Backlund Transformation of the Painlevé Equations

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Introduction

Discovered strictly for mathematical consideration at the beginning of the 20th century, the second Painlevé equation is given by the following **family of nonlinear second order ordinary differential equation**

$$P_{II}(\alpha): \quad \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha, \qquad \alpha \in \mathbb{C}$$

We say that the second Painlevé equation is given by a differential polynomial in variable $y, \frac{dy}{dt}, \frac{d^2y}{dt^2}$ with coefficients in $\mathbb{C}(t)$.

The equation has appeared in many important physical application, such as random matrix theory, statistical physics, plasma physics and many other areas (cf. [1] and [2]). Although, it has been the subject of numerous scholarly work since its discovery, it is extremely difficult to find or compute solutions of the equations. So far, solutions have only been found when $\alpha \in \mathbb{Z}$.

An important feature of the equation is the existence of symmetries that take solutions of P_{II} to solutions of P_{II} possibly with different parameters. These symmetries are called **Backlund Transformations**.

Research Goal

In this project, our aim is to find new Backlund Transformations or rule out the existence of such transformation at a given value of the parameter.

Overview

Consider the second Painlevé equation below with $\alpha = 0$. It is easy to see that, $y = 0$ is a solution of $P_{II}(0)$.

$$\frac{d^2(0)}{dt^2} = 2(0)^3 + t(0) + 0.$$

This is the only known solution. Methods to find all the other solutions are still being developed.

Similarly, for $\alpha = 1$, making use of guesses and computations, it will not be too difficult to see that $y = -\frac{1}{t}$ is a solution

$$\frac{d^2(-\frac{1}{t})}{dt^2} = 2(-\frac{1}{t})^3 + t(-\frac{1}{t}) + 1.$$

Note that it becomes a tedious process to guess, or easily compute solutions of the equation when α is different from $0, 1$ or -1. We will now show how the Backlund Transformation can make this process easier.

Definition

A **Backlund Transformation** of the second Painlevé is a rational function

$$F \in \mathbb{C}(t)(y, y'),$$

such that if $\mathcal{P}$ is a solution of $P_{II}(\alpha)$ then $F(\mathcal{P}, \mathcal{P}')$ is a solution of $P_{II}(\beta)$.

Examples

The following Backlund Transformations, takes a solution of $P_{II}(\alpha)$ to a solution of $P_{II}(\alpha+1)$ and $P_{II}(\alpha-1)$ respectively for any value of α.

$$T_+(\alpha, y) = -y - \frac{\alpha + 1/2}{y' + y^2 + t/2}$$

$$T_-(\alpha, y) = -y + \frac{\alpha - 1/2}{y' - y^2 - t/2}$$

For example, we know that $y = 0$ is a solution of $P_{II}(0)$. So applying T_+ and T_-

$$T_+(0,0) = -0 - \frac{0 + 1/2}{0' + 0^2 + t/2} = -\frac{1}{t}$$

$$T_-(0,0) = -0 + \frac{0 - 1/2}{0' - 0^2 - t/2} = \frac{1}{t}$$

And as we previously stated, $-\frac{1}{t}$ is a solution of $P_{II}(1)$. Observe that $\frac{1}{t}$ is the only known solution of $P_{II}(-1)$.

The Backlund Transformations also allows us to compute more complicated solutions. For example

$$T_+(1, -\frac{1}{t}) = -(-\frac{1}{t}) - \frac{1 + 1/2}{(-\frac{1}{t})' + (-\frac{1}{t})^2 + \frac{t}{2}} = \frac{1}{t} - \frac{1}{2(\frac{t}{2} + \frac{2}{t^2})}$$

is the only known solution of $P_{II}(2)$

Progress so far

We first tackled the problem of the existence of Backlund Transformations of the form

$$F(y) = ay + b, \qquad \text{where } a, b \in \mathbb{C}(t)$$

In other words, we want to find all Backlund Transformations of degree zero in y' and degree one in y.

Result

The only Backlund Transformations of the form $F(y) = ay + b$, where $a, b \in \mathbb{C}(t)$ is either

- the identity (or trivial) transformation, $I(y) = y$; or
- the transformation $S(y) = -y$ which sends a solution of $P_{II}(\alpha)$ to a solution of $P_{II}(-\alpha)$.

The transformation $S(y) = -y$ is well-known in the literature. Notice that it connects the known solutions of $P_{II}(1)$ and $P_{II}(-1)$.

A nice representation of the transformations for $\alpha \in \mathbb{Z}$ is as follows:

$$\begin{tikzcd}
 & P_{II}(1) \arrow[r, "T_+"] \arrow[dd, "S"] & P_{II}(2) \arrow[r, "T_+"] \arrow[dd, "S"] & P_{II}(3) \arrow[r, "T_+"] \arrow[dd, "S"] & \cdots \\
P_{II}(0) \arrow[ur, "T_+"] \arrow[dr, "T_-"'] & & & & \\
 & P_{II}(-1) \arrow[r, "T_-"] & P_{II}(-2) \arrow[r, "T_-"] & P_{II}(-3) \arrow[r, "T_-"] & \cdots
\end{tikzcd}$$

Open Problem [4]

For each $\alpha \in \mathbb{C}$, are **all** the Backlund Transformations of $P_{II}(\alpha)$ obtained from (composing) T_+, T_- and S?

Current Work

We would like to know whether there are any non-trivial Backlund Transformations

$$F: P_{II}(\alpha) \to P_{II}(\alpha).$$

Such function is called an **auto-Backlund Transformation**. If at a given value of α no such auto-Backlund Transformation exists, then T_+, T_- and S generates all the Backlund Transformations of $P_{II}(\alpha) \to P_{II}(\alpha + \mathbb{Z})$. We only study non-trivial **polynomial** auto-Backlund Transformations

Methods

We will use Hamiltonian form of $P_{II}(\alpha)$:

$$S_{II}(\alpha) \begin{cases} y_1' = y_2 - y_1^2 - t/2 \\ y_2' = 2y_1y_2 + \alpha + 1/2. \end{cases}$$

It is not hard to see that $P_{II}(\alpha)$ is equivalent to the first order system $S_{II}(\alpha)$.

Indeed, if $(y_1, y_2) = (\mathcal{P}_1, \mathcal{P}_2)$ is a solution of $S_{II}(\alpha)$, then $y = \mathcal{P}_1$ is a solution of $P_{II}(\alpha)$. Conversely, if $y = \mathcal{P}$ is a solution of $P_{II}(\alpha)$, then $(y_1, y_2) = (\mathcal{P}, \mathcal{P}' + \mathcal{P}^2 + t/2)$ is a solution of $S_{II}(\alpha)$.

An auto-Backlund transformation of $S_{II}(\alpha)$ (and so of $P_{II}(\alpha)$) is

1. an **automorphism** of the affine plane, that is bijective map
$$(y_1, y_2) \mapsto (f_1(y_1, y_2), f_1(y_1, y_2))$$
where f_1, f_2 are polynomial in two variables over $\mathbb{C}(t)$; and such that
2. $(f_1(y_1, y_2), f_1(y_1, y_2))$ is a solution of $S_{II}(\alpha)$ for each (y_1, y_2) solution of $S_{II}(\alpha)$.

We will use the fact that there is a full classification of the automorphisms of the affine plane (cf. [3]).

References

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