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Canonical metrics of commuting maps

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Abstract

Let $\varphi : X \to X$ be a map on an projective variety. It is known that whenever the map $\varphi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ has an eigenvalue $\alpha > 1$, we can build a canonical measure, a canonical height and a canonical metric associated to φ. In the present work, we establish the following fact: if two commuting maps $\varphi, \psi : X \to X$ satisfy these conditions, for eigenvalues α and β and the same eigenvector $\mathcal{L}$, then the canonical metric, the canonical measure, and the canonical height associated to both maps, are identical.

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1 Introduction

Let X be a projective variety defined over a number field K. Suppose that $\varphi : X \to X$ is a map on X, also defined over K. Assume that we can find an ample line bundle $\mathcal{L}$ on X and a number $\alpha > 1$, such that $\mathcal{L}^\alpha \cong \varphi^*\mathcal{L}$. Under this condition, we can build the canonical height $\hat{h}_\varphi$ ([3], theorem 1.1) associated to φ and $\mathcal{L}$. Under the same conditions we can find ([18], proposition 3.1.4) a canonical measure $d\mu_{\varphi,\sigma}$ for every infinite place σ of K. The canonical height and measures satisfy nice properties with respect to the map φ, for example we have $\hat{h}_\varphi \circ \varphi = \alpha \hat{h}_\varphi$ and $\varphi_* \mu_{\varphi,\sigma} = \mu_{\varphi,\sigma}$. Sometimes it happens that a whole set of maps are associated to the same canonical height function and measures. As our first example consider the collection of maps $\phi_k : \mathbb{P}^1_{\bar{\mathbb{Q}}} \to \mathbb{P}^1_{\bar{\mathbb{Q}}}$ on the Riemann Sphere, where ϕ_k is defined as $\phi_k(t) = t^k$ for $k > 1$. The line bundle $\mathcal{L} = \mathcal{O}(1)$ on $\mathbb{P}^1$ satisfies the isomorphism $\phi_k^*\mathcal{L} \cong \mathcal{L}^k$. If one builds the canonical height and measure associated to ϕ_k and $\mathcal{O}(1)$, one obtains:

(i) All ϕ_k have the same canonical height namely, the naive height h_{nv} on $\mathbb{P}^1_{\bar{\mathbb{Q}}}$. The naive height $h_{nv}(P)$ is a refined idea of the function $\sup\{|a_0|, |a_1|\}$, measuring the

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computational complexity of the projective point $P = (a_0 : a_1)$. For a precise definition see later example 2.10.

(ii) All ϕ_k have the same canonical measure, that is, the Haar measure $d\theta$ on the unit circle $S^1 \subset \mathbb{P}^1_{\mathbb{C}}$.

Similar properties are fulfilled by the collection of maps $[n] : E \to E$, representing multiplication by $n > 1$ on an elliptic curve E defined over K. If $\mathcal{L}$ is an ample symmetric line bundle on E, we have the isomorphism $[n]^*\mathcal{L} \cong \mathcal{L}^{n^2}$, along with the properties:

(i) All maps $[n] : E \to E$ share the same canonical height, that is, the Néron-Tate height $\hat{h}_{E,\mathcal{L}}$ on E. In fact this will be our definition (2.11) of the Néron-Tate height on E associated to $\mathcal{L}$. For many other interesting properties we refer to B-4 in [9].

(ii) All maps $[n] : E \to E$ have the same canonical measure, that is, the Haar measure $i/(2\,\mathrm{Im}(\tau))d\omega \wedge d\bar{\omega}$ on $E_\sigma \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$.

We observe that any two maps in each collection commute for the composition of maps. Besides, the line bundle $\mathcal{L} \in \mathrm{Pic}(X)$, suitable to make everything work, is the same within each collection. The present work establish the general fact:

Theorem 1.1. *Let X be a projective variety defined over a number field K. Suppose that two maps $\varphi, \psi : X \to X$ commute ($\varphi \circ \psi = \psi \circ \varphi$) and satisfy the following property: For some ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ and natural numbers $\alpha, \beta > 1$, we have $\varphi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\alpha$ and $\psi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\beta$, then we have $\hat{h}_\varphi = \hat{h}_\psi = \hat{h}_{\varphi\circ\psi}$ and $d\mu_{\varphi,\sigma} = d\mu_{\psi,\sigma} = d\mu_{\varphi\circ\psi,\sigma}$.*

This result is known in dimension one, a proof can found for example in [6]. Also it is a well known fact [9], that commuting maps in a projective variety must share the same canonical height. The main feature of the present work it is to obtain all these results from the equality of the canonical metrics. Given a ample line bundle $\mathcal{L}$ on X, it was an original idea of Arakelov [1] to put (smooth) metrics on $\mathcal{L}_\sigma = \mathcal{L} \otimes_\sigma \mathbb{C}$ over all places σ of K at infinity. This gave rise to heights as intersection numbers and curvature forms at infinity. In was then an idea of Zhang [17] to look for suitable metrics at all places v of K. In presence of the dynamics $\varphi : X \to X$, the line bundle $\mathcal{L}$ on X can be endowed [17] with very special metrics $\|.\|_{\varphi,v}$ on $\mathcal{L}_v$ that satisfy the functional equation

$$\|.\|_{\varphi,v} = (\phi^*\varphi^*\|.\|_{\varphi,v})^{1/\alpha},$$

whenever we have an isomorphism $\phi : \mathcal{L}^\alpha \xrightarrow{\sim} \varphi^*\mathcal{L}$. The canonical height and the canonical measure will be defined (definitions 2.6 and 2.9) depending only on the metric $\|.\|_\varphi$. The equality of canonical heights and measure for commuting maps is a consequence of the following result:

Theorem 1.2. *Suppose that two maps $\varphi, \psi : X \to X$ commute, and for some ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ we have $\varphi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\alpha$ and $\psi^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^\beta$ for some numbers $\alpha, \beta > 1$, then $\|.\|_\varphi = \|.\|_\psi$.*

Towards the end of the paper we discuss maps on $\mathbb{P}^1$ arising as projections of maps on elliptic curves with complex multiplication. We study branch points and present examples of commuting maps on the Riemann sphere.
For the work in the other direction, namely, if the canonical heights h_φ and h_ψ are equal, what can we say about the maps φ and ψ, we refer to the work of Kawaguchi and Silverman [10]. They completely characterized in Theorem 1 and Theorem 2, which functions could be added to each collection of commuting maps at the beginning of this introduction.

2 Canonical heights and canonical measures

2.1 Canonical metrics

Consider the projective variety X defined over a number field K, a map $\varphi : X \to X$ defined over K, and an ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ such that $\phi : \mathcal{L}^\alpha \xrightarrow{\sim} \varphi^*\mathcal{L}$ for some $\alpha > 1$. This situation will be called [18] a polarized dynamical system $(X, \varphi, \mathcal{L}, \alpha)$ on X defined over K.
For each place v of K, denote by K_v the v-adic completion of K. Assume that for every place v of K we have chosen a continuous and bounded metric $\|.\|_v$ on $\mathcal{L}_v = \mathcal{L} \otimes_K K_v$. The following proposition is proposition 2.2 in [17]:

Proposition 2.1. *The sequence defined recurrently by $\|.\|_{v,1} = \|.\|_v$ and $\|.\|_{v,n} = (\phi^*\varphi^*\|.\|_{v,n-1})^{1/\alpha}$ for $n > 1$, converges uniformly on $X(\bar{K}_v)$ to a metric $\|.\|_{v,\varphi}$ on $\mathcal{L}_v$. The metric $\|.\|_{v,\varphi}$ is the unique bounded and continuous metric satisfying the equation $\|.\|_{\varphi,v} = (\phi^*\varphi^*\|.\|_{\varphi,v})^{1/\alpha}$.*

Proof. Denote by h the continuous function $\log \frac{\|.\|_2}{\|.\|_1}$ on $X(\bar{K}_v)$. Then

$$\log \|.\|_n = \log \|.\|_1 + \sum_{k=0}^{n-2} \left(\frac{1}{\alpha} \phi^* \varphi^* \right)^k h.$$

Since $\|(\frac{1}{\alpha}\phi^*\varphi^*)^k h\|_{\sup} \leq (\frac{1}{\alpha})^k \|h\|_{\sup}$, it follows that the series given by the expression $\sum_{k=0}^{\infty} (\frac{1}{\alpha}\phi^*\varphi^*)^k h$, converges absolutely to a bounded and continuous function h^v on $X(\bar{K}_v)$. Let $\|.\|_{\varphi,v} = \|.\|_1 \exp(h^v)$, then $\|.\|_n$ converges uniformly to $\|.\|_{\varphi,v}$ and its not hard to check that $\|.\|_{\varphi,v}$ satisfies $\|.\|_{\varphi,v} = (\phi^*\varphi^*\|.\|_{\varphi,v})^{1/\alpha}$. If another bounded and continuous metric $\|.\|'_{\varphi,v}$ on $\mathcal{L}_v$ satisfies the equation $\|.\|'_{\varphi,v} = (\phi^*\varphi^*\|.\|'_{\varphi,v})^{1/\alpha}$, the bounded function $g = \log(\|.\|_{\varphi,v}/\|.\|'_{\varphi,v})$ will satisfy $g = (\phi^*\varphi^*/\alpha)g$ and therefore $\|g\|_{\sup} = \|g\|_{\sup}/\alpha$ forces $g \equiv 0$. □

Definition 2.2. The metric $\|.\|_{v,\varphi}$ is called the canonical metric on $\mathcal{L}_v$ relative to φ.

Example 2.3. Consider the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n_{\bar{\mathbb{Q}}}$ and the rational map $\phi_k : \mathbb{P}^n_{\bar{\mathbb{Q}}} \to \mathbb{P}^n_{\bar{\mathbb{Q}}}$ given by the expression $\phi_k(T_0 : ... : T_n) = (T_0^k : ... : T_n^k)$. The Fubini-Study metric

$$\|(\lambda_0 T_0 + ... + \lambda_n T_n)(a_0 : ... : a_n)\|_{FS} = \frac{|\sum \lambda_i a_i|}{\sqrt{\sum_i a_i^2}}$$

is a smooth metric on $\mathcal{L}_{\mathbb{C}}$. If we take $\|.\|_1 = \|.\|_{FS}$ as our metric at infinity, the limit metric we obtain is

$$\|(\lambda_0 T_0 + ... + \lambda_n T_n)(a_0 : ... : a_n)\|_{nv} = \frac{|\sum \lambda_i a_i|}{\sup_i(|a_i|)}.$$

Example 2.4. Suppose that $X = E$ is an elliptic curve and assume that $[n] : E \to E$ is denoting the multiplication by $n > 1$ on E. As a consequence of the theorem of the cube, the ample symmetric line bundle $\mathcal{L}$ on E satisfies $\phi : [n]^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}^{n^2}$. The canonical metric is the metric of the cube discussed in [11] and suitable to make ϕ an isomorphism of metrized line bundles.

The following result relates the canonical metrics associated to commuting maps. It represents the main result of this paper.

Theorem 2.5. *Let $(X, \varphi, \mathcal{L}, \alpha)$ and $(X, \psi, \mathcal{L}, \beta)$ be two polarized systems on X defined over K. Suppose that the maps φ and ψ satisfy $\varphi \circ \psi = \psi \circ \varphi$, then $\|.\|_\varphi = \|.\|_\psi$.*

Proof. The key idea is that the canonical metric associated to a morphism does not depend on the metric we start the iteration with, as a consequence of the uniqueness of the canonical metric in proposition 2.1. Let v be a place of K and let $s \in \mathcal{L}_v(x)$ for $x \in X_v$. We are going to consider two metrics $\|.\|_{v,1} = \|.\|_\varphi$ and $\|.\|'_{v,1} = \|.\|_\psi$ on the line bundle $\mathcal{L}_v$. By our definition of canonical metric for φ, we can start with $\|.\|'_{v,1}$ and obtain $\|s(x)\|_\varphi = \lim_{k\to\infty} \|(\phi_k s^{\alpha^k})(\varphi^k(x))\|_\psi^{1/\alpha^k}$, where $\phi_k : \mathcal{L}^{\alpha^k} \xrightarrow{\sim} \varphi^{*k}\mathcal{L}$. Also by our definition of canonical metric for ψ starting with $\|.\|_{v,1} = \|.\|_\varphi$ we get $\|s(x)\|_\psi = \lim_{l\to\infty} \|(\Psi_l s^{\beta^l})(\psi^l(x))\|_{v,1}^{1/\beta^l}$, where $\Psi_l : \mathcal{L}^{\beta^l} \xrightarrow{\sim} \psi^{*l}\mathcal{L}$. Using the uniform convergence and the commutativity of the maps we get,

$$\begin{aligned}\|s(x)\|_\varphi &= \lim_{k\to\infty} \lim_{l\to\infty} \|(\Psi_l(\phi_k s^{\alpha^k})^{\beta^l})(\psi^l \circ \varphi^k(x))\|_{v,1}^{1/\beta^l\alpha^k} \\ &= \lim_{l\to\infty} \lim_{k\to\infty} \|(\phi_k(\Psi_l s^{\beta^l})^{\alpha^k})(\varphi^k \circ \psi^l(x))\|_{v,1}^{1/\alpha^k\beta^l} = \|s(x)\|_\psi,\end{aligned}$$

where the identity $\phi_k(\Psi_l s^{\beta^l})^{\alpha^k} = \Psi_l(\phi_k s^{\alpha^k})^{\beta^l}$ is a consequence of the fact that φ^* and ψ^* are group homomorphisms on $(\mathrm{Pic}(X), \otimes)$ and we have a commutative diagram

$$\begin{array}{ccccccccc} \psi^{*l} \circ \varphi^{*k}\mathcal{L} & \xleftarrow{\phi_k} & (\psi^{*l}\mathcal{L})^{\alpha^k} & \xleftarrow{\alpha^k} & (\psi^{*l}\mathcal{L}) & \xleftarrow{\Psi_l} & \mathcal{L}^{\beta^l} & \xleftarrow{\beta^l} & \mathcal{L} \\ \| & & & & & & & & \| \\ \varphi^{*k} \circ \psi^{*l}\mathcal{L} & \xleftarrow{\Psi_l} & (\varphi^{*k}\mathcal{L})^{\beta^l} & \xleftarrow{\beta^l} & (\varphi^{*k}\mathcal{L}) & \xleftarrow{\Phi_k} & \mathcal{L}^{\alpha^k} & \xleftarrow{\alpha^k} & \mathcal{L} \end{array}$$

□

2.2 Canonical measures

Let X be a n-dimensional projective variety defined over a number field K and suppose that $(X, \varphi, \mathcal{L}, \alpha)$ is a polarized dynamical system defined over K. Let σ be a place of K over infinity. We can consider the morphism $\varphi \otimes_\sigma \mathbb{C} : X_\sigma \to X_\sigma$ on the complex variety $X_\sigma = X \otimes_\sigma \mathbb{C}$. Associated to φ and σ we also have the canonical metric $\|.\|_{\varphi,\sigma}$ and therefore the distribution $c_1(\mathcal{L}, \|.\|_{\varphi,\sigma}) = \frac{1}{(\pi i)}\partial\overline{\partial}\log\|s_1(P)\|_{\varphi,\sigma}$, analogous to the first Chern form in the smooth case. It can be proved that $c_1(\mathcal{L}, \|.\|_{\varphi,\sigma})$ is a positive current in the sense of Lelong, and following [5] we can define the n-product

$$c_1(\mathcal{L}, \|.\|_{\varphi,\sigma})^n = c_1(\mathcal{L}, \|.\|_{\varphi,\sigma})...c_1(\mathcal{L}, \|.\|_{\varphi,\sigma}),$$

which represents a measure on X_σ.

Definition 2.6. The measure $d\mu_{\varphi,\sigma} = c_1(\mathcal{L}_\sigma, \|.\|_{\varphi,\sigma})^n/\mu_{\varphi,\sigma}(X)$, where we are denoting $\mu_{\varphi,\sigma}(X) = \int_X c_1(\mathcal{L}_\sigma, \|.\|_{\varphi,\sigma})^n$, is called the canonical measure associated to φ and σ. Once we have fixed $\mathcal{L}$, it depends only on the metric $\|.\|_{\varphi,\sigma}$.

Example 2.7. Consider the rational map $\phi_k : \mathbb{P}^n_{\overline{\mathbb{Q}}} \to \mathbb{P}^n_{\overline{\mathbb{Q}}}$ given by $\phi_k(T_0 : ... : T_n) = (T_0^k : ... : T_n^k)$. The canonical measure $d\mu_{\phi_k}$ is the normalized Haar measure on the n-torus $S^1 \times ... \times S^1$.

Example 2.8. Let E be an elliptic curve defined over a number field K, $\mathcal{L}$ a symmetric line bundle on E and $[n] : E \to E$ the multiplication by $n > 1$ on E. The canonical measure associated to this map can be proved to be [11] the normalized Haar measure on E_σ.

2.3 Canonical heights as intersection numbers

For a regular projective variety X of dimension n and Z a subvariety of dimension p, the classical theory of intersection ([13], [8]) defines the intersection $c_1(\mathcal{L}_1)...c_1(\mathcal{L}_p)|Z$ of the classes $c_1(\mathcal{L}_i)$ associated to line bundles $\mathcal{L}_i$ on X, when $0 < i \leq p$.
For the purpose of defining the arithmetic intersection, we want to assume that X is an arithmetic variety of dimension $n+1$, that is, given a number field K, there exists a map $f : X \to \mathrm{Spec}(\mathcal{O}_K)$, flat, projective and of finite type over $\mathrm{Spec}(\mathcal{O}_K)$. For a cycle Z of dimension $p+1$ we can define (see for example [4], [2], [14], [1], [15] or [16]) the arithmetic intersection number $c_1(\tilde{\mathcal{L}}_1)...c_1(\tilde{\mathcal{L}}_{p+1})|Z$ of the classes $c_1(\tilde{\mathcal{L}}_i)$ of hermitian line bundles $\tilde{\mathcal{L}}_i = (\mathcal{L}_i, \|.\|)$ on X. The fact that $\tilde{\mathcal{L}}_i$ are hermitian line bundles, means that, for each place σ at infinity, the line bundle $\mathcal{L}_{i,\sigma} = \mathcal{L}_i \otimes_\sigma \mathbb{C}$ is equipped with a smooth and conjugation-invariant metric $\|.\|_{\sigma,i}$ over $X_\sigma = X \otimes_\sigma \mathbb{C}$. The numbers $c_1(\tilde{\mathcal{L}}_1)...c_1(\tilde{\mathcal{L}}_{p+1})|Z$ prove to be the appropriate theory of intersection in the particular case of arithmetic varieties, adding places over infinity allows us to recover the desirable properties of the classical intersection numbers of varieties over fields.

The last step in the theory of intersection is actually the one that plays the more important role in our definition of the canonical height associated to a morphism. Suppose that X is a regular variety of dimension n defined over a number field K, and $\bar{\mathcal{L}}_i = (\mathcal{L}_i, \|.\|_i)_v$ $(i = 1, .., p+1)$ are metrized line bundles on X. Assume also that the $\mathcal{L}_i$ are equipped with semi-positive metrics over all places v (not just at infinity as before) in the sense of [17]. Such line bundles are called adelic metrized line bundles and will be denoted following [17], we can define the adelic intersection number $\hat{c}_1(\bar{\mathcal{L}}_1|Z)...\hat{c}_1(\bar{\mathcal{L}}_{p+1}|Z)$ over a p-cycle Z in X. The adelic intersection number is in fact a limit of classical numbers $c_1(\tilde{\mathcal{L}}_1)...c_1(\tilde{\mathcal{L}}_{p+1})|\bar{Z}$ once the notion of converge is established. The numbers $\hat{c}_1(\bar{\mathcal{L}}_1|Z)...\hat{c}_1(\bar{\mathcal{L}}_{p+1}|Z)$ satisfy again nice properties, they are multilinear in each of the $\bar{\mathcal{L}}_i$ and satisfy a projection formula $\hat{c}_1(f^*\bar{\mathcal{L}}_1|Z)...\hat{c}_1(f^*\bar{\mathcal{L}}_{p+1}|Z) = \hat{c}_1(\bar{\mathcal{L}}_1|f_*(Z))...\hat{c}_1(\bar{\mathcal{L}}_{p+1}|f_*(Z))$, whenever we have a map $f : Y \to X$ and Z is a p-cycle in Y. We are interested in a particular case of this situation. Suppose that we are in the presence of a polarized dynamical system $(X, \varphi, \mathcal{L}, \alpha)$, in this situation the canonical metric $\|.\|_\varphi$ of 2.1 represent a semipositive adelic metric on $\mathcal{L}$, (again we refer to [17]) and we can define the canonical height associated to $(\mathcal{L}, \|.\|_\varphi)$ as an arithmetic intersection number.

Definition 2.9. Let K' be an extension of K. The canonical height $\hat{h}_\varphi(Z)$ of a p-cycle Z in $X(K')$ is defined as

$$\hat{h}_\varphi(Z) = \frac{\hat{c}_1(\bar{\mathcal{L}}_{K'}|Z)^{p+1}}{[K' : \mathbb{Q}](\dim(Z) + 1)c_1(\mathcal{L}_{K'}|Z)^p}.$$

It depends only on $(\mathcal{L}, \|.\|_\varphi)$, where $\|.\|_\varphi$ is actually representing a collection of canonical metrics over all places of K.

Example 2.10. Consider the map $\phi_k : \mathbb{P}^n_{\bar{\mathbb{Q}}} \to \mathbb{P}^n_{\bar{\mathbb{Q}}}$ given by the formula $\phi_k(T_0 : ... : T_n) = (T_0^k : ... : T_n^k)$. Assuming that $k > 1$, the canonical height associated to ϕ_k and $\mathcal{L} = \mathcal{O}(1)$ is called the naive height h_{nv} on $\mathbb{P}^n_{\bar{\mathbb{Q}}}$. If $P = [t_0 : ... : t_n]$ is a point in $\mathbb{P}^n_{K'}$ we have,

$$h_{nv}([t_0 : ... : t_n]) = \frac{1}{[K' : \mathbb{Q}]} \log \prod_{\text{places } v \text{ of } K'} \sup(|t_0|_v, ..., |t_n|_v)^{\mathrm{N}_v},$$

where $\mathrm{N}_v = [K'_v : \mathbb{Q}_w]$ and w is the place of $\mathbb{Q}$ such that $v \mid w$.

Definition 2.11. Let E be an elliptic curve and $\mathcal{L}$ an ample symmetric line bundle on E. The canonical height associated to $[n] : E \to E$ and $\mathcal{L}$ is called the Néron-Tate height $\hat{h}_{E,\mathcal{L}}$ associated to $\mathcal{L}$ on E. The fact that it is independent of n, will be a consequence of proposition 2.12.

The collection of maps $\{\phi_k\}_{k>1}$ on $\mathbb{P}^n$ and the collection $\{[n]\}_{n>1}$ on a given elliptic curve E, share two important properties: the maps within each collection commute, and share the same canonical height and canonical measure. The following result establishes a general fact about canonical heights and canonical measures of commuting maps on a projective variety X.

Theorem 2.12. *Let $(X, \varphi, \mathcal{L}, \alpha)$ and $(X, \psi, \mathcal{L}, \beta)$ be two polarized systems on X defined over K. Suppose that the maps φ and ψ satisfy $\varphi \circ \psi = \psi \circ \varphi$, then $\hat{h}_\varphi = \hat{h}_\psi = \hat{h}_{\varphi\circ\psi}$ and $d\mu_{\varphi,\sigma} = d\mu_{\psi,\sigma} = d\mu_{\varphi\circ\psi,\sigma}$ for all σ.*

Proof. This is a consequence of our definitions of canonical measure 2.6, canonical height as intersection numbers 2.9 and proposition 2.5. □

Corollary 2.13. *Suppose that two maps $\varphi, \psi : \mathbb{P}^1 \to \mathbb{P}^1$, satisfy the hypothesis of the previous proposition, then the two maps have the same Julia set.*

Proof. The Julia set of a map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is nothing but the closure in $\mathbb{P}^1$ of the set of repelling periodic points. For details we refer to definition 2.2 in [12]. Now, the corollary is a consequence of proposition 2.12 and proposition 7.2 in [12]. For a similar result on $\mathbb{P}^n$ we refer to [7]. □

3 Elliptic Curves and examples

This section illustrates examples of commuting maps on $\mathbb{P}^1$. They all share one thing in common: being induced in some sense by endomorphisms on elliptic curves. Consider an elliptic curve E defined over the number field K by a Weierstrass equation $y^2 = p(x)$. Suppose that E admits multiplication by the algebraic number λ. Let's denote by $N(\lambda)$ the norm of λ and let $E_\sigma = E \otimes_\sigma \mathbb{C}$ for each place σ at infinity. It is a classical fact of the theory of elliptic functions and complex multiplication, that there exist polynomials $P(z)$ and $Q(z)$, with $\deg(P) = \deg(Q) + 1 = N(\lambda)$, such that $\wp(\lambda z) = P(\wp(z))/Q(\wp(z))$, where $\wp$ is denoting the Weierstrass $\wp$-function on E_σ. We have therefore a map φ_λ on $\mathbb{P}^1$ and a commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\lambda} & E \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \mathbb{P}^1 & \xrightarrow{\varphi_\lambda} & \mathbb{P}^1 \end{array}$$

where π is denoting the quotient map $E \to \mathbb{P}^1$. The line bundle $\mathcal{L}_0 = \mathcal{O}(1)$ on $\mathbb{P}^1$ satisfy $\varphi_\lambda^* \mathcal{L}_0 \xrightarrow{\sim} \mathcal{L}^{N(\lambda)}$ and the same equality holds for the ample symmetric line bundle $\pi^*\mathcal{L}_0$ on E.

Proposition 3.1. *Consider an elliptic curve E defined over the number field K by a Weierstrass equation $y^2 = p(x)$. Suppose that E admits multiplication by the algebraic number λ of norm $N(\lambda) > 1$, then*

(i) $\hat{h}_{E,\pi^*\mathcal{L}_0}(P) = \hat{h}_{E,\pi^*\mathcal{L}_0,\lambda}(P) = \hat{h}_{\varphi_\lambda}(\pi(P))$ *for any point P on E.*

(ii) Suppose that $E_\sigma \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, then the canonical measure on $\mathbb{P}^1$ associated to φ_λ and σ is

$$d\mu_{\varphi_\lambda}(z) = \frac{idz \wedge d\bar{z}}{2\operatorname{Im}(\tau)|p(z)|}.$$

Proof. The commutativity of the maps $[\lambda] : E \to E$ and $[2] : E \to E$, together with theorem 2.12, give the equality $\hat{h}_{E,\pi^*\mathcal{L}_0} = \hat{h}_{E,\pi^*\mathcal{L}_0,\lambda}$. The equality $\hat{h}_{E,\pi^*\mathcal{L}_0,\lambda}(P) = \hat{h}_{\varphi_\lambda}(\pi(P))$ is a consequence of the projection formula for the intersection numbers and the definition of the canonical height.
For (ii) consider the Haar measure $i/2d\omega \wedge d\bar{\omega}$ on E, normalized by $\operatorname{Im}(\tau)$. If $\wp$ denote the Weierstrass function and $z = \wp(\omega)$, we have

$$\frac{id\omega \wedge d\bar{\omega}}{2\operatorname{Im}(\tau)} = \frac{idz \wedge d\bar{z}}{2|\wp'(w)|^2 \operatorname{Im}(\tau)} = \frac{idz \wedge d\bar{z}}{2|y^2| \operatorname{Im}(\tau)} = \frac{idz \wedge d\bar{z}}{2|p(z)| \operatorname{Im}(\tau)},$$

which gives the result we wanted to prove. □

Remark 3.2. If the elliptic curve E admits multiplication by the numbers λ and δ, then $\varphi_\lambda \circ \varphi_\delta = \varphi_\delta \circ \varphi_\lambda$.

Remark 3.3. The height $\hat{h}_E = \hat{h}_{E,\pi^*\mathcal{L}_0}$ is also characterized by being the Weil height associated to $\pi^*\mathcal{L}_0 \cong \mathcal{O}_E(2[0])$ satisfying $\hat{h}_E(\lambda.(x,y)) = N(\lambda)\hat{h}_E(x,y)$. Condition (i) can be proved by checking that $\hat{h}_{\varphi_\lambda} \circ \pi$ satisfies this characterization.

Example 3.4. Consider an elliptic curve E given by Weierstrass equation $E : y^2 = p(x)$. For $\lambda = 2$ we have

$$\varphi_2(z) = \frac{(p'(z))^2 - 8zp(z)}{4p(z)}.$$

Example 3.5. Let's consider some examples of elliptic curves with complex multiplication:

$\operatorname{Im}(\tau) = 1$: The elliptic curve $E_1 : y^2 = x^3 + x$ admits multiplication by $\mathbb{Z}[i]$. The multiplication by i morphism can be written in x, y coordinates as $[i](x,y) = (-x, iy)$. The two maps

$$\varphi_{1+i}(z) = \frac{1}{(1+i)^2}\frac{z^2+1}{z} \qquad \varphi_{1-i}(z) = -\frac{1}{(1+i)^2}\frac{z^2+1}{z}$$

commute, and their composition satisfies

$$\varphi_{1+i}(\varphi_{1-i}(z)) = \varphi_{1-i}(\varphi_{1+i}(z)) = \varphi_2(z) = \frac{z^4 - 2z^2 + 1}{4(z^3+z)}.$$

The canonical height and measure are:

$$\hat{h}(z) = h_{E_1}(z, \pm\sqrt{z^3+z}) \qquad d\mu_{E_1}(z) = \frac{idz \wedge d\bar{z}}{2|z^3+z|}$$

Other examples of maps attached to E_1 are

$$\varphi_{1+2i}(z)=\frac{(-3-4i)z(z^2+1+2i)^2}{(5z^2+1-2i)^2}\quad \varphi_{1-2i}(z)=\frac{(3+4i)z(z^2+1+2i)^2}{(5z^2+1-2i)^2}$$
$$\varphi_{2+i}(z)=\frac{(3-4i)z(z^2+1-2i)^2}{(5z^2+1+2i)^2}\quad \varphi_{2-i}(z)=\frac{(-3+4i)z(z^2+1-2i)^2}{(5z^2+1+2i)^2}.$$

$\mathrm{Im}(\tau)=\sqrt{3}/2$: The curve $E_2 : y^2=x^3+1$ admits multiplication by the ring $\mathbb{Z}[\rho]$ where $\rho=(\sqrt{-3}-1)/2$. The multiplication by ρ can be expressed in x,y coordinates as $[\rho](x,y)=(\rho x,y)$. An example of commuting maps coming from E_2 is

$$\varphi_{\sqrt{-3}}(z)=\frac{-(z^3+4)}{3z^2}\qquad \varphi_{\sqrt{-3}\rho}(z)=\frac{-\rho(z^3+4)}{3z^2}$$
$$\varphi_{\sqrt{-3}}\circ\varphi_{\sqrt{-3}\rho}(z)=\varphi_{\varepsilon}(z)=\frac{(z^9-96z^6+48z^3+64)}{9\rho z^2(z^3+4)^2},$$

where $\varepsilon=(-3\sqrt{-3}+3)/2$. The canonical measure associated to the three maps is

$$d\mu_{E_2}(z)=\frac{\sqrt{3}idz\wedge d\bar{z}}{3|z^3+1|}.$$

The branch points of the maps φ_λ are closely related to the 2-torsion points on the elliptic curve E.

Lemma 3.6. *A branch point for φ_λ belongs to the image by π of the 2-torsion points on E.*

Proof. Let P be a points on E such that $\pi(P)$ is a branch point of the map φ_λ. Then the set $\varphi_\lambda^{-1}(\pi(P))=\{\pi(Q)|\lambda Q=\pm P\}$ has cardinality strictly smaller than $N(\lambda)$. Therefore there are two points $Q\neq -Q\in E$ such that $\lambda Q=-\lambda Q$ and consequently $0=2\lambda Q=2P$. □

The image by π of a 2-torsion is not necessarily a branch point of φ_λ. Let d be a positive square free integer. Assume that the elliptic curve $\mathbb{C}/\mathbb{Z}+\sqrt{-d}\mathbb{Z}$, admits multiplication by $\lambda=a+b\sqrt{-d}$ for integers a,b. Suppose that $P_0=0$, $P_1=1/2$, $P_2=\sqrt{-d}/2$ and $P_3=1/2+\sqrt{-d}/2$ denote the 2-torsion points on E. Denote by r_j the amount of pre-images of the point $\pi(P_j)$, that is, the cardinality of the set $\varphi_\lambda^{-1}(\pi(P_j))$. Also denote by s_j the amount of 2-torsion points that hit P_j, that is, the cardinality of the intersection $\{Q\in E|Q=-Q\}\cap\{Q\in E|\lambda Q=P_j\}$.

Lemma 3.7. $r_j=(N(\lambda)+s_j)/2$.

Proof. Fix a 2-torsion point of $P_j\in E$. If $Q\in E$ is a solution of the equation $\lambda Q=P_j$, then $-Q$ is also a solution. So, up to torsion points, by counting the solutions of $\lambda Q=P_j$, we count twice the elements of the set $\varphi_\lambda^{-1}(\pi(P_j))=\{\pi(Q)|\lambda Q=P_j=-P_j\}$. As a consequence $r_j=(N(\lambda)-s_j)/2+s_j=(N(\lambda)+s_j)/2$. □

One can observe that for $\lambda = 2$, the point $\pi(P_0)$ is not a branch point of φ_2, in fact we have $\lambda P_j = P_0$ for all $j = 0, .., 3$. On the other hand for the multiplication by $\lambda = 1 + 2i$ on E_1, all points $\pi(P_0), \pi(P_1), \pi(P_2), \pi(P_3)$ are branch points of φ_{1+2i}. The results show that the image by π of a 2-torsion points is usually a branch point for the map φ_λ.

$$\lambda P_1 = \frac{a + b\sqrt{-d}}{2} = \begin{cases} P_0 = 0, & \text{if } (a,b) \equiv (0,0) \pmod 2; \\ P_2 = \sqrt{-d}/2, & \text{if } (a,b) \equiv (0,1) \pmod 2; \\ P_1 = 1/2, & \text{if } (a,b) \equiv (1,0) \pmod 2; \\ P_3 = (1 + \sqrt{-d})/2 & \text{if } (a,b) \equiv (1,1) \pmod 2; \end{cases}$$

$$\lambda P_2 = \frac{a\sqrt{-d} - bd}{2} = \begin{cases} P_0 = 0, & \text{if } (a,bd) \equiv (0,0) \pmod 2; \\ P_2 = \sqrt{-d}/2, & \text{if } (a,bd) \equiv (1,0) \pmod 2; \\ P_1 = 1/2, & \text{if } (a,bd) \equiv (0,1) \pmod 2; \\ P_3 = (1 + \sqrt{-d})/2 & \text{if } (a,bd) \equiv (1,1) \pmod 2; \end{cases}$$

$$\lambda P_3 = \frac{a - bd + (a+b)\sqrt{-d}}{2} = \begin{cases} P_0 = 0, & \text{if } (a,b) \equiv (0,0) \pmod 2; \\ P_3 = (1 + \sqrt{-d})/2, & \text{if } (a,b) \equiv (1,0) \pmod 2; \\ P_3 = (1 + \sqrt{-d})/2, & \text{if } (a,bd) \equiv (0,1) \pmod 2; \\ P_0 = 0, & \text{if } (a,bd) \equiv (1,1) \pmod 2; \\ P_3 = 1/2, & \text{if } (a,b,d) \equiv (1,1,0) \pmod 2; \\ P_2 = \sqrt{-d}/2, & \text{if } (a,b,d) \equiv (0,1,0) \pmod 2; \end{cases}$$

$a,b,d \pmod 2$	$\lambda P_j, j=0,2$	$\lambda P_j, j=1,3$	$r_j, j=0,2$	$r_j, j=1,3$
$a \equiv d \equiv 0 \pmod 2$	$\lambda P_0 = P_0$	$\lambda P_1 = P_2$	$r_0 = N(\lambda)/2+1$	$r_1 = N(\lambda)/2$
$b \equiv 1 \pmod 2$	$\lambda P_2 = P_0$	$\lambda P_3 = P_2$	$r_2 = N(\lambda)/2+1$	$r_3 = N(\lambda)/2$
$a \equiv d \equiv 0 \pmod 2$	$\lambda P_0 = P_0$	$\lambda P_1 = P_0$	$r_0 = N(\lambda)/2+2$	$r_1 = N(\lambda)/2$
$b \equiv 0 \pmod 2$	$\lambda P_2 = P_0$	$\lambda P_3 = P_0$	$r_2 = N(\lambda)/2$	$r_3 = N(\lambda)/2$
$a \equiv b \equiv 0 \pmod 2$	$\lambda P_0 = P_0$	$\lambda P_1 = P_0$	$r_0 = N(\lambda)/2+2$	$r_1 = N(\lambda)/2$
$d \equiv 1 \pmod 2$	$\lambda P_2 = P_0$	$\lambda P_3 = P_0$	$r_2 = N(\lambda)/2$	$r_3 = N(\lambda)/2$
$d \equiv b \equiv 1 \pmod 2$	$\lambda P_0 = P_0$	$\lambda P_1 = P_2$	$r_0 = (N(\lambda)+1)/2$	$r_1 = (N(\lambda)+1)/2$
$a \equiv 0 \pmod 2$	$\lambda P_2 = P_1$	$\lambda P_3 = P_3$	$r_2 = (N(\lambda)+1)/2$	$r_3 = (N(\lambda)+1)/2$
$d \equiv b \equiv 0 \pmod 2$	$\lambda P_0 = P_0$	$\lambda P_1 = P_1$	$r_0 = (N(\lambda)+1)/2$	$r_1 = (N(\lambda)+1)/2$
$a \equiv 1 \pmod 2$	$\lambda P_2 = P_2$	$\lambda P_3 = P_3$	$r_2 = (N(\lambda)+1)/2$	$r_3 = (N(\lambda)+1)/2$
$a \equiv d \equiv 1 \pmod 2$	$\lambda P_0 = P_0$	$\lambda P_1 = P_1$	$r_0 = (N(\lambda)+1)/2$	$r_1 = (N(\lambda)+1)/2$
$b \equiv 0 \pmod 2$	$\lambda P_2 = P_2$	$\lambda P_3 = P_3$	$r_2 = (N(\lambda)+1)/2$	$r_3 = (N(\lambda)+1)/2$
$a \equiv b \equiv 1 \pmod 2$	$\lambda P_0 = P_0$	$\lambda P_1 = P_3$	$r_0 = (N(\lambda)+1)/2$	$r_1 = (N(\lambda)+1)/2$
$d \equiv 0 \pmod 2$	$\lambda P_2 = P_2$	$\lambda P_3 = P_1$	$r_2 = (N(\lambda)+1)/2$	$r_3 = (N(\lambda)+1)/2$
$d \equiv b \equiv 1 \pmod 2$	$\lambda P_0 = P_0$	$\lambda P_1 = P_3$	$r_0 = N(\lambda)/2+1$	$r_1 = N(\lambda)/2$
$a \equiv 1 \pmod 2$	$\lambda P_2 = P_3$	$\lambda P_3 = P_0$	$r_2 = N(\lambda)/2$	$r_3 = N(\lambda)/2+1$

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