D-RATIO OF A CONFIGURATION

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Abstract. We extend the notion of D-ratio to a general configuration of lines and study its properties.

1. Succesive blow-ups for resolving the indeterminacy of rational maps

Let $\varphi : X \dashrightarrow X'$ be rational map between algebraic varieties. The map φ is actually a well defined morphism $\varphi : U = X \setminus I(\varphi) \to X'$ outside the closed set $I(\varphi)$ of indeterminacy of the map. The technique of blowing-up an ideal sheaf can be used to extend the rational map φ to a well defined morphism $\tilde{\varphi} : \tilde{X} \to X'$. First we need some definitions. We follow the presentation in [4].

Definition 1.1. *Let $\pi : \tilde{X} \longrightarrow X$ be a birrational morphism. We say that a closed subscheme $\mathcal{J}$ is the center of π if*

$$\tilde{X} = \mathrm{Proj}(\bigoplus_{d\geq 0} S^d),$$

where S is the ideal sheaf associated to $\mathcal{J}$.

Definition 1.2. *We say that $\pi : \tilde{X} \longrightarrow X$ is a monoidal transformation if its center is a smooth irreducible subvariety of X. We say that $\pi : \tilde{X} \longrightarrow X$ is a successive blow-up if it is the composition of monoidal transformations.*

Theorem 1.3. *(Hironaka) Let $\varphi : X \dashrightarrow X'$ be a rational map between algebraic varieties with X non-singular. Then, there is a sequence of proper varieties $X_0, \ldots, X_r$ such that*

(1) $X_0 = X$
(2) *$\pi_i : X_i \longrightarrow X_{i-1}$ is a monoidal transformation.*
(3) *φ extends to a morphism $\tilde{\varphi} : X_r \longrightarrow Y$.*
(4) *The underlying subvariety $Z(S)$ of the center $\mathcal{J}$ of the composition $\pi : X_r \longrightarrow X$ of all monoidal transformation is exactly the indeterminacy locus $I(\varphi)$ of the map φ.*

Proof. See Main Theorem II in [2]. □

Definition 1.4. *Let $\varphi : X \dashrightarrow X$ be a rational map. We say that a pair $(\tilde{X}_\varphi, \pi)$ is a resolution of singularities of φ when $\tilde{X}_\varphi$ is a successive blow-up of X and $\pi : \tilde{X}_\varphi \longrightarrow X$ is such that $\varphi \circ \pi : \tilde{X} \longrightarrow X$ extends to a morphism $\tilde{\varphi} : \tilde{X}_\varphi \longrightarrow X$. We call $\tilde{\varphi}$ a resolving morphism for φ.*

Definition 1.5. *Let $\pi : \tilde{X} \longrightarrow X$ be a birrational morphism with center $\mathcal{J}$. Let D be an irreducible divisor on X, we define the proper transform of D by $\pi^\sharp D = \overline{\pi^{-1}(D \cap U)}$, where $U = X \setminus Z(\mathcal{J})$.*

Proposition 1.6. *Let X a non-singular projective variety and $\varphi : X \dashrightarrow X'$ a rational map. Suppose that $\tilde{X}$ is a successive blowup of X as composition of monoidal transformations $\pi_i : X_i \longrightarrow X_{i-1}$. Denote by F_i the exceptional divisor of $\pi_i : X_i \longrightarrow X_{i-1}$ and take $\pi_i' = \pi_{i+1} \circ \cdots \circ \pi_r : \tilde{X} \longrightarrow X_i$. Let $E_i = (\pi_i')^\sharp F_i$ be the proper transform of F_i under π_i'. Then $\mathrm{Pic}(\tilde{X})$ is the module*

$$\mathrm{Pic}(\tilde{X}) = \mathrm{Pic}(X) \oplus E_1\mathbb{Z} \oplus \cdots \oplus E_r\mathbb{Z}.$$

Proof. The case of $r = 1$ is part of exercise II.7.9 in [1]. In general for X a noetherian scheme, and $\mathcal{E}$ a locally coherent sheaf of rank ≥ 2 on X, we have $\mathrm{Pic}(P(\mathcal{E})) = \mathrm{Pic}(\mathrm{Proj}(\oplus_{i=1}^\infty Sym^i\mathcal{E})) \backsimeq \mathrm{Pic}(X) \oplus \mathbb{Z}$. More precisely we can write, for a monoidal transformation $\pi : X' \longrightarrow X$ with exceptional divisor F, $\mathrm{Pic}(X') = \pi^\sharp \mathrm{Pic}(X) \oplus (\pi^\sharp F)\mathbb{Z}$.. Then we proceed by induction. □

2. Rational maps on the projective plane

Ins this section we are going to build the intersection matrix associated to the resolution of indeterminacy of certain maps on the projective plane. Consider a fix plane $H \subset \mathbb{P}^2$ and let $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a ratinal map on the projective plane $\mathbb{P}^2$, whose indeterminacy locus $I(\varphi)$ is contained in H. In this situation, a base for $\mathrm{Pic}(\mathbb{P}^2)$ can be taking to be $\{H\}$ and if we need r successive blow-ups to resolve the map φ, we will get in the notation of the previous section:

$$\mathrm{Pic}(\widetilde{\mathbb{P}^2}_\varphi) = \pi^\sharp H\mathbb{Z} \oplus E_1\mathbb{Z} \oplus \cdots \oplus E_r\mathbb{Z}.$$

Definition 2.1. *(Joey) Suppose that $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a rational map of degree d with indeterminacy locus $I(\varphi)$ contained in the hyperplane H. Consider a resolution of the indeterminacy $\pi : \widetilde{\mathbb{P}^2}_\varphi \longrightarrow \mathbb{P}^2$ and a resolving morphism $\tilde{\varphi} : \widetilde{\mathbb{P}^2}_\varphi \longrightarrow \mathbb{P}^2$. Assume that*

$$\pi^* H = \pi^\sharp H + a_1E_1 + \cdots + a_rE_r \quad \textit{and} \quad \tilde{\varphi}^* H = d\pi^\sharp H + a_1'E_1 + \cdots + a_r'E_r.$$

If $a'_i \neq 0$ for all $a_i \neq 0$, the D-ratio $r(H,\varphi)$ is defined as

$$r(H,\varphi) = \sup_i \{a_i/a'_i\}.$$

Remark 2.2. For the coefficients 1 and d for $\pi^\sharp H$ in $\pi^* H$ and $\tilde{\varphi}^* H$ respectively see proposition 4.5 in [4]. The defination of r is taken in such a way that we obtain an effective divisor $r\tilde{\varphi}^* H - \pi^* H$.

2.1. **One blow-up.** In the case that we need only one blow-up at a point to resolve the indeterminacy of a rational map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of degree $d > 1$ and we have intersection conditions:

$$(E_1, E_1) = -1, \quad \text{and} \quad (\pi^\sharp H, E_1) = 1.$$

Using intersection with E_1, we can solve a linear system in whole numbers to determine a and a' in the pullbacks:

$$\pi^* H = \pi^\sharp H + aE_1, \qquad \tilde{\varphi}^* H = d\pi^\sharp H + a'E_1$$

$$\begin{aligned} 0 &= 1 - a \\ 1 &= d - a', \end{aligned}$$

So $\pi^* H = \pi^\sharp H + E_1$ and $\tilde{\varphi}^* H = d\pi^\sharp H + (d-1)E_1$. The D-ratio is $r(H,\varphi) = 1/d - 1$.

2.2. **The case of two blow-ups.** Suppose that we need two successive blow-ups at points to resolve the indeterminacy of the map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of degree $d > 1$ with intersection conditions:

$$(E_2, E_2) = -1, \quad (\pi^\sharp H, E_2) = 1,$$
$$(E_1, E_1) = -q, \quad (E_2, E_1) = 1, \quad (\pi^\sharp H, E_1) = 0.$$

Using intersection with E_2 and E_1, we can solve a system in whole numbers to determine q, a, a', b and b' in the pullbacks:

$$\pi^* H = \pi^\sharp H + aE_1 + bE_2, \qquad \tilde{\varphi}^* H = d\pi^\sharp H + a'E_1 + b'E_2$$

$$\begin{aligned} 0 &= 1 + a - b, \\ 0 &= -qa + b, \\ 1 &= d + a' - b', \\ 0 &= -qa' + b', \end{aligned}$$

and we get $q = 2, a = 1, b = 2, a' = d - 1$ and $b' = 2(d-1)$, giving

$$\pi^* H = \pi^\sharp H + E_1 + 2E_2, \quad \text{and} \quad \tilde{\varphi}^* H = d\pi^\sharp H + (d-1)E_1 + 2(d-1)E_2.$$

The D-ratio of the map is $r(H,\varphi) = 1/d - 1$.

2.3. **The case of three blow-ups.** Suppose that we need three successive blow-ups at points to resolve the indeterminacy of the map $\varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of degree $d > 1$ with intersection conditions:

$$(E_3, E_3) = -1, \quad (\pi^\sharp H, E_3) = 1, \quad (E_2, E_2) = -q_2,$$
$$(E_3, E_2) = 1, \quad (E_1, E_3) = 0, \quad (E_1, E_1) = -q_1,$$
$$(\pi^\sharp H, E_1) = 0, \quad (\pi^\sharp H, E_2) = 0, \quad (E_2, E_1) = 1.$$

Using intersection respectively with E_3, E_2 and E_1, we can solve a system in whole numbers to determine $q_1, q_2, a, b, c, a', b'$ and c' in the pullbacks:

$$\pi^* H = \pi^\sharp H + aE_1 + bE_2 + cE_3, \quad \text{and} \quad \tilde{\varphi}^* H = d\pi^\sharp H + a'E_1 + b'E_2 + c'E_3,$$

$$\begin{aligned}
0 &= 1 + b - c, \\
0 &= a - q_2 b + c, \\
0 &= -q_1 a + b, \\
1 &= d + b' - c', \\
0 &= a' - q_2 b' + c', \\
0 &= -q_1 a' + b',
\end{aligned}$$

and we get two possibilities:

$$(q_1, q_2, a, b, c, a', b', c') = (2, 2, 1, 2, 3, d-1, 2(d-1), 3(d-1))$$
$$(q_1, q_2, a, b, c, a', b', c') = (1, 3, 1, 1, 4, d-1, d-1, 4(d-1))$$

The first case gives:

$$\begin{aligned}
\pi^* H &= \pi^\sharp H + E_1 + 2E_2 + 3E_3, \\
\tilde{\varphi}^* H &= d\pi^\sharp H + (d-1)E_1 + 2(d-1)E_2 + 3(d-1)E_3.
\end{aligned}$$

The second case gives:

$$\begin{aligned}
\pi^* H &= \pi^\sharp H + E_1 + E_2 + 4E_3, \\
\tilde{\varphi}^* H &= d\pi^\sharp H + (d-1)E_1 + (d-1)E_2 + 4(d-1)E_3.
\end{aligned}$$

In any case the D-ratio is again $r(H, \varphi) = 1/d - 1$.

3. The D-ratio of a configuration

In this section we extend the notion of D-ratio to a configuration of intersection. We expect to study the properties of the D-ratio in this general context and its implications for the resolution of indeterminacy of maps on $\mathbb{P}^2$.

Definition 3.1. *Suppose that A is a symmetric matrix with integer coefficients, satisfying the conditions:*

(i) $a_{i,i} < 0$ *fpr* $i > 0$ *and* $a_{0,0} = 1$,
(ii) $a_{i,j} \in \{0, 1\}$ *for* $i \neq j$.

The D-ratio associated A *and integral vectors* $c = (c_0, c_1, \ldots, c_n)$ *and* $b = (b_0, b_1, \ldots, b_n$ *is given by*

$$r = r(A, c, b) = \sup_{0 \leq i \leq n} \{a_i/a'_i\}$$

For solutions $a = (x_0, \ldots, x_n)$ *and* $x' = (x'_0, \ldots, x'_n)$ *respectively of the systems:*

$$Ax = b, \qquad\qquad Ax = c$$

Remark 3.2. In the way we have defined the r, if we consider divisors $D = x_0 D_0 + x_1 D_2 + \cdots + x_r D_r$ and $D' = x'_0 D_0 + x'_1 D_2 + \cdots + x'_r D_r$ on a projective variety X, the divisor will be effective $rD' - D \succeq 0$.

Example 3.3. For the configuration associated to the resolution of the singularity of one map of degree d using one blow-up of a point we have vectors $b = (2, 0)$ and $c = (2d - 1, 1)$ as well as the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

. The solution as before $(x_0, x_1) = (1, 1)$; $(x'_0, x'_1) = (d, d - 1)$ and the D-ratio $1/(d - 1)$.

Example 3.4. The configuration for two successive blow-ups of points to resolve the indeterminacy of a map of degree d gives self-intersection $-q_1 = -2$ and vectors $b = (3, 0, 0)$ and $c = (3d - 2, 0, 1)$. The intersection matrix is

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

. The solution as before

$$(x_0, x_1, x_2) = (1, 1, 2) \qquad (x'_0, x'_1, x'_2) = (d, d - 1, 2(d - 1))$$

and the D-ratio $1/(d - 1)$.

Example 3.5. When we need three blow-ups along points and the degree of the map is d, we can choose two sets of vectors depending on the possible values for q_1 and q_2. For $(q_1, q_2) = (2, 2)$ we get $b = (4, 0, 0, 0)$ and $c = (4d - 3, 0, 0, 1)$ and for $(q_1, q_2) = (1, 3)$ we have $b = (5, 0, 0, 0)$ and $c = (5d - 4, 0, 0, 1)$. The intersection matrix depending also on the self-intersection of E_1 and E_2 is given bellow:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -q_1 & 1 & 0 \\ 0 & 1 & -q_2 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix}$$

The D-ratio is $1/(d-1)$ for both systems of solutions

$$(x_0, x_1, x_2, x_3, x_0', x_1', x_2', x_3') = (1, 1, 2, 3, d, d-1, 2(d-1), 3(d-1))$$

and

$$(x_0, x_1, x_2, x_3, x_0', x_1', x_2', x_3') = (1, 1, 1, 4, d-1, d-1, d-1, 4(d-1)).$$

Question 3.6. *We will like to study the D-ratio $r = r(A, c, b)$. For example, we would like to ask when is the D-ratio less than one ? Under which conditions is the D-ratio more than one?*

Lemma 3.7. *If $r(A, b, c) < 1$, then $b_0 < c_0$. If in addition we know that $x_0 = 1$ and $x_0' = d$, then $b_0 < c_0 - d + 1$.*

Proof. If $r(A, b, c) < 1$ we will have $x_i < x_i'$ for all $i = 0, \ldots, n$ and therefore

$$b_0 = \sum_{i=0}^{n} a_{0.i} x_i < \sum_{i=0}^{n} a_{0,i} x_i' = c_0.$$

If we have the extra information $x_0 = 1$ and $x_0' = d$, then

$$b_0 + d - 1 = 1 + \sum_{i=1}^{n} a_{0.i} x_i + d - 1 < d + \sum_{i=1}^{n} a_{0,i} x_i' = c_0.$$

□

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