of generators of $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2)$. The key ingredient in finding the dynamical degree will be proposition 2.5 that describes the particular action induced on divisors by dominant rational maps that represent involutions in some open set.

2. Four dimensional Varieties with three involutions

Let $\mathbf{L}^A \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ be a family of varieties defined over a field K by a single equation linear on each variable,

$$\mathbf{L}^A = \{P \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(x,y,z) = \sum_{i,j,k=0}^{2} a_{i,j,k} x_i y_j z_k = 0\},$$

where $A = (a_{ijk})_{0\leq i,j,k\leq 2}$. A member of the family $\mathbf{L}$ comes equipped with projections

$$p_3 = p_{xy} : \mathbf{L} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2,$$

$$p_2 = p_{xz} : \mathbf{L} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2,$$

$$p_1 = p_{yz} : \mathbf{L} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2,$$

and the $\mathrm{Pic}(\mathbf{L}) \cong \mathbb{Z}^3$ from the embedding $\mathbf{L} \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Using the adjunction formula we can get its canonical line bundle

$$\omega_{\mathbf{L}} \cong \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2}(-3,-3,-3)\otimes\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2}(\mathbf{L}) = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2}(-2,-2,-2).$$

By choosing a section $Q = Q^A$ of $\mathcal{O}_{\mathbf{L}}(2,2,2)$ and consider the variety $X = \mathrm{Var}(Q)$ we get a variety with trivial canonical divisor $K_X \sim 0$. Besides, by the weak lefschetz theorem, we have an injective map $\mathbb{Z}^3 \cong \mathrm{Pic}(L) \hookrightarrow \mathrm{Pic}(X)$ and we will get three distinct classes even in $NS(X)$ and therefore a Picard number $p(X) \geq 3$.
By varying the coefficients A, B one obtains a family $\{X^{A,B}\}_{A,B}$ defined in $\mathbb{P}^2_K \times \mathbb{P}^2_K \times \mathbb{P}^2_K$ by equations

$$L(x,y,z) = \sum_{i,j,k=0}^{2} a_{i,j,k} x_i y_j z_k = 0,$$
$$Q(x,y,z) = \sum_{i,j,k,l,m,n=0}^{2} b_{i,j,k,l,m,n} x_i x_l y_j y_m z_k z_n = 0,$$

where $A = (a_{ijk})$, $B = (b_{i,j,k,l,m,n})$ and all indices are moving in the set $\{0,1,2\}$. The projections p_1, p_2, p_3 restricted to X represent generically $(2:1)$ coverings of $\mathbb{P}^2 \times \mathbb{P}^2$. Indeed when we fix two of the variables we get the intersection on $\mathbb{P}^2$ of a quadric and a line, which is general, will give two points $P_i, P_i' \in X$ for $i = 1,2,3$ and will determine involutions $\sigma_1, \sigma_2, \sigma_3 : X \dashrightarrow X$. The involutions σ_i for $i = 1,2,3$, will not be in general morphisms but just rational maps defined on certain open sets $U_i \subset X$. We are interesting in studying the dynamics of the maps σ_i, but first we should devote some time to get familiar with the geometry of $X = X^{A,B}$. We collect the coefficients of our variables using the following notation for i, j, k in the set $\{0, 1, 2\}$

$$L_k^{x,y}(x,y) = \sum_{i,j=0}^{2} a_{i,j,k} x_i y_j, \quad Q_{k,n}^{x,y}(x,y) = \sum_{i,j,l,m=0}^{2} b_{i,j,k,l,m,n} x_i x_l y_j y_m,$$
$$L_j^{x,z}(x,z) = \sum_{i,k=0}^{2} a_{i,j,k} x_i z_k, \quad Q_{i,l}^{y,z}(y,z) = \sum_{j,k,m,n=0}^{2} b_{i,j,k,l,m,n} y_j y_m z_k z_n,$$
$$L_i^{y,z}(y,z) = \sum_{j,k=0}^{2} a_{i,j,k} y_j z_k, \quad Q_{j,m}^{x,z}(x,z) = \sum_{i,k,l,n=0}^{2} b_{i,j,k,l,m,n} x_i x_l z_k z_n.$$

Suppose, with the above notation in mind, that we want to study the action of σ_3 computing the solutions $(z_0, z_1, 1)$ of the system

$$0 = L_0^{x,y} z_0 + L_1^{x,y} z_1 + L_2^{x,y},$$
$$0 = Q_{0,0}^{x,y} z_0^2 + Q_{1,1}^{x,y} z_1^2 + Q_{2,2}^{x,y} + Q_{0,1}^{x,y} z_0 z_1 + Q_{0,2}^{x,y} z_0 + Q_{1,2}^{x,y} z_1,$$

assuming that $L_1^{x,y} \neq 0$ and replacing $z_1 = \frac{-L_2^{x,y} - L_0^{x,y} z_0}{L_1^{x,y}}$ in the second equation gives $G_0^{x,y} + H_{0,2}^{x,y} z_0 + G_2^{x,y} z_0^2 = 0$ where,

$$G_0^{x,y} = (L_1^{x,y})^2 Q_{2,2}^{x,y} - L_1^{x,y} L_2^{x,y} Q_{1,2}^{x,y} + (L_2^{x,y})^2 Q_{1,1}^{x,y},$$
$$G_2^{x,y} = (L_1^{x,y})^2 Q_{0,0}^{x,y} - L_1^{x,y} L_0^{x,y} Q_{0,1}^{x,y} + (L_0^{x,y})^2 Q_{1,1}^{x,y},$$
$$H_{0,2}^{x,y} = 2 L_0^{x,y} L_2^{x,y} Q_{1,1}^{x,y} - L_0^{x,y} L_1^{x,y} Q_{1,2}^{x,y} - L_2^{x,y} L_1^{x,y} Q_{0,1}^{x,y} + (L_1^{x,y})^2 Q_{0,2}^{x,y},$$

and the map σ_3 that sends $(z_0, z_1, 1) \mapsto (z_0', z_1', 1)$ will be defined unless all the three coefficients $G_0^{x,y}, H_{0,2}^{x,y}, G_2^{x,y}$ vanish. So, we are forced, by a codimension checking, to work with rational maps $\sigma_i : X \dashrightarrow X$ and our first task will be, to locate where are these maps well defined morphisms.

Motivated by the above discussion we define for any permutation (i, j, k) of $(0, 1, 2)$ the $(4, 4)$-bi-homogeneous forms

$$G_k^{x,y} = (L_i^{x,y})^2 Q_{j,j}^{x,y} - L_i^{x,y} L_j^{x,y} Q_{i,j}^{x,y} + (L_j^{x,y})^2 Q_{i,i}^{x,y},$$
$$G_k^{y,z} = (L_i^{y,z})^2 Q_{j,j}^{y,z} - L_i^{y,z} L_j^{y,z} Q_{i,j}^{y,z} + (L_j^{y,z})^2 Q_{i,i}^{y,z},$$
$$G_k^{x,z} = (L_i^{x,z})^2 Q_{j,j}^{x,z} - L_i^{x,z} L_j^{x,z} Q_{i,j}^{x,z} + (L_j^{x,z})^2 Q_{i,i}^{x,z},$$
$$H_{i,j}^{x,y} = 2 L_i^{x,y} L_j^{x,y} Q_{kk}^{x,y} - L_i^{x,y} L_k^{x,y} Q_{jk}^{x,y} - L_j^{x,y} L_k^{x,y} Q_{ik}^{x,y} + (L_k^{x,y})^2 Q_{ij}^{x,y},$$
$$H_{i,j}^{x,z} = 2 L_i^{x,z} L_j^{x,z} Q_{kk}^{x,z} - L_i^{x,z} L_k^{x,z} Q_{jk}^{x,z} - L_j^{x,z} L_k^{x,z} Q_{ik}^{x,z} + (L_k^{x,z})^2 Q_{ij}^{x,z},$$
$$H_{i,j}^{y,z} = 2 L_i^{y,z} L_j^{y,z} Q_{kk}^{y,z} - L_i^{y,z} L_k^{y,z} Q_{jk}^{y,z} - L_j^{y,z} L_k^{y,z} Q_{ik}^{y,z} + (L_k^{y,z})^2 Q_{ij}^{y,z},$$

For any $a, b, c \in \mathbb{P}^2_K$, the fibres of the projections p_1, p_2 and p_3 will be defined as $X^z_{a,b} = p_3^{-1}(a, b) = L^z_{a,b} \cap Q^z_{a,b}$, $X^x_{b,c} = p_1^{-1}(b, c) = L^x_{b,c} \cap Q^x_{b,c}$ and $X^y_{a,c} = p_2^{-1}(a, c) = L^y_{a,c} \cap Q^y_{a,c}$; where

$$L^z_{a,b} = \{(a, b, z) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(a, b, z) = 0\},$$
$$Q^z_{a,b} = \{(a, b, z) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : Q(a, b, z) = 0$$
$$L^x_{b,c} = \{(x, b, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(x, b, c) = 0\},$$
$$Q^x_{b,c} = \{(x, b, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : Q(x, b, c) = 0$$
$$L^y_{a,c} = \{(a, y, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(a, y, c) = 0\},$$
$$Q^y_{a,c} = \{(a, y, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : Q(a, y, c) = 0.$$

Definition 2.1. *We say that a fibre $X^z_{a,b}, X^x_{b,c}$ or $X^y_{a,c}$ is degenerate if it has positive dimension.*

If the fibres $X^z_{a,b}, X^x_{b,c}$ or $X^y_{a,c}$ are non-degenerate at (a, b, c), they will consist of two points and the maps σ_1, σ_2 and σ_3 will be well defined morphisms at $(a, b, c) \in X$. Following the outline of [6] we have the following result characterizing the degenerate fibres.

Proposition 2.2. *Let $[a, b, c] \in X$.*

(1) *$X^z_{a,b}$ is degenerate if and only if*

$$G^{x,y}_0(a, b) = G^{x,y}_1(a, b) = G^{x,y}_2(a, b) = H^{x,y}_{0,1}(a, b) = H^{x,y}_{0,2}(a, b) = H^{x,y}_{1,2}(a, b) = 0.$$

(2) *$X^y_{a,c}$ is degenerate if and only if*

$$G^{x,z}_0(a, c) = G^{x,z}_1(a, c) = G^{x,z}_2(a, c) = H^{x,z}_{0,1}(a, c) = H^{x,z}_{0,2}(a, c) = H^{x,z}_{1,2}(a, c) = 0.$$

(3) *$X^x_{b,c}$ is degenerate if and only if*

$$G^{y,z}_0(b, c) = G^{y,z}_1(b, c) = G^{y,z}_2(b, c) = H^{y,z}_{0,1}(b, c) = H^{y,z}_{0,2}(b, c) = H^{y,z}_{1,2}(b, c) = 0.$$

Proof. The proof is identical to the proof of proposition 1.4 in [6]. We do the proof of (1). When we substitute $z_0 = (L - L^{x,y}_1 z_1 - L^{x,y}_2 z_2)/L^{x,y}_0$, $z_1 = (L - L^{x,y}_0 z_0 - L^{x,y}_2 z_2)/L^{x,y}_1$ and $z_2 = (L - L^{x,y}_1 z_1 - L^{x,y}_0 z_0)/L^{x,y}_2$ into Q respectively we get formulas:

$$(L^{x,y}_0)^2 Q(x, y, z) \equiv G^{x,y}_2 z_1^2 + H^{x,y}_{1,2} z_1 z_2 + G^{x,y}_1 z_2^2 \quad (mod L(x, y, z)),$$
$$(L^{x,y}_1)^2 Q(x, y, z) \equiv G^{x,y}_2 z_0^2 + H^{x,y}_{0,2} z_0 z_2 + G^{x,y}_0 z_2^2 \quad (mod L(x, y, z)),$$
$$(L^{x,y}_2)^2 Q(x, y, z) \equiv G^{x,y}_1 z_0^2 + H^{x,y}_{0,1} z_0 z_1 + G^{x,y}_0 z_1^2 \quad (mod L(x, y, z)).$$

Now, the proof is divided into two parts, depending on whether or not for the point $[a, b, c] \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ we have $L(a, b, z) \equiv 0$.
If $L(a, b, z) \equiv 0$, then $X^z_{a,b} = Q^z_{a,b}$ and the fibre is degenerate. In this case $L^{a,b}_0 = L^{a,b}_1 = L^{a,b}_2 = 0$ will force $H^{x,y}_{i,j}(a, b) = G^{x,y}_k(a, b) = 0$ and the proof is finished.

If $L(a,b,z) \neq 0$, one of the $L_i^{x,y}(a,b) \neq 0$ and the fact that $G_0^{x,y}(a,b) = G_1^{x,y}(a,b) = G_2^{x,y}(a,b) = H_{0,1}^{x,y}(a,b) = H_{0,2}^{x,y}(a,b) = H_{1,2}^{x,y}(a,b) = 0$ forces $Q(a,b,z) \equiv 0 \quad (modL(a,b,z))$ and hence $X_{a,b}^z$ is degenerate containing the entire line $L_{a,b}^z$.
If $L(a,b,z) \neq 0$ and the fibre $X_{a,b}^z$ is degenerate we must have $L_{a,b}^z \subset Q_{a,b}^z$. We are going to proof that $G_0^{x,y}(a,b) = G_1^{x,y}(a,b) = G_2^{x,y}(a,b) = H_{0,1}^{x,y}(a,b) = H_{0,2}^{x,y}(a,b) = H_{1,2}^{x,y}(a,b) = 0$. First let's do $G_0^{x,y}(a,b) = 0$. If $L_1^{x,y}(a,b) = L_2^{x,y}(a,b) = 0$, this follows from the definition, otherwise $(0, L_2^{x,y}(a,b), -L_1^{x,y}(a,b)) \in L_{a,b}^z$ and therefore must belong to $Q_{a,b}^z$, when we evaluate we get

$$0 = Q_{1,1}^{x,y}(a,b)(L_2^{x,y}(a,b))^2 - Q_{1,2}^{x,y} L_2^{x,y}(a,b) L_1^{x,y}(a,b) + Q_{2,2}^{x,y}(L_1^{x,y}(a,b))^2$$

So $G_0^{x,y}(a,b) = 0$. In a similar way we do $G_1^{x,y}(a,b) = G_2^{x,y}(a,b) = 0$. The substitution of the results $G_i^{x,y}(a,b) = 0$ in the equations and evaluations at $x = a, y = b$ will give

$$H_{1,2}^{x,y}(a,b) z_1 z_2 = H_{0,2}^{x,y}(a,b) z_0 z_2 = H_{1,0}^{x,y}(a,b) z_1 z_0 = 0$$

for all points $(z_0, z_1, z_2) \in L^z(a,b)$. If $L^z(a,b)$ is the line $z_1 = 0$, then $L_0^{x,y}(a,b) = L_2^{x,y}(a,b) = 0$ and $H_{1,2}^{x,y}(a,b) = 0$ using the definition. If $L^z(a,b)$ is the line $z_2 = 0$, then $L_1^{x,y}(a,b) = L_2^{x,y}(a,b) = 0$ and $H_{1,2}^{x,y}(a,b) = 0$ will be again equal to zero. Otherwise if $L_{a,b}^z$ is none of the lines $z_1 = 0$ or $z_2 = 0$, then $H_{1,2}^{x,y}(a,b) = 0$ from the previous line. The other cases for $H_{i,j}^{x,y}(a,b) = 0$ are solved similarly. □

We can now define open sets U_1, U_2, U_3 in such a way that the dominant rational maps $\sigma_i : X \dashrightarrow X$ are bijective morphisms

$$\sigma_i : U_i \longrightarrow U_i.$$

$$\begin{aligned} U_1 = X - \{(a,b,c) \in X : & \ G_0^{y,z}(b,c) = G_1^{y,z}(b,c) = G_2^{y,z}(b,c) = 0 \\ & H_{0,1}^{y,z}(b,c) = H_{0,2}^{y,z}(b,c) = H_{1,2}^{y,z}(b,c) = 0\}, \\ U_2 = X - \{(a,b,c) \in X : & \ G_0^{x,z}(a,c) = G_1^{x,z}(a,c) = G_2^{x,z}(a,c) = 0 \\ & H_{0,1}^{x,z}(a,c) = H_{0,2}^{x,z}(a,c) = H_{1,2}^{x,z}(a,c) = 0\}, \\ U_3 = X - \{(a,b,c) \in X : & \ G_0^{x,y}(a,b) = G_1^{x,y}(a,b) = G_2^{x,y}(a,b) = 0 \\ & H_{0,1}^{x,y}(a,b) = H_{0,2}^{x,y}(a,b) = H_{1,2}^{x,y}(a,b) = 0\}. \end{aligned}$$

The maps $\sigma_1, \sigma_2, \sigma_3$ induce maps on divisors: Let's consider Y a closed subvariety of codimension one and $\sigma_i^* Y = \overline{\sigma_i^{-1} Y}$, the Zariski closure of the pre-image. In this way we induce maps on Weil divisors, that respect linear and numerical equivalence and descend to maps

$$\sigma_i^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X) \qquad \tilde{\sigma}_i^* : NS(X)_{\mathbb{Q}} \longrightarrow NS(X)_{\mathbb{Q}}.$$

To study the action of the σ_i^* on $\mathrm{Pic}(X)$ we denote by H, H' hyperplane sections representing the two fundamental classes in $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2)$,

$$H = \{((a_0 : a_1 : a_2), (b_0 : b_1 : b_2)) \in \mathbb{P}^2 \times \mathbb{P}^2 : a_0 = 0\},$$
$$H' = \{((a_0 : a_1 : a_2), (b_0 : b_1 : b_2)) \in \mathbb{P}^2 \times \mathbb{P}^2 : b_0 = 0\}.$$

and the divisors D_x, D_y, D_z on X defined by:

$$D_x = \{P \in X : x_0 = 0\}, \quad D_y = \{P \in X : y_0 = 0\}, \quad D_z = \{P \in X : z_0 = 0\}.$$

The pullbacks of H, H' by the different projections give back the D_x, D_y, D_z,

$$p_{xy}^* H = p_3^* H = D_x, \quad p_{xy}^* H' = p_3^* H' = D_y,$$
$$p_{xz}^* H = p_2^* H = D_x, \qquad p_{xz}^* H' = p_2^* H' = D_z,$$
$$p_{yz}^* H = p_1^* H = D_y, \qquad p_{yz}^* H' = p_1^* H' = D_z.$$

Lemma 2.3. *We have the following equivalences of divisors in* $\mathrm{div}(X)$*:*
(a) $p_{1*}p_2^* H \sim 4H + 4H'$; (*b*) $p_{2*}p_1^* H \sim 4H + 4H'$;
(c) $p_{3*}p_1^* H' \sim 4H + 4H'$.

Proof. The prove of all parts will be analogous and straightforward from the definition of H, H' and the p_i's. Let's see for example the proof of (a). The pull-back $p_2^* H = \{P \in X : x_0 = 0\}$ is given by the two equations

$$L_1^{y,z} x_1 + L_2^{y,z} x_2 = 0, \qquad Q_{1,1}^{y,z} x_1^2 + Q_{1,2}^{y,z} x_1 x_2 + Q_{2,2}^{y,z} x_2^2 = 0.$$

When we project onto (y, z) we eliminate x_1, x_2 and get the equation

$$G_0^{y,z} = (L_1^{y,z})^2 Q_{2,2}^{y,z} - L_1^{y,z} L_2^{y,z} Q_{1,2}^{y,z} + (L_2^{y,z})^2 Q_{1,1}^{y,z} = 0.$$

where $G_0^{y,z}$ is a $(4,4)$-bihomogeneous form in y and z, and therefore $p_{1*}p_2^* H \sim 4H + 4H'$. □

Applying lemma 2.3 we obtain the push-forwards:

$$p_{1*}(D_x) \sim 4H + 4H', \qquad p_{2*}(D_y) \sim 4H + 4H', \qquad p_{3*}(D_z) \sim 4H + 4H',$$

and the action of the σ_i^*'s on the divisors D_x, D_y, D_z:

$$\sigma_1^*(D_x) = p_1^* p_{1*} D_x - D_x \sim 4D_y + 4D_z - D_x,$$
$$\sigma_1^*(D_y) = \sigma_1^* p_1^* H = (p_1 \circ \sigma_1)^* H = D_y,$$
$$\sigma_1^*(D_z) = \sigma_1^* p_1^* H' = (p_1 \circ \sigma_1)^* H' = D_z,$$
$$\sigma_2^*(D_x) = \sigma_2 p_2^* H = (p_2 \circ \sigma_2)^* H' = D_x,$$
$$\sigma_2^*(D_y) = p_2^* p_{2*} D_y - D_y \sim 4D_x + 4D_z - D_y,$$
$$\sigma_2^*(D_z) = \sigma_2^* p_2^* H' = (p_2 \circ \sigma_2)^* H' = D_z,$$
$$\sigma_3^*(D_x) = \sigma_3^* p_3^* H = (p_3 \circ \sigma_3)^* H = D_x,$$
$$\sigma_3^*(D_y) = \sigma_3^* p_3^* H' = (p_3 \circ \sigma_3)^* H' = D_y,$$

$$\sigma_3^*(D_z) = p_3^* p_{3*} D_z - D_z \sim 4D_x + 4D_y - D_z.$$

Using the actions of the σ_i^* we can get a polarizations by a very ample line bundle for the system of involutions $\sigma_1, \sigma_2, \sigma_3$.

Proposition 2.4. *Suppose that r_x, r_y, r_z are positive real numbers and we have the polarization by three maps*

$$\sum_i \sigma_i^*(r_x D_x + r_y D_y + r_z D_z) \sim d(r_x D_x + r_y D_y + r_z D_z),$$

in $\mathrm{Pic}(X) \otimes \mathbb{R}$. *Then $d = 9$ and $r_x = r_y = r_z = 1$.*

Proof. When we add up the actions of σ_i^* on $r_x D_x + r_y D_y + r_z D_z$, and equal that to $d(r_x D_x + r_y D_y + r_z D_z)$ for some $d > 3$, we get the system of linear equations:

$$\begin{aligned} r_x + 4r_y + 4r_z &= dr_x, \\ 4r_x + r_y + 4r_z &= dr_y, \\ 4r_x + 4r_y + r_z &= dr_z. \end{aligned}$$

The determinant is $(9-d)(3+d)^3$ and the value of $d = 9$ gives $r_x = r_y = r_z = 1$. □

Proposition 2.5. *The maps σ_i and $\sigma_{ij} = \sigma_i \circ \sigma_j$, for $i, j \in \{0, 1, 2\}$, satisfy the properties:*

(1) $(\sigma_i \circ \sigma_j)^* = \sigma_j^* \circ \sigma_i^*$,
(2) $(\sigma_{ij}^n)^* = (\sigma_{ij}^*)^n$.

Proof. In general, given two rational maps $\tau : X \dashrightarrow X$ and $\tau' : X \dashrightarrow X$ defining involutions $\tau : U_\tau \longrightarrow U_\tau$ and $\tau' : U_{\tau'} \longrightarrow U_{\tau'}$ on open sets U_τ and $U_{\tau'}$ respectively, we will have $(\tau \circ \tau')^* = \tau'^* \circ \tau^*$. Let Y be an irreducible subvariety. If $P \in \overline{\tau(Y \cap U_\tau)} \cap U_{\tau'}$, there exist a sequence $P_n \to P$, with $P_n \in \tau(Y \cap U_\tau) \cap U_{\tau'}$. Therefore $\tau'(P_n) \to \tau'(P)$ and $\tau'(P) \in \overline{\tau'(\tau(Y \cap U_\tau)) \cap U_{\tau'}}$. In other words $\tau'(\overline{\tau(Y \cap U_\tau)} \cap U_{\tau'}) \subset \overline{\tau'(\tau(Y \cap U_\tau) \cap U_{\tau'})}$, so this two sets must be equal and $(\tau \circ \tau')^* = \tau'^* \circ \tau^*$. For the first part of the theorem we take $\sigma_i = \tau$ and $\sigma_j = \tau'$. For the second part we proceed by induction and use the result to proof the induction step. If we suppose that $(\sigma_{ij}^n)^* = (\sigma_{ij}^*)^n$ is true, then $(\sigma_{ij}^*)^{n+1} = \sigma_{ij}^*((\sigma_{ij}^*)^n) = \sigma_{ij}^*((\sigma_{ij}^n)^*)$ By our result above with $\tau = \sigma_{ij}$ and $\tau' = \sigma_{ij}^n$, the last equals to $(\sigma_{ij}^{n+1})^*$. □

2.1. **Computation of dynamical degree.** In this subsection we study the action induced by the maps $\sigma_{ij} = \sigma_i \circ \sigma_j$ on the subspace $V = \mathrm{Span}(D_x, D_y, D_z)$ of $\mathrm{Pic}(X) \otimes \mathbb{R}$. As an application we will be able to get the dynamical degree of those maps for members of the family with Picard number $p(X) = 3$.

Theorem 2.6. *Let σ_{ij} be the rational dominant map $\sigma_i \circ \sigma_j : X \dashrightarrow X$. Let V be the subspace of* $\mathrm{Pic}(X) \otimes \mathbb{R}$ *spanned by D_x, D_y, D_z and consider the action of $\sigma_{ij}^{*n} : V \longrightarrow V$. The eigenvalues of $\sigma_{ij}^{*n}|V$ belong to the set $\{1, \beta^n, \beta'^n\}$, where $\beta = 7 + 4\sqrt{3}$ and $\beta' = \frac{1}{\beta}$.*

Proof. The action of the maps σ_{12}^*, σ_{31}^*, σ_{31}^*, σ_{32}^*, σ_{13}^* and σ_{23}^* with respect to that base $\{D_x, D_y, D_z\}$ is given respectively by the matrices

$$\sigma_{12}^* = \begin{pmatrix} -1 & -4 & 0 \\ 4 & 15 & 0 \\ 4 & 20 & 1 \end{pmatrix} \qquad \sigma_{13}^* = \begin{pmatrix} 15 & 0 & 4 \\ 20 & 1 & 4 \\ -4 & 0 & -1 \end{pmatrix}$$

$$\sigma_{12}^* = \begin{pmatrix} 15 & 4 & 0 \\ -4 & -1 & 0 \\ 20 & 4 & 1 \end{pmatrix} \qquad \sigma_{23}^* = \begin{pmatrix} 1 & 20 & 4 \\ 0 & 15 & 4 \\ 0 & -4 & -1 \end{pmatrix}$$

$$\sigma_{31}^* = \begin{pmatrix} -1 & 0 & -4 \\ 4 & 1 & 20 \\ 4 & 0 & 15 \end{pmatrix} \qquad \sigma_{32}^* = \begin{pmatrix} 1 & 4 & 20 \\ 0 & -1 & -4 \\ 0 & 4 & 15 \end{pmatrix}$$

With the help of SAGE we find that the six matrices are sharing the same characteristic polynomial $p = -(\lambda - 1)(\lambda^2 - 14\lambda + 1)$. The roots of $p(\lambda)$ are $\{1, \beta, \beta'\}$ with $\beta = 7 + 4\sqrt{3}$ and $\beta' = 1/\beta$, therefore all the six matrices are diagonalizable and the eigenvalues of the the powers are from the set $\{1, \beta^n, \beta'^n\}$. □

Corollary 2.7. *Suppose that the Picard number $p(X) = 3$, then the first dynamical degree $\delta_{\sigma_{ij}}$ of σ_{ij} is $\delta_{\sigma_{ij}} = \beta$.*

Proof. The divisors D_x, D_y, D_z represent three distinct classes in $NS(X)_{\mathbb{Q}}$. If the Picard number $p(X) = 3$, then we have $NS(X)_{\mathbb{Q}} \cong V_{\mathbb{Q}}$. The first dynamical degree of any of the maps σ_{ij} is:

$$\delta_{\sigma_{ij}} = \limsup_{n\to\infty} \rho((\sigma_{ij}^n)^*)^{1/n} = \limsup_{n\to\infty} \rho((\sigma_{ij}^*)^n)^{1/n} = \limsup_{n\to\infty} (\beta^n)^{1/n} = \beta,$$

where the last step comes from proposition 2.5. □

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