ARITHMETIC SEQUENCES AND BOOKING PROBLEM

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Abstract. We look to use some results on arithmetic progressions and properties of the greatest common divisor of numbers to implement a booking system that checks for conflicting reservations.

1. Project Description

Given two integers a and b, the greatest common divisor, denoted $\gcd(a, b)$ is the largest positive integer that divide both, a and b.

Example 1.1. The $\gcd(12, 16) = 4$ because the divisors of 12 are $\{\pm1, \pm2, \pm3, \pm4, \pm6, \pm12\}$ and the divisors or 16 are $\{\pm1, \pm2, \pm4, \pm8, \pm16\}$. We check that the largest number **in both lists** is $d = 4$.

Some properties of the greatest common divisor $\gcd(a, b)$ are:

(1) The greatest common divisor is a commutative function $\gcd(a, b) = \gcd(b, a)$.
(2) If a number c divides evenly both a and b, then c also divides the greatest common divisor $\gcd(a, b)$.
(3) If r denotes the remainder of a when divided by b, then $\gcd(a, b) = \gcd(b, r)$. This property lies at the basis of the **Euclidean Algorithm**, a very efficient way to find the common divisor.
(4) The **Euclidean Algorithm** can be used to find the greatest common divisor of two integers and to find integers x and y satisfying the **Bezout identity**: $ax + by = d$. For example in example 1.1, the $\gcd(12, 16) = 4$ and we find $12(-1) + 16(1) = 4$, or $x = -1$ and $y = 1$. There may be other solutions x, y, for example $12(3) + 16(-2) = 4$. The result of Bezout guarantees at least one pair (x, y), but we can find in fact infinitely many of such pairs.
(5) The greatest common divisor is the smallest positive d number that can be written in the form $ax + by = d$, where x and y are integers. **The only numbers** that can be written this way are the multiples of the $\gcd(a, b)$. For example $\gcd(12, 16) = 4$, therefore the equation

$$12x + 16y = n$$

has solution in the integers if and only if n is a multiple of 4.
(6) Let n be an integer multiple of the $\gcd(a, b)$. If x, y is a solution to the linear equation $ax + by = n$. Then, for any integer $t = 0, 1, -1, 2, -2, 3, -3, \ldots$, the pair $(x + bt, y - at)$ is also a solution of the equation. We can check directly:

$$a(x + bt) + b(y - at) = ax + abt + by - abt = ax + by = n.$$

This provides a precise description of the infinitely many integral solutions of the linear equation $ax + by = n$, **when n is divisible by the greatest common divisor** $\gcd(a, b)$.
(7) In general, given numbers a, b and n. the steps to solve an equation of the form $ax + by = n$ are:
   (a) If n **is not a multiple** of $d = \gcd(a, b)$, there is **no possible solutions in integers**.
   (b) If n is indeed a multiple of $d = \gcd(a, b)$, we obtain the quotient $q = n/d$ and we make use of the extended Euclid's algorithm to find x', y' such that $ax' + by' = d$.
   (c) A solution to $ax + by = n$ is then given by $(x, y) = (qx', qy')$ and in general every pair $(x, y) = (qx' + at, qx' - bt)$ will be a solution for any integer value $t = 0, 1, -1, 2, -2, \ldots$

Example 1.2. For the equation $6x + 9y = 10$, the greatest common divisor $d = \gcd(6, 9) = 3$ and 10 **is not** a multiple of 3, therefore, there are not integer numbers x, y that satisfy the equation.

On the other hand, for $6x + 9y = 15$, the 15 is a multiple of $d = \gcd(6, 9) = 3$ and we have infinitely many solutions. To find the solutions we follow the steps above. First, we find solutions x', y' (using the extended Euclidean algorithm) of the equation $6x' + 9y' = 3$, they are for instance, $x' = 2$ and $y = -1$. Now to find a solution of $6x + 9y = 15$, we multiply x', y' by 5 because $q = n/d = 15/3 = 5$ and obtain $x = 10, y = -5$. All solutions of $6x + 9y = 15$ are given by:

$$(x, y) = (10 + bt, -5 - at) = (10 + 9t, -5 - 6t) \quad \text{for} \quad t = 0, 1, -1, 2, -2, \ldots.$$

Definition 1.3. *Two numbers a, b are said to be **relatively prime** or coprime if the greatest common divisor* $\gcd(a, b) = 1$*. In other words, no positive number other that* 1*, divides evenly both: a and b.*

Example 1.4. The $\gcd(16, 45) = 1$, because the list of divisors are $\pm\{1, 2, 4, 8, 16\}$ and $\pm\{1, 3, 5, 9, 15, 45\}$ and the only positive number in the two sets is $d = 1$.

(6) Given integers a, b, the numbers $a/\gcd(a, b)$ and $b/\gcd(a, b)$ are relatively prime. The explanation is, when two numbers are divided by their highest divisor in common, the remainder two numbers, has no other positive common divisor except $d = 1$,
(7) (Bezout identity for relatively prime numbers) If a, b are relatively prime, there exist integers x, y such that $ax + by = 1$. For instance in example 1.4, the greatest divisor is 1 and we have $16(-14) + 45(5) = 1$ and $x = -14$ and $y = 5$. Because any number is a multiple of 1, **when a, b are relatively prime, every integer d can be expressed as linear combination** $ax + by = d$.

Definition 1.5. *A sequence of numbers is an **arithmetic progression** or **arithmetic sequence** if the difference between consecutive numbers is constant, or, in other words, the numbers are of the from*

$$\{a_1, a_1 + D, a_1 + 2D, a_1 + 3D, \ldots\} = \{a_1 + Dk \,|\, k = 0, 1, \ldots\}$$

for an initial value a_1 and a constant difference D.

Example 1.6. The sequence $\{5, 8, 11, 14, \ldots\}$ is an arithmetic sequence with initial value $a_1 = 5$ and difference $D = 3$.

Practice Questions:

(1) Compute $\gcd(323, 437)$ and write $\gcd(323, 437) = 323x + 437y$ for some integer numbers $x, y \in \mathbb{Z}$.
(2) Compute $\gcd(1437, 345)$ and write $\gcd(1437, 345) = 1437x + 345y$ for some $x, y \in \mathbb{Z}$.
(3) Show that the numbers $a = 811$ and $b = 2260$ are relatively prime (coprime). Find integers x, y such that $ax + by = 1$.
(4) For the arithmetic sequence $\{10, 25, 40, \ldots\}$. Find the initial term a_1 and the difference D. What number will in the position 100th of the sequence?
(5) Design and implement an algorithm to compute the $\gcd(a, b)$ and to find Bezout coefficients x, y.
(6) Design and implement an algorithm to determine if two finite sets in different arithmetic sequences:

$$L_1 = \{a_1 + Dk \,|\, k = 0, 1, \ldots, N\} \qquad L_2 = \{a'_1 + D'k \,|\, k = 0, 1, \ldots, M\}$$

have an element in common. There are several way this could be done. You could construct the two lists and check if there is a number belonging to both. We can also use ideas of arithmetic sequences together with the Bezout identity to determine if two lists have an element in common.
(7) Integrate your function from the previous part into a booking system that avoids conflicts, i.e. two customers are not allowed to select the same day for a reservation or allocation of a resource.

References

[1] Aitken, *Bezout Identity and Euclidean algorithm* available at *https://public.csusm.edu/aitken_html/m422/Handout1.pdf*
[2] J. Silverman, *A friendly introduction to Number Theory (third edition)* Pearson, (2005)
[3] Wikipedia, *Greatest Common divisor* available at *https://en.wikipedia.org/wiki/Greatest_common_divisor*
[4] Wikipedia, *Arithmetic progression* available at *https://en.wikipedia.org/wiki/Arithmetic_progression*