Model theory and differential equations

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Introduction

The model theoretic approach to the study of differential equations has a long and rich history. The theory of differentially closed fields of characteristic 0, $DCF_0$, has been studied intensively and has played an important role in the internal development of geometric model theory. It is also behind one of the most spectacular applications of logic to number theory; namely, E. Hrushovski’s celebrated proof of the function field Mordell-Lang conjecture. Furthermore, the study of the theory $DCF_0$ has led to substantial development in a Galois theory for differential equations and its applications.

Nevertheless, only very recently has the techniques from model theory been used to study classical differential equations. First in the work of the author and A. Pillay on the Painlevé transcendents [NP14], [NP17] and then in that of J. Freitag and T. Scanlon [FS18] on the differential equation satisfied by the modular j-function. More recently, in joint work with G. Casale and J. Freitag, the author has also studied the differential equations satisfied by the Fuchsian automorphic functions and in the process proved an old claim of P. Painlevé (1895).

In this article, we give an overview of those recent applications of model theory to the study of differential equations. The focus will be on the role of the classification problem for strongly minimal sets and results in functional transcendence. Unavoidably, many other interesting and important aspects of the interaction between model theory and differential algebra will be omitted.

Differential Algebraic Geometry

Differential algebraic geometry, which has its origin at the beginning of the 1930’s, was founded by J. Ritt and E. Kolchin. Although not widely known, it gives a general algebraic setting for the study of differential equations and the approach is similar to that of the study of polynomial equations in algebraic geometry. We will in this article focus on ordinary differential equations. Moreover, we will say a few words at the end about the partial and difference context. The standard reference for this section is Kolchin’s book [Kol73]. All fields will be assumed to be of characteristic 0.

**Definition 1.** A differential field $(K, \delta)$ is a field $K$ equipped with a derivation $\delta: K \to K$, i.e., an additive homomorphism satisfying the Leibniz rule $\delta(xy) = x\delta(y) + y\delta(x)$.

The field of constants $C_K$ of $K$ is defined set theoretically as $\{x \in K : \delta(x) = 0\}$. We usually write $x'$ for $\delta(x)$ and $x^{(n)}$ for $\underbrace{\delta \ldots \delta}_{n}(x)$.

**Example 1.** $\mathbb{C}(t), d/dt)$ the field of rational functions over $\mathbb{C}$ in a single indeterminate, where in this case, the field of constants is $\mathbb{C}$.

Associated with a differential field $(K, \delta)$, is the ring of differential polynomials $K\{X\}$ in $m$ differential variables $\overline{X} = (X_1, \ldots, X_m)$. An element of $K\{\overline{X}\}$ is called a differential polynomial over $K$ and is simply a regular polynomial with coefficients in $K$ but in...
variables $\overline{X}, \overline{X'}, \overline{X}^{(2)}, \ldots$. We use here the notation $\overline{X}^{(n)} = (X_1^{(n)}, \ldots, X_m^{(n)})$. If $f \in K\{\overline{X}\}$, then the order of $f$, denoted $\text{ord}(f)$, is the largest $n$ such that for some $i$, $X_i^{(n)}$ occurs in $f$.

**Example 2.** $f(X) = (X')^2 - 4X^3 - tX$ is a differential polynomial in $C(t)\{X\}$ and $\text{ord}(f) = 1$.

As one can see, if $f \in K\{\overline{X}\}$, then $f(\overline{X}) = 0$ is an ordinary (algebraic) differential equation. More generally, by a Kolchin closed subset of $K^n$, we mean the common zero set of a finite system of differential polynomial equations, i.e., a set of the form

$$V(K) = \{y \in K^n : f(y) = 0 \text{ for all } f \in S\}$$

where $S \subset K\{\overline{X}\}$ is a finite subset. The Kolchin closed sets are the basic closed sets in the Kolchin topology and are the analogues of the basic closed sets in the Zariski topology. A Kolchin constructible set is simply a boolean combination of Kolchin closed sets.

The analogue of algebraically closed fields in the differential context and which gives a notion of fields that contain enough solutions of ordinary differential equations is defined as follows

**Definition 2.** A differential field $(K, \delta)$ is said to be differentially closed if for every $f, g \in K\{X\}$ such that $\text{ord}(f) > \text{ord}(g)$, there is $y \in K$ such that $f(y) = 0$ and $g(y) \neq 0$.

Differential algebraic geometry as developed by Kolchin, studies Kolchin closed sets in a differentially closed field. At this point, let us mention that Kolchin closed sets can have very rich algebraic structure. Take for example, the field of constants: if $K$ is differentially closed, then from Definition 2 we see that $C_K$ is an algebraically closed field. Less obvious is that $C_K$ is indeed the only algebraically closed subfield of $K$ that is given by a differential equation.

Another interesting, yet somehow well-known, example is that of an homogeneous linear differential polynomial

$$f(X) = X^{(n)}a_{n-1}X^{(n-1)} + \cdots + a_1X' + a_0X, \quad a_i \in K.$$

One has that associated Kolchin closed set (in a differentially closed field $K$) is a vector space over $C_K$.

Kolchin's approach has been instrumental in the development of a Galois theory for differential equations that solidifies and extends the Picard-Vessiot theory for linear differential equations. For example, it generalizes the fact that in a differentially closed field $K$, the Galois group of a linear differential equation is a linear algebraic group defined over $C_K$, by putting algebraic groups at the forefront.

**Model theory**

There are many great introductory books in model theory. Moreover, for the topics covered in this article, we recommend D. Marker’s book [Mar02].

A major focus of model theory is the study definable sets in models of a first order theory. Here, by a first order theory $T$ we mean a set of axioms (or more accurately first order sentences) in a fixed language $L$. The language $L$ is simply a set of constant
symbols, function symbols and relation symbols. We assume throughout that the language is countable.

**Example 3.** A familiar example is $T_G$ the theory for groups which consists of the usual axioms for groups expressed using the language $L_G = \{1, *, ^{-1}\}$ together with other logical symbols such as $\exists$ and $\forall$.

A *structure* for a language $L$, or an $L$-structure for short is a set together with interpretations for each symbol in $L$. A *model* of a theory $T$ is simply a $L$-structure in which the axioms are true. In Example 3, we see that both $(\mathbb{N}, 0, +, -)$ and $(\mathbb{Z}, 0, +, -)$ are $L_G$-structure, moreover only the latter is a model of $T_G$. Now for the definition of a definable set in a model (or structure) $M$.

**Definition 3.** A *definable set* $Y \subset M^n$ is a set of the form

$$Y = \{\bar{y} \in M^n : \phi(\bar{y}) \text{ is true}\}$$

where $\phi$ is a formula in $L$ with $n$ free variables.

The notion of a (well-formed) formula extends that of an axiom, whereby free variables, that is those not quantified upon, are allowed. Continuing with Example 3, we see that a well-formed formula is $\phi(X) = \forall Y (X * Y = Y * X)$ and for a model $G$ (i.e., a group) the formula $\phi(X)$ defines the center of $G$.

**Remark 1.** For any subset $A \subset M$ of a model, one can extend the language $L$ by adding a constant symbol for each element $a \in A$. One usually denotes the new language obtained by $L_A$. If in Definition 3 one replace $L$ by $L_A$ for some $A \subset M$, then one obtains the definition of an $A$-definable set or more precisely a definable set with *parameters* from the set $A$.

So for a fix theory $T$ a major goal is to study all definable sets in some/any model of $T$. This of course would be hopeless unless one could identify classes of structures where there are some control over the definable sets. In model theory, this leads the distinction between “tame” and “wild” structures or theories. In this article we discuss two notion of tameness, namely quantifier elimination and $\omega$-stability. There are many more “tame” versus “wild” distinctions and some are illustrated in Figure 2.

A theory $T$ is said to have quantifier elimination if for every formula $\phi(\bar{X})$ there is a quantifier-free formula $\psi(\bar{X})$ such that the two define the same definable set. It hence follows that for theories with quantifier elimination the definable sets are defined using “simple” formulas.

A theory $T$ is $\omega$-stable if every definable set $X$ can be given an intrinsic ordinal valued dimension called the Morley Rank, denoted by $RM(X)$. In rough terms, the inductive definition is as follows: $RM(X) = 0$ if $X$ is finite, and $RM(X) \geq \alpha + 1$ if there are pairwise disjoint definable subsets $X_i$ of $X$ for $i = 1, 2, \ldots$ such that each $RM(X_i) \geq \alpha$ for all $i < \omega$ (one extends the definition naturally to limit ordinals). We set $RM(X) = \alpha$ if $RM(X) \geq \alpha$ but not $\geq \alpha + 1$. Using this rank, one can define in $T$ a good notion of independence and dimension analogous to the notion of linear independence and basis in the study of vector space.

The theory of algebraically closed field of characteristic zero $ACF_0$ with the obvious axioms given in the language of rings $L_R = (+, -, \cdot, 0, 1)$ has both quantifier elimination and is $\omega$-stable. In this setting quantifier elimination is equivalent to Chevalley’s theorem that over an algebraically closed field
the projection of a constructible set is constructible. The Morley rank of a definable set (so a constructible set) corresponds to the transcendence degree of a generic point, while the independence notion is equivalent to algebraic independence.

The Theory $DCF_0$

Let us bring together the ideas of the first two sections. We refer the reader to [Mar96] for additional details. In the context of differentially closed fields, the relevant language is $L_δ = (+, -, \cdot, 0, 1)$, the language of differential rings and we denote by $DCF_0$ the theory of differential fields of characteristic zero. The axioms of $DCF_0$ consist of the axioms for fields and the axioms for the derivation (expressed using $δ$).

Now, for each $n, d_1$ and $d_2 \in \mathbb{N}$, one can write down an axiom (in $L_δ$) that asserts that if $f$ is a differential polynomial of order $n$ and degree at most $d_1$ and $g$ is a nonzero differential polynomial of order less than $n$ and degree at most $d_2$, then there is a solution to $f(X) = 0$ and $g(X) \neq 0$. The theory obtained by adding to $DCF_0$ all these axioms is called the theory of differentially closed fields of characteristic zero, $DCF_0$. This theory is very "tame" in the sense described above. For example

**Theorem 1.** The theory $DCF_0$ eliminates quantifiers and is $ω$-stable.

For the remainder of the article $\mathcal{U}$ will denote a fixed model$^1$ of $DCF_0$, i.e., a differentially closed field.

Quantifier elimination means that a definable set $Y \subseteq \mathcal{U}^n$, definable over a differential subfield $K$ of $\mathcal{U}$, is nothing other than a Kolchin constructible set over $K$. On the other hand, as discussed above, $ω$-stability means (among other things) that any definable set has a well-defined ordinal-valued Morley rank. The independence notion in $\mathcal{U}$ is as follows: $\pi$ is independent from $\bar{a}$ over $K$ if $K(\bar{a})$ is algebraically disjoint from $K(\bar{b})$ over $K$. Here $K(\bar{a}) = K(\bar{a}, \pi(1), \pi(2), \ldots)$ denotes the differential field generated by $\pi$ over $K$.

Along with the Morley rank, we also have another invariant for definable sets call the order. For $\pi \in \mathcal{U}^n$ and $K < \mathcal{U}$, we define $\text{ord}(\pi/K)$ to be the transcendence degree of the field $K(\pi)$ over $K$. If $Y \subseteq \mathcal{U}^n$ is definable over $K$, we define the $\text{ord}(Y) = \sup\{\text{ord}(\pi/K) : \pi \in Y\}$. One can show that $\text{RM}(Y)$ is always less than or equal to $\text{ord}(Y)$. Furthermore, $\text{RM}(Y) < ω$ if and only if $\text{ord}(Y) < ω$. We will later see examples of Kolchin closed sets for which the Morley rank is strictly less than the order.

**Definition 4.** Let $Y \subseteq \mathcal{U}^n$ be a definable set.

1. $Y$ is said to be finite dimensional (or rank) if it has finite order, i.e., $\text{ord}(Y) < ω$.

2. $Y$ is said to be strongly minimal if it is infinite and for every definable subset $Z \subseteq Y$, either $Z$ or $Y \setminus Z$ is finite.

If $Y$ is strongly minimal then it has Morley rank one. Strongly minimal sets determine, in a precise manner (not to be discussed in this article), the structure of all finite dimensional definable sets. This fact, which follows from very general model theoretic considerations, holds in any $ω$-stable theory and is obtained in part using the robust notion of independence.

Notice that if $Y$ is a definable set with $\text{ord}(Y) = n$, then $Y$ is strongly minimal if and if $Y$ can not be written as the disjoint union of definable sets of order $n$, and for any differential field $K$ over which $Y$ is defined, and element $y \in Y$, either $y \in K^{alg}$ or $\text{tr.deg}(K(\bar{y})/K)$ is 0 or $n$. Here, for a field $F$, we denote its algebraic closure by $F^{alg}$.

**Example 4.** The field of constants $C_\mathcal{U}$ is strongly minimal.

**Example 5.** If $f$ is an absolutely irreducible polynomial over $\mathcal{U}$ in 2 variables then the subset $Y$ of $\mathcal{U}$ defined by $f(y, y') = 0$ is strongly minimal, of order 1.

It is a quite a difficult task to show that the set defined by a given differential equation is strongly minimal. Indeed, except for limited or special cases, no general tools are available. For example, we refer

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$^1$We actually need the more technical assumption that the model is saturated.
the reader to Section 5.17 of [Mar96], for the (tedious)
calculations involved in showing that the subset of \( U \)
deﬁned by \( \{ y y'' = y', y' \neq 0 \} \) is strongly minimal, of
order 2.

Nevertheless, the goal of understanding all deﬁnable sets in \( DCF_0 \), goes through a complete under-
standing of the strongly minimal sets. A considerable
amount of work, beginning in the 1990’s, has been de-
vo ted to just that. The deepest result in that direc-
tion, due to E. Hrushovski and Z. Sokolovic [HS94],
concerns the classiﬁcation of strongly minimal sets
that have “non-trivial” structures.

**Deﬁnition 5.** Let \( Y \) be strongly minimal set de-
ﬁned over a differential ﬁeld \( K \). Then \( Y \) is said to be
g e o m e t r i c a l l y t r i v i a l if for any \( y, y_1, \ldots, y_n \in Y \) if
\( y \in K \langle y_1, \ldots, y_n \rangle_{alg} \), then there is \( 1 \leq i \leq n \) such
that \( y \in K \langle y_i \rangle_{alg} \).

In essence, a geometrically trivial set can have at
most a ‘binary’ structure. The ﬁeld of constants \( C_U \)
is not geometrically trivial. The same is true of de-
ﬁnable groups (i.e., deﬁnable sets equipped with de-
ﬁnable group structures).

Figure 3: Presence of a deﬁnable group: in non-geometrically trivial strongly minimal
sets one can ﬁnd a group conﬁguration.

The work of Hrushovski and Sokolovic did not at-
ttempt to classify geometrically trivial strongly min-
imal sets. On the other hand, a key step in their
work and which builds on those of A. Bui um [Bui92],
was the identiﬁcation of some ‘exotic’ differential al-
gebraic groups (i.e., deﬁnable groups where the un-
derlying deﬁnable set is Kolchin closed).

**Theorem 2.** Let \( A \) be an abelian variety over \( U \). We
identify \( A \) with its set \( A(U) \) of \( U \)-points. Let \( A^2 \) be the
Kolchin closure of the torsion subgroup of \( A \). Then

1. \( A^2 \) is a differential algebraic group and is Zariski
dense in \( A \).

2. If \( A \) is a simple abelian variety that does not
descend to \( C_U \), then \( A^2 \) is strongly minimal.

The group \( A^2 \) is called the Manin kernel of \( A \). One
remarkable property of \( A^2 \) is that any of its de-
ﬁnable subset is a ﬁnite Boolean combination of cosets
of some of its deﬁnable subgroups. The result of
Hrushovski and Sokolovic is that up to equivalence,
the ﬁeld of constants \( C_U \) and the groups \( A^2 \) cover all
the non geometrically trivial examples!

**Theorem 3 (The trichotomy theorem).** If \( Y \in
U^n \) is strongly minimal, then exactly one of the fol-
lowing hold

1. \( Y \) is geometrically trivial, or

2. (Group-like) \( Y \) is non-orthogonal to the Manin
kernel \( A^2 \) of some simple abelian variety \( A \) that
does not descend to \( C_U \), or

3. (Field-like) \( Y \) is non-orthogonal to the ﬁeld of
constants \( C_U \).

We say that \( Y \) and \( Z \) (both strongly minimal) are
non orthogonal if there is some inﬁnite deﬁnable re-
lation \( R \subset Y \times Z \) such that \( \pi_{Y|R} \) and \( \pi_{Z|R} \) are
ﬁnite-to-one functions. Here \( \pi_Y : Y \times Z \to Y \) and
\( \pi_Z : Y \times Z \to Z \) denote the projections to \( Y \) and \( Z \)
respectively. It is not hard to see that nonorthogonality
is indeed an equivalence relation on strongly mini-
mal sets. Furthermore, if \( Y \) and \( Z \) are nonorthogonal
strongly minimal sets, then \( ord(Y) = ord(Z) \).

The work of Hrushovski and Sokolovic was never
published. Moreover, an alternate proof of the char-
acterization of the ﬁeld-like strongly minimal sets - a
key step - has appeared in the work of A. Pillay and

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A good summary of the proof of Theorem 3 can be found in [NP17, Section 2.1].

There are other interesting and important consequences of the trichotomy theorem that are not apparent but worth mentioning. Firstly, if $A^♯$ is the Manin Kernel of a simple abelian variety $A$ that does not descend to $C_U$, then $\text{ord}(A^♯) \geq 2$. Hence, strongly minimal sets of order 1 are either geometrically trivial or non-orthogonal to $C_U$. Secondly, strongly minimal sets that are defined over $C_U$ and of order $\geq 2$ are geometrically trivial! This surprising fact was somewhat forgotten for a while but now plays a crucial role in some of the applications of the theory to functional transcendence and beyond.

**Trivial Pursuits and Applications**

As we have seen the trichotomy theorem, which gives a very general classification theorem for strongly minimal sets, has nothing to say about geometrically trivial strongly minimal sets. Understanding these strongly minimal sets, or trivial pursuits (as coined by D. Maker), is one of the most important open problems in the study of $DCF_0$. But to this date very little progress has been made.

For a while it was conjectured that all geometrically trivial strongly minimal sets would have no (or very little) structure: for any element $y$ of a trivial strongly minimal set $Y$ only finitely many other elements of $Y$ are interalgebraic with $y$. More precisely

**Definition 6.** Let $Y$ be strongly minimal set defined over a differential field $K$. Then $Y$ is said to be $\omega$-categorical if for any tuple $\bar{y}$ from $U$, the set $K \langle \bar{y} \rangle^{\text{alg}} \cap Y$ is finite.

If a strongly minimal set is $\omega$-categorical, then it is geometrically trivial. A beautiful result of Ehud Hrushovski [Hru95] is that the converse holds for order-1 strongly minimal sets (cf. [Pil02, Cor 1.82]) and [FM17] for a generalization:

**Theorem 4.** Let $Y \subset U^n$ be an order 1 geometrically trivial strongly minimal set. Then $Y$ is $\omega$-categorical.

This result of Hrushovski had given rise to a conjecture about geometrically trivial strongly minimal sets of arbitrary order: In differentially closed fields, every geometrically trivial strongly minimal set is $\omega$-categorical. This was proven to be false at this level of generality in [FS18] using the order 3 differential equation satisfied by the modular $j$-function (see below). The following interesting question remains.

**Question 1.** Are all order 2 geometrically trivial strongly minimal sets $\omega$-categorical?

At this point, it is worth noting that establishing the $\omega$-categoricity of a Kolchin closed set corresponds to obtaining an important number theoretic/functional transcendence type result for the solutions of the corresponding differential equations. As such a positive answer to the above question is of great interest. We will now illustrate this by looking at several recent applications of the model theoretic approach, in particular the trivial pursuits, to some classical differential equations.

**The generic Painlevé Transcendents**

The Painlevé equations are second order ordinary differential equations and come in six families $P_I-P_VI$, where $P_I$ consists of the single equation

$$\frac{d^2y}{dt^2} = 6y^2 + t,$$

and $P_{II}-P_{VI}$ come with some complex parameters. They were isolated in the early part of the 20th century, by P. Painlevé, with refinements by B. Gambier and R. Fuchs, as those ODE’s of the form $y'' = f(y, y', t)$ (where $f$ is rational over $\mathbb{C}$) which have the Painlevé property: any local analytic solution extends to a meromorphic solution on the universal cover of $\mathbb{P}^1(\mathbb{C}) \setminus S$, where $S$ is the finite set of singularities of the equation. The equations have arisen in a variety of important physical applications including, for example, statistical mechanics, general relativity and fibre optics.

**Example 6.** The second Painlevé equation $P_{II}(\alpha)$ is given by

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

This equation has been used to study integrable systems and has connections to other areas of mathematics and physics.
where $\alpha \in \mathbb{C}$. The equation appears quite prevalently in random matrix theory (cf. [FW15]).

Painlevé believed that, at least for general values of the parameters, the set defined by the equations would be strongly minimal\(^2\). This was proven to be true in a series of papers by K. Okamoto, K. Nishioka, M. Noumi, H. Umemura and H. Watanabe (cf. [Oka99] for a survey). In particular, the first Painlevé equations is strongly minimal and in the case of the second Painlevé equation, they proved that $P_{II}(\alpha)$ is strongly minimal if and only if $\alpha \notin \frac{1}{2} + \mathbb{Z}$. By a generic Painlevé equation we mean one equation among the family $P_I - P_{VI}$, such that all the corresponding complex parameters are transcendental and algebraically independent over $\mathbb{Q}$. So $P_{II}(\pi)$ is a generic equation. The works of Watanabe and others hence give that all the generic Painlevé equations are strongly minimal. They left wide open the question of the fine structure of the definable sets. We now have a full answer.

**Theorem 5.** Suppose $y_1, \ldots, y_n$ are distinct solutions of one of the generic Painlevé equations. Then $y_1, y_1', \ldots, y_n, y_n'$ are algebraically independent over $\mathbb{C}(t)$, i.e.,

$$tr.deg(\mathbb{C}(t)(y_1, y_1', \ldots, y_n, y_n')/\mathbb{C}(t)) = 2n.$$ 

In particular the generic Painlevé equations are all $\omega$-categorical. K. Nishioka [Nis04] proved the result for $P_I$ using differential algebra. However his calculations and techniques does not seem to generalize to the other equations. The author, in [Nag20] and before that in joint work with Pillay in [NP14], proved the result for all the other equations using model theory. The proofs heavily relies on earlier work [NP17] in which the trichotomy is used to show that the generic Painlevé equations are all geometrically trivial.

The model theoretic approach has also allowed us to show that the generic equations from most distinct Painlevé families are orthogonal. Work is currently underway towards obtaining a full classification of algebraic relations between solutions of the Painlevé equations. As of now, except for the second Painlevé equations (where for example the author showed geometrically triviality holds if and only if $\alpha \notin \frac{1}{2} + \mathbb{Z}$) the study of the non-generic Painlevé equations is wide open. The following is an example of the most basic question one would like to answer.

**Question 2.** For which values of the parameters of a fixed Painlevé equation is it true that if $y_1, \ldots, y_n$ are distinct solutions (not in $\mathbb{C}(t)^{alg}$), then

$$tr.deg(\mathbb{C}(t)(y_1, y_1', \ldots, y_n, y_n')/\mathbb{C}(t)) = 2n?$$

**Fuchhian Automorphic Functions**

We now consider the most natural generalizations of the trigonometric and elliptic functions (i.e., the periodic functions).

Let $\Gamma < PSL_2(\mathbb{R})$ be a Fuchhian group\(^3\) of first kind and genus zero. An automorphic function $f$ for $\Gamma$ is a function on the complex upper half plane $\mathbb{H}$, such that

$$f(g \cdot \tau) = f(\tau) \quad \text{for all } g \in \Gamma \text{ and } \tau \in \mathbb{H},$$

and such that $f$ is meromorphic at every cusp of $\Gamma$. The collection $\mathcal{A}_0(\Gamma)$ of all automorphic functions for $\Gamma$ is a field and is generated (over $\mathbb{C}$) by some univalent automorphic function called an hauptmodul or uniformizer for $\Gamma$. We will denote by $j_\Gamma(t)$ one such fixed hauptmodul.

It is a classical fact that $j_\Gamma(t)$ satisfy a third order ordinary differential equation of Schwarzian type

$$S_t(y) + (y')^2 R_{j_\Gamma}(y) = 0. \quad (\star)$$

Here $S_t(y) = \left(\frac{w''}{w'}\right)' - \frac{1}{2} \left(\frac{w''}{w'}\right)^2$ denotes the Schwarzian derivative ($t = \frac{d}{dt}$) and

$$R_{j_\Gamma}(y) = \frac{1}{2} \sum_{i=1}^{r} \frac{1 - \alpha_i^{-2}}{(y - a_i)^2} + \sum_{i=1}^{r} \frac{\beta_i}{y - a_i}.$$ 

\(^2\)In Painlevé’s terminology, the equations should be “irreducible” in the sense that none of its solutions are “new” special functions. It took the work of Nishioka and Umemura, after about 80 years, to clarify his notion of irreducibility and the work of the author and Pillay, after another 30 years, to make the connection with strong minimality.

\(^3\)Throughout $g \cdot \tau$ will denote the action of an element of $GL_2(\mathbb{C})$ by linear fractional transformation.
with $a_1, \ldots, a_n$ and $\beta_1, \ldots, \beta_n$ real numbers depending on $\Gamma$ and $j_\Gamma$. Every solution in $\mathcal{U}$ of the Schwarzian equation ($\ast$) can be taken to be of the form $j_\Gamma(g \cdot t)$ for some $g \in GL_2(\mathbb{C})$.

**Example 7.** If $\Gamma = PSL_2(\mathbb{Z})$, then the classical modular $j$-function

$$ j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots, $$

where $q = e^{2\pi i \tau}$, is an hauptmodul. In this case the differential equation is given with

$$ R_j(y) = \frac{y^2 - 1968y + 2654208}{y^2(y - 1728)^2} $$

P. Painlevé in 1895, again claimed that the set defined by equation ($\ast$) would be strongly minimal. K. Nishioka proved that the Hauptmodul $j_\Gamma$ does not satisfy any algebraic differential equation of order two or less over $\mathbb{C}(t, e^{\lambda t})$, for any $\lambda \in \mathbb{C}$. He also obtained a very weak form of Painlevé assertion in the case of triangle groups. The first real progress was made by J. Freitag and T. Scanlon [FS18] in their work on the modular $j$-function (they did not know of Painlevé’s claim then).

**Theorem 6.** Let $\Gamma = PSL_2(\mathbb{Z})$. Then the set defined by the Schwarzian equation ($\ast$) is strongly minimal, geometrically trivial but not $\omega$-categorical.

Their proof relies on a deep functional transcendence result of J. Pila [Pil13] called the modular Ax-Lindemann-Weierstrass theorem with derivatives (see below).

**Remark 2.** Granted that strong minimality holds, it is rather unsurprising that the definable set in the case of the $j$-function is not $\omega$-categorical. Indeed, for each $n \in \mathbb{N}$ we have the classical modular polynomials $\Phi_n(X,Y) \in \mathbb{Z}[X,Y]$ that relates solution of the equation for $j$: if $g_1$ and $g_2$ are in the same coset of $GL_2(\mathbb{Q})$, then $\Phi (j(g_1 \cdot t), j(g_1 \cdot t)) = 0$ for some $n$.

For a while the result of Freitag and Scanlon seemed to have shut the door on a possible classification of geometrically trivial strongly minimal sets. However, it turned out that studying the Schwarzian equation ($\ast$) in its full generality has allowed us to place the case $\Gamma = PSL_2(\mathbb{Z})$ in context. A natural and key question is the following: is there a way to explain the existence of the modular polynomials? The answer is again very classical and is brought to light through the notion of commensurability.

Recall that two subgroups $G$ and $H$ of $PSL_2(\mathbb{R})$ are commensurable, denoted by $G \sim H$, if their intersection $G \cap H$ has finite index in both $G$ and $H$. For a Fuchsian group $\Gamma$, let $\text{Comm}(\Gamma)$ be the commensurator of $\Gamma$, namely

$$ \text{Comm}(\Gamma) = \{ g \in PSL_2(\mathbb{R}) : g\Gamma g^{-1} \sim \Gamma \}. $$

If $g \in \text{Comm}(\Gamma) \setminus \Gamma$, then one has that the intersection $\Gamma_g = g\Gamma g^{-1} \cap \Gamma$ is a Fuchsian group of first kind and with the same set of cusps as $\Gamma$. But the functions $j_\Gamma(t)$ and $j_\Gamma(g^{-1}t)$ are respective uniformizers for $\Gamma$ and $g\Gamma g^{-1}$. It follows that they also are automorphic functions for $\Gamma_g$. The work of H. Poincaré gives that any two automorphic functions for a Fuchsian group are algebraically dependent over $\mathbb{C}$. So there is a polynomial $\Phi_g(X,Y) \in \mathbb{C}[X,Y]$, such that $\Phi_g(j_\Gamma(t), j_\Gamma(g \cdot t)) = 0$. Such polynomial is called a $\Gamma$-special polynomial.

So if $\Gamma$ has infinite index in $\text{Comm}(\Gamma)$, then there are infinitely many $\Gamma$-special polynomials. In particular, if one can prove strong minimality, then non-$\omega$-categoricity would follow. It turns out that groups $\Gamma$ having this ‘infinite index’ property are well known in group theory.

Let $F$ be a totally real number field and let $A$ be a quaternion algebra over $F$ that is ramified at exactly one infinite place. Let $\rho$ be the unique embedding of $A$ into $M_2(\mathbb{R})$ and let $\mathcal{O}$ be an order in $A$. The
image $\rho(O^1)$ of the norm-one group of $O$ under $\rho$ is a discrete subgroup of $SL_2(\mathbb{R})$. We denote by $\Gamma(A,O)$ the Fuchsian group obtained under the projection in $PSL_2(\mathbb{R})$ of the group $\rho(O^1)$.

**Definition 7.** A Fuchsian group $\Gamma$ is said to be arithmetic if it is commensurable with a group of the form $\Gamma(A,O)$.

The modular group $PSL_2(\mathbb{Z})$ and its finite index subgroups are the most well-known examples of arithmetic groups. We have the following deep result of G. Margulis.

**Theorem 7.** The group $\Gamma$ is arithmetic if and only if it has infinite index in $Comm(\Gamma)$ and so there are infinitely many $\Gamma$-special polynomials.

The work of the author with G. Casale and J. Freitag [CFN18], completely proves Painlevé’s claim and provides a striking connection between categoricity and arithmeticity.

**Theorem 8.** Let $\Gamma$ be a Fuchsian group of first kind and genus zero and let $X_\Gamma$ be the set defined by the Schwarzian equation $(\ast)$. Then

1. $X_\Gamma$ is strongly minimal and (so) geometrically trivial.

2. $X_\Gamma$ is $\omega$-categorical if and only if $\Gamma$ is non-arithmetic.

The following question can be seen as the next major challenge in the classification of geometrically trivial strongly minimal sets in differentially closed fields.

**Question 3.** In $DCF_0$, are there non-$\omega$-categorical strongly minimal sets that do not arise from arithmetic Fuchsian groups?

Finally, let us mention that the above work on fully classifying the structure of the definable sets associated with the Schwarzian equation $(\ast)$ has been used to give a proof of the Ax-Lindemann-Weierstrass Theorem with derivatives for $\Gamma$: Let $V \subset \mathbb{A}^n$ be an irreducible algebraic variety defined over $\mathbb{C}$ such that $V(\mathbb{C}) \cap \mathbb{H}^n \neq \emptyset$ and $V$ projects dominantly to each of its coordinates (each coordinate function is nonconstant). Let $t_1, \ldots, t_n$ be the functions on $V$ induced by the canonical coordinate functions on $\mathbb{A}^n$. We say that $t_1, \ldots, t_n$ are $\Gamma$-geodesically independent if there are no relations of the form $t_i = g \cdot t_j$ where $i \neq j$ and $g \in Comm(\Gamma)$.

**Theorem 9.** With the notation (and assumption $V(\mathbb{C}) \cap \mathbb{H}^n \neq \emptyset$) as above, suppose that $t_1, \ldots, t_n$ are $\Gamma$-geodesically independent. Then the $3n$ functions

$$j_\Gamma(t_1), j_\Gamma'(t_1), j_\Gamma''(t_1), \ldots, j_\Gamma(t_n), j_\Gamma'(t_n), j_\Gamma''(t_n)$$

(defined locally) on $V(\mathbb{C})$ are algebraically independent over $\mathbb{C}(V)$.

As mentioned earlier, J. Pila [Pil13] had already proved the result for $PSL_2(\mathbb{Z})$. J. Freitag and T. Scanlon [FS18] established the same for arithmetic subgroups of $PSL_2(\mathbb{Z})$. The Ax-Lindemann-Weierstrass (mostly without derivatives) has also been proved by various authors in the more general context of Shimura varieties. The work in [CFN18] differs from all the above in that it does not use a tool called o-minimality (originating in model theory) and also tackles the non-arithmetic groups as well as the derivatives of the functions all at once.

Beyond $DCF_0$

We end by saying a few words about the partial differential and the difference equations settings. We denote by $DCF_{0,m}$ the theory of differentially closed field of charateristic 0 with $m$ commuting derivations (partial context) and by $ACFA$ the theory of algebraically closed field with an automorphism (difference context). The theory $DCF_{0,m}$ is also $\omega$-stable and has quantifier elimination. However, strongly minimal sets do not fully capture the complexity of all definable sets. There are so called infinite rank regular types that do so. The trichotomy theorem is yet to be fully established in that setting. On the other hand, $ACFA$ is not $\omega$-stable but is rather a so-called

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4One such infinite rank regular type also exists for $DCF_0$. Moreover the finite rank part of the theory is where most of the complexity lies.
simple theory (characterized by existence of a good notion of independence). Furthermore, although definable sets are still given by simple enough formulas, ACFA does not have full quantifier elimination. A version of the trichotomy theorem does hold in that setting and the study of ACFA has been very successfully used to obtain new results in number theory and algebraic dynamics.

However in both cases, except for few examples, applications to the study of classical equations is yet to be undertaken. There are obvious candidates that would mirror the situation of DCF. In DCF, tackling the generalized Schwarzian equations for uniformizers for Shimura varieties is of great interest. In ACFA, proving that the q-Painlevé equations are rank 1 is a challenge. These difference equations are discrete analogues of the classical Painlevé equations. In fact, in many real world problems, the Painlevé equations arise from a limiting process, starting with the q-Painlevé equations. We expect that as with DCF, important model theoretic questions about the structure of definable sets can be formed and answered by studying these concrete differential and difference equations.

References


