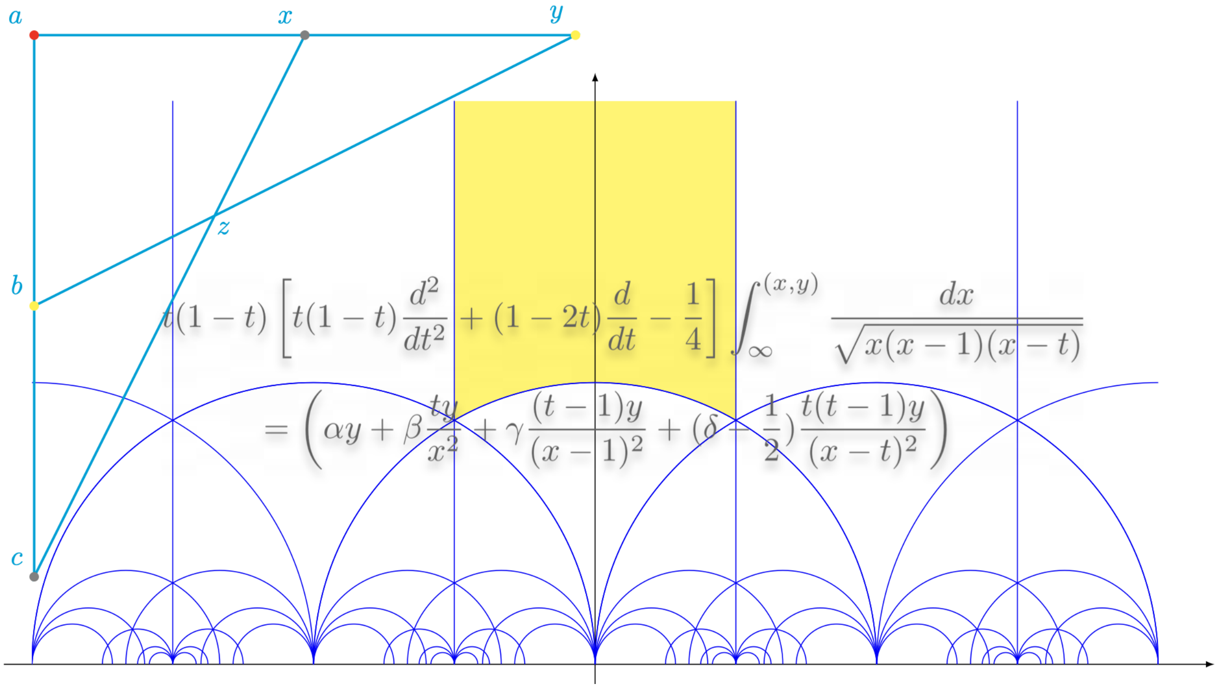
Model theory and differential equations



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Introduction

The model theoretic approach to the study of differential equations has a long and rich history beginning with A. Robinson [Rob59]. The theory of differentially closed fields of characteristic 0, DCF_0, has been studied intensively and has played an important role in the internal development of geometric model theory. It is also behind one the most spectacular application of logic to number theory; namely, E. Hrushovski's celebrated proof of the function field Mordell-Lang conjecture. Furthermore, the study of the theory DCF_0 has lead to substantial development in a Galois theory for differential equations and its applications.

Nevertheless, only very recently have the techniques from model theory been used to study *classical* differential equations. First in the work of the author and A. Pillay on the Painlevé transcendents

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[NP14], [NP17] and then in that of J. Freitag and T. Scanlon [FS18] on the differential equation satisfied by the modular j-function. More recently, in joint work with G. Casale and J. Freitag [CFN18], the author has also studied the differential equations satisfied by the Fuchsian automorphic functions and in the process proved an old claim of P. Painlevé (1895).

In this article, we give an overview of those recent applications of model theory to the study of differential equations. The focus will be on the role of the classification problem for strongly minimal sets and on results in functional transcendence. Unavoidably, many other interesting and important aspect of the interaction between model theory and differential algebra will be omitted.

Differential Algebraic Geometry

Differential algebraic geometry, which has its origin at the beginning of the 1930's, was founded by J. Ritt and E. Kolchin. Although not widely known, it gives a general *algebraic* setting for the study of differential equations and the approach is similar to that of the study of polynomial equations in algebraic geometry. We will in this article focus on *ordinary* differential equations. Moreover, we will say a few words at the end about the setting of partial differential and difference equations. The standard reference for this section is Kolchin's book [Kol73]. All fields will be assumed to be of characteristic 0.

Definition 1. A *differential field* (K, δ) is a field K equipped with a derivation $\delta : K \to K$, i.e., an additive group homomorphism satisfying the Leibniz rule $\delta(xy) = x\delta(y) + y\delta(x)$.

The field of *constants* C_K of K is defined set theoretically as $\{x \in K : \delta(x) = 0\}$. We usually write x' for $\delta(x)$ and $x^{(n)}$ for $\underbrace{\delta \dots \delta\delta}_{n}(x)$.

Example 1. $(\mathbb{C}(t), d/dt)$ the field of rational functions over $\mathbb{C}$ in a single indeterminate, where in this case, the field of constants is $\mathbb{C}$.

Associated with a differential field (K, δ), is the *ring of differential polynomials* $K\{\mathbf{X}\}$ in m differential variables $\mathbf{X} = (X_1, \dots, X_m)$. An element of $K\{\mathbf{X}\}$ is called a *differential polynomial* over K and is simply a regular polynomial with coefficients in K but in variables $\mathbf{X}, \mathbf{X}', \mathbf{X}^{(2)}, \dots$. We use here the notation $\mathbf{X}^{(n)} = (X_1^{(n)}, \dots, X_m^{(n)})$. If $f \in K\{\mathbf{X}\}$, then the order of f, denoted $ord(f)$, is the largest n such that for some i, $X_i^{(n)}$ occurs in f.

Example 2. $f(X) = (X')^2 - 4X^3 - tX$ is a differential polynomial in $\mathbb{C}(t)\{X\}$ and $ord(f) = 1$.

As one can see, if $f \in K\{\mathbf{X}\}$, then $f(\mathbf{X}) = 0$ is an ordinary (algebraic) differential equation. More generally, by a *Kolchin closed* subset of K^n, we mean the common zero set of a finite system of differential polynomial equations, i.e., a set of the form

$$V(S) = \{y \in K^n : f(y) = 0 \text{ for all } f \in S\}$$

where $S \subset K\{\mathbf{X}\}$ is a finite subset. The Kolchin closed sets are the basic closed sets in the Kolchin topology and are the analogues of the basic closed sets in the Zariski topology. A *Kolchin constructible set* is simply a boolean combination of Kolchin closed sets.

Given a differential field (K, δ), it follows that the derivation δ uniquely extends to the algebraic closure K^{alg} of K. However, in order for Kolchin closed sets to necessarily have points whose coordinates are from the underlying field, a much stronger condition than algebraic closedness is needed.

Definition 2. A differential field (K, δ) is said to be *differentially closed* if for every $f, g \in K\{X\}$ such that $ord(f) > ord(g)$, there is $y \in K$ such that $f(y) = 0$ and $g(y) \neq 0$.

Differential algebraic geometry as developed by Kolchin, studies Kolchin closed sets in a differentially closed field. At this point, let us mention that Kolchin closed sets can have very rich algebraic structure. Take for example, the field of constants: if K is differentially closed, then from Definition 2 we see that C_K is an algebraically closed field. Less obvious is that C_K is indeed the only algebraically closed subfield of K that is given by a differential equation. Another interesting well-known example is that of an homogeneous linear differential polynomial

$$f(X) = X^{(n)} + a_{n-1}X^{(n-1)} + \cdots + a_1X' + a_0X, \quad a_i \in K.$$

One has that the associated Kolchin closed set (in a differentially closed field K) is a vector space over C_K.

Kolchin's approach has been instrumental in the development of a Galois theory for differential equations that solidifies and extends the Picard-Vessiot theory for linear differential equations. For example, the fact that in a differentially closed field K, the Galois group of a linear differential equation is a linear algebraic group defined over C_K, has been generalized in Kolchin's strongly normal theory using algebraic groups as the Galois group of so-called logarithmic equations.

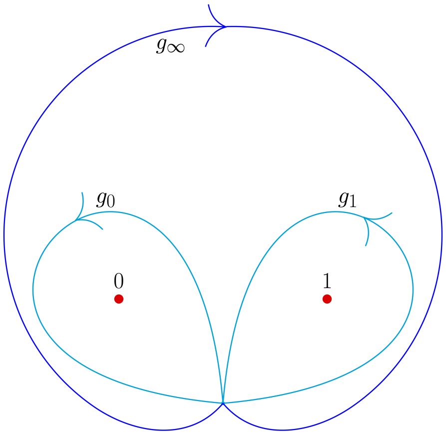

Figure 1: Loops g_0,g_1,g_∞ in the complex plane around the singularities $0, 1, \infty$ of the hypergeometric equation. The differential Galois group is the Zariski closure in $GL_2(\mathbb{C})$ of the monodromy group of the equation.

Kolchin's Galois theory for differential equations has helped answer questions related to the study of the structure of Kolchin constructible sets. Moreover, it can be argued that the point of view of Kolchin's theory coincides with that of the model theory of differentially closed fields. This is of course our approach to studying differential equations.

Model theory

For the topics covered in this article, we recommend D. Marker's book [Mar02]. The starting point in model theory is the notion of a model of a first order theory. Here by a *first order theory* T we mean a set of axioms (or more accurately first order sentences) in a fixed language L. The language L is simply a set of constant symbols, function symbols and relation symbols. We assume throughout that the language is countable.

Example 3. A familiar example is T_G the theory for groups which consists of the usual axioms for groups expressed using the language $L_G = (e, *)$ together with the logical symbols $=$, (,), $\exists$ and $\forall$.

A *structure* for a language L, or an L-*structure* for short is a set together with interpretations for each symbol in L. A *model* of a theory T is simply an L-structure in which the axioms are true. In Example 3, we see that both $(\mathbb{N}, 0, +)$ and $(\mathbb{Z}, 0, +)$ are L_G-structures, moreover only the latter is a model of T_G.

The notion of a (well-formed) formula extends that of an axiom, whereby free variables, that is those not quantified upon, are allowed. Continuing with Example 3, we see that a well-formed formula with free variable X is $\phi(X) := \forall\, Y(X * Y = Y * X)$. For a model G (i.e., a group), if $C(G)$ denotes the set of elements of G which satisfy the formula $\phi(X)$, then we have that $C(G)$ is the center of G. The center, $C(G)$, is an example of a definable set:

Definition 3. A *definable set* $Y \subset M^n$ is a set of the form

$$Y = \{\mathbf{y} \in M^n : \phi(\mathbf{y}) \text{ is true}\}$$

where ϕ is a formula in L with with n free variables.

Remark 1. For any subset $A \subset M$ of a model, one can extend the language L by adding a constant symbol for each element $a \in A$. One usually denotes the new language obtained by L_A. If in Definition 3 one replaces L by L_A for some $A \subset M$, then one obtains the definition of an A-definable set or more precisely a definable set with *parameters* from the set A.

For a fixed theory T a major goal of model theory is to study *all* definable sets in some/any model of T. This of course would be hopeless unless one could identify classes of structures where there are some control over the definable sets. In model theory, this leads the distinction between "tame" and "wild" structures or theories. In this article we discuss two notion of tameness, namely quantifier elimination and ω-stability. There are many more natural "tame" versus "wild" distinctions and some are illustrated in Figure 2.

A theory T is said to have *quantifier elimination* if for every formula $\phi(\mathbf{X})$ there is a quantifier-free formula $\psi(\mathbf{X})$ such that the two define the same definable set. It hence follows that for theories with quantifier elimination the definable sets are defined using "simple" formulas.

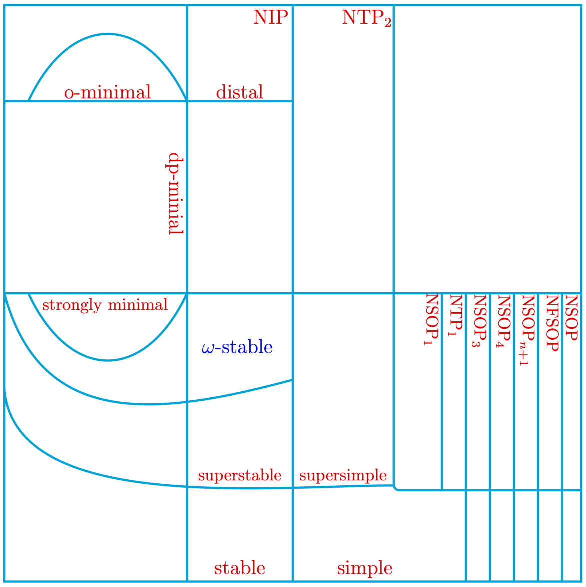

Figure 2: The model theory universe as described at *forkinganddividing.com*. In the sea of theories, an ω-stable theory is ideally placed in the center left of the natural 'tame/wild' divide.

A theory T is ω-stable if every definable set X can be given an intrinsic ordinal valued dimension called the Morley Rank, denoted by $RM(X)$. In rough terms, the inductive definition is as follows: $RM(X) = 0$ if X is finite, and $RM(X) \geq \alpha + 1$ if there are pairwise disjoint definable subsets X_i of X for $i = 1, 2, \ldots$ such that each $RM(X) \geq \alpha$ for all $i < \omega$ (one extends the definition naturally to limit ordinals). We set $RM(X) = \alpha$ if $RM(X) \geq \alpha$ but not $\geq \alpha + 1$. Using this rank, one can define in T a good notion of independence and dimension analogous to the notion of linear independence and basis in the study of vector space.

The theory of algebraically closed field of characteristic zero ACF_0 with the obvious axioms given in the language of rings $L_R = (+, -, \cdot, 0, 1)$ has both quantifier elimination and is ω-stable. In this setting quantifier elimination is equivalent to the Chevalley-Tarski theorem that over an algebraically closed field the projection of a constructible set is constructible. The Morley rank of a definable set (so a constructible set) corresponds to the transcendence degree of a generic point, while the independence notion is equivalent to algebraic independence.

The Theory DCF_0

Let us bring together the ideas of the first two sections. We refer the reader to [Mar96] for historical background and additional details. In the context of differentially closed fields, the relevant language is $L_\delta = (+, -, \cdot, \delta, 0, 1)$, the language of differential rings and we denote by DF_0 the theory of differential fields of characteristic zero. The axioms of DF_0 consist of the axioms for fields and the axioms for the derivation δ.

Now, for each n, d_1 and $d_2 \in \mathbb{N}$, one can write down an axiom (in L_δ) that asserts that if f is a differential polynomial of order n and degree at most d_1 and g is a nonzero differential polynomial of order less than n and degree at most d_2, then there is a solution to $f(X) = 0$ and $g(X) \neq 0$. The theory obtained by adding to DF_0 all these axioms is called the theory of differentially closed fields of characteristic 0, DCF_0. This theory sits on the tame side of many of the most important dividing lines in model theory as shown by Blum [Blu69]:

Theorem 1. *The theory DCF_0 eliminates quantifiers and is ω-stable.*

For the remainder of the article $\mathcal{U}$ will denote a saturated[1] model of DCF_0.

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[1] Saturation is a notion of 'largeness' which mimics the idea that an algebraically closed field of uncountable transcendence degree over the prime field is large/rich.

Quantifier elimination means that a definable set $Y \subseteq \mathcal{U}^n$, definable over a differential subfield K of $\mathcal{U}$, is nothing more than a Kolchin constructible set over K. On the other hand, as discussed above, ω-stability means (among other things) that any definable set has a well-defined ordinal-valued Morley rank. The independence notion in $\mathcal{U}$ is as follows: $\mathbf{a}$ is *independent* from $\mathbf{b}$ over K if $K\langle \mathbf{a} \rangle$ is algebraically disjoint from $K\langle \mathbf{b} \rangle$ over K. Here $K\langle \mathbf{a} \rangle = K(\mathbf{a}, \mathbf{a}', \mathbf{a}^{(2)}, \ldots)$ denotes the differential field generated by $\mathbf{a}$ over K.

Along with the Morley rank, we also have another invariant for definable sets called the *order*. For $\mathbf{a} \in \mathcal{U}^n$ and $K < \mathcal{U}$, we define $ord(\mathbf{a}/K)$ to be the transcendence degree of the field $K\langle \mathbf{a} \rangle$ over K. If $Y \subseteq \mathcal{U}^n$ is definable over K, we define the $ord(Y) = sup\{ord(\mathbf{a}/K) : \mathbf{a} \in Y\}$. One can show that $RM(Y)$ is always less than or equal to $ord(Y)$. Furthermore, $RM(Y) < \omega$ if and only if $ord(Y) < \omega$. We will later see examples of Kolchin closed sets for which the Morley rank is strictly less than the order.

Definition 4. Let $Y \subseteq \mathcal{U}^n$ be a definable set.

1. Y is said to be *finite dimensional (or rank)* if it has finite order, i.e., $ord(Y) < \omega$.
2. Y is said to be *strongly minimal* if it is infinite and for every definable subset $Z \subseteq Y$, either Z or $Y \setminus Z$ is finite.

If Y is strongly minimal then it has Morley rank one. Strongly minimal sets determine, in a precise manner (not to be discussed in this article), the structure of *all* finite dimensional definable sets. This fact, which follows from very general model theoretic considerations, holds in any ω-stable theory and is obtained in part using the robust notion of independence.

Notice that if Y is a definable set with $ord(Y) = n$, then Y is strongly minimal if and only if Y can not be written as the disjoint union of definable sets of order n, and for *any* differential field K over which Y is defined, and element $y \in Y$, then $tr.deg(K\langle y \rangle /K) = 0$ or n.

Example 4. The field of constants $C_{\mathcal{U}}$ is strongly minimal.

Example 5. If f is an absolutely irreducible polynomial over $\mathcal{U}$ in 2 variables then the subset Y of $\mathcal{U}$ defined by $f(y, y') = 0$ is strongly minimal, of order 1.

It is a quite a difficult task to show that the set defined by a given differential equation is strongly minimal. Indeed, except for limited or special cases, no general tools are available. For example, we refer the reader to Section 5.17 of [Mar96], for the (tedious) calculations involved in showing that the subset of $\mathcal{U}$ defined by $\{yy'' = y', y' \neq 0\}$ is strongly minimal, of order 2.

Nevertheless, the goal of understanding all definable sets in DCF_0, goes through a complete understanding of the strongly minimal sets. A considerable amount of work, beginning in the 1990's, has been devoted to just that. The deepest result in that direction, due to E. Hrushovski and Z. Sokolovic [HS94], concerns the classification of strongly minimal sets that have "non-trivial" structures.

Definition 5. Let Y be strongly minimal set defined over a differential field K. Then Y is said to be *geometrically trivial* if for any $y, y_1, \ldots, y_n \in Y$ if $y \in K\langle y_1, \ldots, y_n \rangle^{alg}$, then there is $1 \leq i \leq n$ such that $y \in K\langle y_i \rangle^{alg}$.

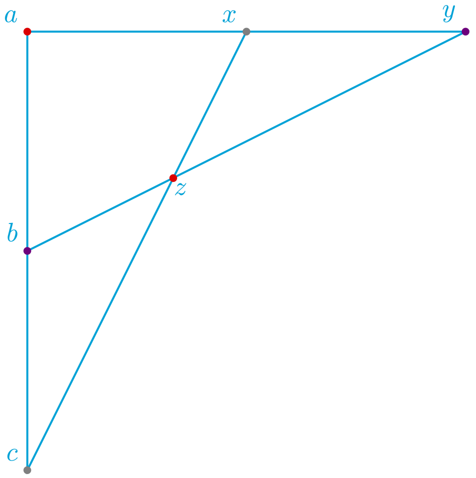

Figure 3: Presence of a definable group: in a non-geometrically trivial strongly minimal set one can find a group configuration. Each point has (Morley) rank 1, each line rank 2 and any three non-collinear points are independent.

In essence, a geometrically trivial set can have at most a 'binary' structure. The field of constants $C_{\mathcal{U}}$ is not geometrically trivial. The same is true of definable groups (i.e., definable sets equipped with definable group structures).

The work of Hrushovski and Sokolovic did not attempt to classify geometrically trivial strongly minimal sets. On the other hand, a key step in their work and which builds on those of A. Buium [Bui92], was the identification of some 'exotic' differential algebraic groups (i.e., definable groups where the underlying definable set is Kolchin closed not Zariski closed).

Theorem 2. *Let A be an abelian variety over $\mathcal{U}$. We identify A with its set $A(\mathcal{U})$ of $\mathcal{U}$-points. Let $A^\sharp$ be the Kolchin closure of the torsion subgroup of A. Then*

1. *$A^\sharp$ is a differential algebraic group and is Zariski dense in A.*
2. *If A is a simple abelian variety that does not descend to $C_{\mathcal{U}}$, then $A^\sharp$ is strongly minimal.*

The group $A^\sharp$ is called the *Manin kernel* of A. One remarkable property of $A^\sharp$ is that every definable subset of it is a finite Boolean combination of cosets of definable subgroups. The result of Hrushovski and Sokolovic is that up to equivalence, the field of constants $C_{\mathcal{U}}$ and the groups $A^\sharp$ cover all the non geometrically trivial examples!

Theorem 3 (The trichotomy theorem). *If $Y \in \mathcal{U}^n$ is strongly minimal, then exactly one of the following hold*

1. *Y is geometrically trivial, or*
2. *(Group-like) Y is non-orthogonal to the Manin kernel $A^\sharp$ of some simple abelian variety A that does not descend to $C_{\mathcal{U}}$, or*
3. *(Field-like) Y is non-orthogonal to the field of constants $C_{\mathcal{U}}$.*

We say that Y and Z (both strongly minimal) are *nonorthogonal* if there is some infinite definable relation $R \subset Y \times Z$ such that $\pi_{Y\restriction R}$ and $\pi_{Z\restriction R}$ are finite-to-one functions. Here $\pi_Y : Y \times Z \to Y$ and $\pi_Z : Y \times Z \to Z$ denote the projections to Y and Z respectively. It is not hard to see that nonorthogonality is indeed an equivalence relation on strongly minimal sets. Furthermore, if Y and Z are nonorthogonal strongly minimal sets, then $ord(Y) = ord(Z)$.

The work of Hrushovski and Sokolovic was never published. Moreover, an alternate proof of the characterization of the field-like strongly minimal sets - a key step - has appeared in the work of A. Pillay and M. Zeigler [PZ03]. A good summary of the proof of Theorem 3 can be found in [NP17, Section 2.1].

There are other interesting and important consequences of the trichotomy theorem that are not apparent but worth mentioning. Firstly, if $A^\sharp$ is the Manin Kernel of a simple abelian variety A that does not descend to $C_{\mathcal{U}}$, then $ord(A^\sharp) \geq 2$. Hence, strongly minimal sets of order 1 are either geometrically trivial or non-orthogonal to $C_{\mathcal{U}}$. Secondly, strongly minimal sets that are defined over $C_{\mathcal{U}}$ and of order ≥ 2 are geometrically trivial! This surprising fact was somewhat forgotten for a while but now plays a crucial role in some of the applications of the theory to functional transcendence as we shall see in the next section.

Finally, it is worth mentioning that strong minimality is closely related to Painlevé's notion of *irreducibility* of differential equations. Roughly speaking, a differential equation is irreducible if none of it solutions are "known" special functions. Establishing irreducibility, which goes through establishing strong minimality, has been part of long-standing open conjectures in the theory of non-linear special function.

Trivial Pursuits and Applications

As we have seen the trichotomy theorem, which gives a very general classification theorem for strongly minimal sets, has nothing to say about geometrically trivial strongly minimal sets. Understanding these strongly minimal sets, or trivial pursuits (as coined by J. Baldwin and L. Harrington), is one of the most important open problems in the study of DCF_0. But to this date very little progress has been made.

For a while it was conjectured that all geometrically trivial strongly minimal sets would have no (or very little) structure: for any element y of a trivial strongly minimal set Y only finitely many other elements of Y are interalgebraic with y. More precisely

Definition 6. Let Y be strongly minimal set defined over a differential field K. Then Y is said to be *ω-categorical* if for any tuple $\mathbf{b}$ from $\mathcal{U}$, the set $K\langle\mathbf{b}\rangle^{alg}\cap Y$ is finite.

If a strongly minimal set is ω-categorical, then it is geometrically trivial. A beautiful result of E. Hrushovski [Hru95] is that the converse holds for order 1 strongly minimal sets (cf. [Pil02, Cor 1.82]] and [FM17] for a generalization):

Theorem 4. *Let $Y\subset\mathcal{U}^n$ be an order 1 geometrically trivial strongly minimal set. Then Y is ω-categorical.*

This result of Hrushovski gave rise to a conjecture about geometrically trivial strongly minimal sets of arbitrary order: *In differentially closed fields, every geometrically trivial strongly minimal set is ω-categorical.* This was proven to be false at this level of generality in [FS18] using the order 3 differential equation satisfied by the modular j-function (see below). The following interesting question remains.

Question 1. Are all order 2 geometrically trivial strongly minimal sets ω-categorical?

At this point, let us mention that if a strongly minimal set Y has $ord(Y)=n$ and is defined over K, then ω-categoricity can be translated to the following strong transcendence statement: there is a $m\in\mathbb{N}$ such that if $y_1,\ldots,y_k\in Y$ are distinct and satisfy $tr.deg(K\langle y_1,\ldots,y_k\rangle/k)=nk$, then for any other $y\in Y$, except for at most mk, we have that $tr.deg(K\langle y_1,\ldots,y_k,y\rangle/k)=n(k+1)$. It follows that establishing strong minimality, geometric triviality and ω-categoricity can be seen as part of a strategy to tackle number theoretic/functional transcendence type result for the solutions of the differential equations. As such a positive answer to the above question is of great interest. We will now illustrate this by looking at several recent applications of the model theoretic approach, in particular the trivial pursuits, to some classical differential equations.

The generic Painlevé Transcendents

The Painlevé equations are second order ordinary differential equations and come in six families P_I-P_{VI}, where P_I consists of the single equation

$$\frac{d^2y}{dt^2}=6y^2+t,$$

and $P_{II}-P_{VI}$ come with some complex parameters. They were isolated in the early part of the 20^{th} century, by P. Painlevé, with refinements by B. Gambier and R. Fuchs, as those ODE's of the form $y''=f(y,y',t)$ (where f is rational over $\mathbb{C}$) which have the Painlevé property: any local analytic solution extends to a meromorphic solution on the universal cover of $P^1(\mathbb{C})\setminus S$, where S is the finite set of singularities of the equation. The equations have arisen in a variety of important physical applications including, for example, statistical mechanics, general relativity and fibre optics.

Example 6. The second Painlevé equation $P_{II}(\alpha)$ is given by

$$\frac{d^2y}{dt^2}=2y^3+ty+\alpha$$

where $\alpha\in\mathbb{C}$. The equation appears quite prevalently in random matrix theory (cf. [FW15])

Painlevé believed that, at least for general values of the parameters, the set defined by the equations would be strongly minimal. This was proven to be true in a series of papers by K. Okamoto, K. Nishioka, M. Noumi, H. Umemura and H. Watanabe (cf. [Oka99] for a survey). In particular, the first Painlevé equations is strongly minimal and in the case of the second Painlevé equation, they proved that $P_{II}(\alpha)$ is strongly minimal if and only if $\alpha\notin\frac{1}{2}+\mathbb{Z}$. By a *generic* Painlevé equation we mean one equation among the family P_I-P_{VI}, such that all the corresponding complex parameters are transcendental and algebraically independent over $\mathbb{Q}$. So $P_{II}(\pi)$ is a generic equation. The works of Watanabe and others hence give that all the generic Painlevé equations are strongly minimal. They left wide open the question of the fine structure of the definable sets. We now have a full answer.

Theorem 5. *Suppose $y_1, \dots, y_n$ are distinct solutions of one of the generic Painlevé equations. Then $y_1, y_1', \dots, y_n, y_n'$ are algebraically independent over $\mathbb{C}(t)$, i.e.,*

$$tr.deg(\mathbb{C}(t)(y_1, y_1', \dots, y_n, y_n')/\mathbb{C}(t)) = 2n.$$

In particular the generic Painlevé equations are all ω-categorical. K. Nishioka [Nis04] proved the result for P_I using differential algebra. However his calculations and techniques does not seem to generalize to the other equations. The author, in [Nag20] and before that in joint work with Pillay in [NP14], proved the result for all the other equations using model theory. The proofs rely heavily on earlier work [NP17] in which the trichotomy is used to show that the generic Painlevé equations are all geometrically trivial.

The model theoretic approach has also allowed us to show that the generic equations from most distinct Painlevé families are orthogonal. Work is currently underway towards obtaining a full classification of algebraic relations between solutions of the Painlevé equations. As of now, except for the second Painlevé equations (where for example the author showed geometrically triviality holds if and only if $\alpha \notin \frac{1}{2} + \mathbb{Z}$) the study of the non-generic Painlevé equations is wide open. The following is an example of the most basic question one would like to answer.

Question 2. For which values of the parameters of a fixed Painlevé equation is it true that if $y_1, \dots, y_n$ are distinct solutions (not in $\mathbb{C}(t)^{alg}$), then

$$tr.deg(\mathbb{C}(t)(y_1, y_1', \dots, y_n, y_n')/\mathbb{C}(t)) = 2n?$$

Fuchshian Automorphic Functions

We now consider the most natural generalizations of the trigonometric and elliptic functions (i.e., the periodic functions).

Let $\Gamma \subset PSL_2(\mathbb{R})$ be a Fuchsian group, that is, assume that Γ is a discrete subgroup of $PSL_2(\mathbb{R})$. A point $\tau \in \mathbb{H} \cup \mathbf{P}^1(\mathbb{R})$ is said to be a cusp if its stabilizer group $\Gamma_\tau = \{g \in \Gamma \ : \ g \cdot \tau = \tau\}$ has infinite order. We also assume throughout that Γ is of first kind (*i.e.*, its limit set is $\mathbf{P}^1(\mathbb{R})$) and of genus zero (*i.e.*, $\Gamma \backslash \mathbb{H}$ can be compactified to a compact Riemann surface of genus 0). An *automorphic function* f for Γ is a function on the complex upper half plane $\mathbb{H}$, such that[2]

$$f(g \cdot \tau) = f(\tau) \quad \text{for all} \ \ g \in \Gamma \text{ and } \tau \in \mathbb{H},$$

and such that f is meromorphic at every cusp of Γ. The collection $\mathcal{A}_0(\Gamma)$ of all automorphic functions for Γ is a field and is generated (over $\mathbb{C}$) by some automorphic function called an *hauptmodul* or *uniformizer* for Γ. We will denote by $j_\Gamma(t)$ one such fixed hauptmodul.

It is a classical fact that $j_\Gamma(t)$ satisfy a third order ordinary differential equation of Schwarzian type

$$S_t(y) + (y')^2 R_{j_\Gamma}(y) = 0. \qquad (\star)$$

Here $S_t(y) = \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2$ denotes the Schwarzian derivative ($' = \frac{d}{dt}$) and

$$R_{j_\Gamma}(y) = \frac{1}{2}\sum_{i=1}^{r} \frac{1 - \alpha_i^{-2}}{(y - a_i)^2} + \sum_{i=1}^{r} \frac{\beta_i}{y - a_i}$$

with $a_1, \dots, a_n$ and $\beta_1, \dots, \beta_n$ real numbers depending on Γ and j_Γ. Every solution in $\mathcal{U}$ of the Schwarzian equation $(\star)$ can be taken to be of the form $j_\Gamma(g \cdot t)$ for some $g \in GL_2(\mathbb{C})$.

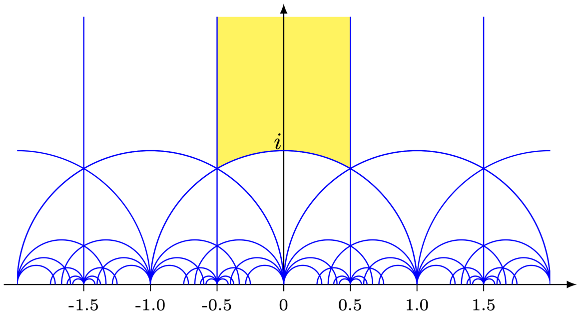

Figure 4: The fundamental domain for the action of a Fuchsian group of the first kind - here $PSL_2(\mathbb{Z})$ - has a finite number of generators and is of finite volume.

Example 7. If $\Gamma = PSL_2(\mathbb{Z})$, then the classical modular j-function

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots,$$

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[2] Throughout $g \cdot \tau$ will denote the action of an element of $GL_2(\mathbb{C})$ by linear fractional transformation.

where $q = e^{2\pi i \tau}$, is an hauptmodul. In this case the differential equation is given with

$$R_j(y) = \frac{y^2 - 1968y + 2654208}{y^2(y-1728)^2}$$

P. Painlevé in 1895, again claimed that the set defined by equation ($\star$) would be strongly minimal. K. Nishioka proved that the Hauptmodul j_Γ does not satisfy any algebraic differential equation of order two or less over $\mathbb{C}(t, e^{\lambda t})$, for any $\lambda \in \mathbb{C}$. He also obtained a very weak form of Painlevé assertion in the case of triangle groups. The first real progress was made by J. Freitag and T. Scanlon [FS18] in their work on the modular j-function (they did not know of Painlevé's claim then).

Theorem 6. *Let $\Gamma = PSL_2(\mathbb{Z})$. Then the set defined by the Schwarzian equation ($\star$) is strongly minimal, geometrically trivial but not ω-categorical.*

Their proof relies on a deep functional transcendence result of J. Pila [Pil13] called the modular Ax-Lindemann-Weierstrass theorem with derivatives (see below).

Remark 2. Granted that strong minimality holds, it is rather unsurprising that the definable set in the case of the j-function is not ω-categorical. Indeed, for each $n \in \mathbb{N}$ we have the classical modular polynomials $\Phi_n(X,Y) \in \mathbb{Z}[X,Y]$ that relates solution of the equation for j: if g_1 and g_2 are in the same coset of $GL_2(\mathbb{Q})$, then $\Phi_n\left(j(g_1 \cdot t), j(g_1 \cdot t)\right) = 0$ for some n.

For a while the result of Freitag and Scanlon seemed to have shut the door on a possible classification of geometrically trivial strongly minimal sets. However, it turned out that studying the Schwarzian equation ($\star$) in its full generality has allowed us to place the case $\Gamma = PSL_2(\mathbb{Z})$ in context. A natural and key question is the following: is there a way to explain the existence of the modular polynomials? The answer is again very classical and is brought to light through the notion of commensurability.

Recall that two subgroups G and H of $\mathrm{PSL}_2(\mathbb{R})$ are commensurable, denoted by $G \sim H$, if their intersection $G \cap H$ has finite index in both G and H. For a Fuchsian group Γ, let $\mathrm{Comm}(\Gamma)$ be the commensurator of Γ, namely

$$\mathrm{Comm}(\Gamma) = \{g \in PSL_2(\mathbb{R}) \ : \ g\Gamma g^{-1} \sim \Gamma\}.$$

If $g \in \mathrm{Comm}(\Gamma) \setminus \Gamma$, then one has that the intersection $\Gamma_g = g\Gamma g^{-1} \cap \Gamma$ is a Fuchsian group of first kind and with the same set of cusps as Γ. But the functions $j_\Gamma(t)$ and $j_\Gamma(g^{-1}t)$ are respective uniformizers for Γ and $g\Gamma g^{-1}$. It follows that they also are automorphic functions for Γ_g. The work of H. Poincaré gives that any two automorphic functions for a Fuchsian group are algebraically dependent over $\mathbb{C}$. So there is a polynomial $\Phi_g \in \mathbb{C}[X,Y]$, such that $\Phi_g\left(j_\Gamma(t), j_\Gamma(g \cdot t)\right) = 0$. Such polynomial is called a Γ-*special polynomial.*

So if Γ has infinite index in $\mathrm{Comm}(\Gamma)$, then there are infinitely many Γ-special polynomials. In particular, if one can prove strong minimality, then non-ω-categoricity would follow. It turns out that groups Γ having this 'infinite index' property are well known in group theory.

Let F be a totally real number field and let A be a quaternion algebra over F that is ramified at exactly one infinite place. Let ρ be the unique embedding of A into $M_2(\mathbb{R})$ and let $\mathcal{O}$ be an order in A. The image $\rho(\mathcal{O}^1)$ of the norm-one group of $\mathcal{O}$ under ρ is a discrete subgroup of $SL_2(\mathbb{R})$. We denote by $\Gamma(A, \mathcal{O})$ the Fuchsian group obtained under the projection in $PSL_2(\mathbb{R})$ of the group $\rho(\mathcal{O}^1)$.

Definition 7. A Fuchsian group Γ is said to be arithmetic if it is commensurable with a group of the form $\Gamma(A, \mathcal{O})$.

The modular group $PSL_2(\mathbb{Z})$ and its finite index subgroups are the most well-known examples of arithmetic groups. We have the following deep result of G. Margulis.

Theorem 7. *The group Γ is arithmetic if and only if it has infinite index in $Comm(\Gamma)$ and so there are infinitely many Γ-special polynomials.*

The work of the author with G. Casale and J. Frietag [CFN18], completely proves Painlevé's claim and provides a striking connection between categoricity and arithmeticity.

Theorem 8. *Let Γ be a Fuchsian group of first kind and genus zero and let X_Γ be the set defined by the Schwarzian equation ($\star$). Then*

1. *X_Γ is strongly minimal and (so) geometrically trivial.*
2. *X_Γ is ω-categorical if and only if Γ is non-arithmetic*

The techniques in the proof of Theorem 8 relies on differential Galois theory, monodromy of linear differential equations, the study of algebraic and Liouvillian solutions, differential algebraic work of Nishioka towards the Painlevé irreducibility of certain Schwarzian equations, and considerable machinery from the model theory of differentially closed fields. The following question can be seen as the next major challenge in the classification of geometrically trivial strongly minimal sets in differentially closed fields.

Question 3. In DCF_0, does every non-ω-categorical strongly minimal set arise from an arithmetic Fuchsian group in this way?

Finally, let us mention that the above work on fully classifying the structure of the definable sets associated with the Schwarzian equation ($\star$) has been used in [CFN18] to give a proof of the Ax-Lindemann-Weierstrass Theorem with derivatives for Γ: Let $V \subset \mathbb{A}^n$ be an irreducible algebraic variety defined over $\mathbb{C}$ such that $V(\mathbb{C}) \cap \mathbb{H}^n \neq \emptyset$ and V projects dominantly to each of its coordinates (each coordinate function is nonconstant). Let $t_1, \ldots, t_n$ be the functions on V induced by the canonical coordinate functions on $\mathbb{A}^n$. We say that $t_1, \ldots, t_n$ are Γ-*geodesically independent* if there are no relations of the form $t_i = g \cdot t_j$ where $i \neq j$ and $g \in \mathrm{Comm}(\Gamma)$.

Theorem 9. *With the notation (and assumption $V(\mathbb{C}) \cap \mathbb{H}^n \neq \emptyset$) as above, suppose that $t_1, \ldots, t_n$ are Γ-geodesically independent. Then the 3n functions*

$$j_\Gamma(t_1), j'_\Gamma(t_1), j''_\Gamma(t_1) \ldots, j_\Gamma(t_n), j'_\Gamma(t_n), j''_\Gamma(t_n)$$

(defined locally) on $V(\mathbb{C})$ are algebraically independent over $\mathbb{C}(V)$.

As mentioned earlier, J. Pila [Pil13] had already proved the result for $PSL_2(\mathbb{Z})$. J. Freitag and T. Scanlon [FS18] established the same for arithmetic subgroups of $PSL_2(\mathbb{Z})$. The Ax-Lindemann-Weierstrass (mostly without derivatives) has also been proved by various authors in the more general context of Shimura varieties. The work in [CFN18] differs from all the above in that it does not use a tool called *o-minimality* (originating in model theory) and also tackles the non-arithmetic groups as well as the derivatives of the functions all at once.

Beyond DCF_0

We end by saying a few words about the partial differential and the difference equations settings. We denote by $DCF_{0,m}$ the theory of differentially closed field of charateristic 0 with m commuting derivations (partial context) and by $ACFA$ the theory of algebraically closed field with a generic automorphism (difference context). The theory $DCF_{0,m}$ is also ω-stable and has quantifier elimination. However, strongly minimal sets do not fully capture the complexity of all definable sets. There are so called infinite rank regular types[3] that do so. The trichotomy theorem is yet to be fully established in that setting. On the other hand, $ACFA$ is not ω-stable but is rather a so-called *simple theory* (characterized by existence of a good notion of independence). Furthermore, although definable sets are still given by simple enough formulas, $ACFA$ does not have full quantifier elimination. A version of the trichotomy theorem does hold in that setting and the study of $ACFA$ has been very successfully used to obtain new results in number theory and algebraic dynamics.

However in both cases, except for few examples, applications to the study of classical equations is yet to be undertaken. There are obvious candidates that would mirror the situation of DCF_0. In $DCF_{0,m}$, tackling the generalized Schwarzian equations for uniformizers for Shimura varieties is of great interest. In $ACFA$, proving that the q-Painlevé equations are

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[3] One such infinite rank regular type also exists for DCF_0. However the finite rank part of the theory is where most of the complexity lies.

rank 1 is a challenge. These difference equations are discrete analogues of the classical Painlevé equations. In fact, in many real world problems, the Painlevé equations arise from a limiting process, starting with the q-Painlevé equations. We expect that as with DCF_0, important model theoretic questions about the structure of definable sets can be formed and answered by studying these concrete differential and difference equations.

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