Open Problems

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Context

I am interested in systems of polynomial differential equations :

$$\left\{\begin{array}{l} y_1(0) = 0 \\ y_2(0) = 1 \\ y_3(0) = 0 \end{array}\right. \qquad \left\{\begin{array}{l} y_1' = y_2 \\ y_2' = -2y_3y_2^2 \\ y_3' = 1 \end{array}\right.$$

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where $y_0 \in \mathbb{R}^n$, $p : \mathbb{R}^n \to \mathbb{R}^n$ vector of polynomial. By Cauchy-Lipschitz, there exists a unique maximal solution $y : I \to \mathbb{R}^n$.

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My interests :

- study the class of solutions (and its multivariate extensions),
- compute with them (model of computation, talk on Friday),
- understand the impact of coefficients
- study the series generated this way

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Non-examples :

- non-analytic functions
- Riemann Γ and ζ

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Open problem

What is $\mathbb{R}_G$? Is it the case that $\mathbb{R}_G = \mathbb{R}_P$?

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- If $(a_n)_n \in \mathbb{Z}^{\mathbb{N}}$ generable, is $(a_n \mod 2)_n$ generable ? Or $(\operatorname{sgn} a_n)_n$? Something of that nature needed to encode conputations.