Solving difference equations in sequences

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Systems of algebraic difference equations

$$\sigma^5(y)\sigma^4(z)^2 - \sigma^4(y)^2\sigma^4(z)^2 + \sigma^2(y)z^3 - \sigma(y)^2z^3 + \sigma^4(z)^4 + z^5 = 0$$

$$\sigma(z) - 2yz = 0$$

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Solution: $(y, z) = (\cos(x), \sin(x)), \quad \sigma(f(x)) = f(2x).$

$$\sigma(z) - 2yz = 0 \leftrightsquigarrow \sin(2x) - 2\sin(x)\cos(x) = 0$$

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$$y_{n+5}z_{n+4}^2 - y_{n+4}^2z_{n+4}^2 + y_{n+2}z_n^3 - y_{n+1}^2z_n^3 + z_{n+4}^4 + z_n^5 = 0$$

$$z_{n+1} - 2y_nz_n = 0$$

Classical difference algebra (J. Ritt and R. Cohn)

k σ-field

$$F \subseteq k\{y\} = k[y_1, \ldots, y_n, \sigma(y_1), \ldots, \sigma(y_n), \sigma^2(y_1), \ldots, \sigma^2(y_n), \ldots]$$

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$$\mathbb{I}(\mathbb{V}(F)) := \left\{ f \in k\{y\} \;\middle|\; \begin{array}{c} f \text{ vanishes on all solutions of } F \\ \text{in all } \sigma\text{-field extension of of } k \end{array} \right\}$$

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$\{F\}$ = smallest perfect σ-ideal of $k\{y\}$ that contains F
I perfect: $f\sigma(f) \in I \Rightarrow f \in I$.

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F has a solution in some σ-field extension of $k \Leftrightarrow 1 \notin \{F\}$.

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Decidability

Given $f \in k\{y\}$ and $F \subseteq k\{y\}$ finite, there exists an algorithm that decides if $f \in \{F\}$. In particular, the question if F has a solution in some σ-field extension of k is decidable.

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σ-fields versus sequences

$$\sigma(y) + y = 1$$

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$$y_{n+1} + y_n = 1$$
$$y_n y_{n+1} = 0$$

has solution $(y_n)_{n\in\mathbb{N}} = (0, 1, 0, 1, 0, 1, \ldots)$.

Solutions in sequences (A. Ovchinnikov, G. Pogudin, T. Scanlon, 2020)

k algebraically closed σ-field, $F \subseteq k\{y\} = k\{y_1, \ldots, y_n\}$ finite

$\sigma \colon k^{\mathbb{N}} \to k^{\mathbb{N}}, \ \sigma((a_n)_{n\in\mathbb{N}}) = (a_{n+1})_{n\in\mathbb{N}}$
$k \to k^{\mathbb{N}}, \ \lambda \mapsto (\sigma^n(\lambda))_{n\in\mathbb{N}}.$

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Counterexample

The strong Nullstellensatz fails for $k = \overline{\mathbb{Q}}$.

Strong Nullstellensatz for arbitrary k

k σ-field, $F \subseteq k\{y\} = k\{y_1, \ldots, y_n\}$

$$I(V(F)) := \left\{ f \in k\{y\} \;\middle|\; \begin{array}{c} f \text{ vanishes on all solutions of } F \text{ in } K^{\mathbb{N}} \\ \text{for all field extensions } K \text{ of } k \end{array} \right\}$$

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Strong Nullstellensatz

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Recall: Decidability (Cohn)

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Given $F \subseteq k\{y\}$ finite, there exists an algorithm that decides if $1 \in \sqrt{[F]}$.

Undecidability

Given $f \in k\{y\}$ and $F \subseteq k\{y\}$ finite, the problem

$$\text{“}f \in \sqrt{[F]}\ ?\text{”}$$

is undecidable.

More undecidability

Recall: Decidability (Ovchinnikov, Pogudin, Scanlon)

Given $F \subseteq \mathbb{C}\{y\}$ finite, there exists an algorithm that decides if F has a solution in $\mathbb{C}^{\mathbb{N}}$.

More undecidability

Recall: Decidability (Ovchinnikov, Pogudin, Scanlon)

Given $F \subseteq \mathbb{C}\{y\}$ finite, there exists an algorithm that decides if F has a solution in $\mathbb{C}^{\mathbb{N}}$.

Undecidability

Given $F \subseteq \mathbb{R}\{y\}$ finite, the problem

"Does F have a solution in $\mathbb{R}^{\mathbb{N}}$?"

is undecidable.

Partial difference equation

$$\sigma_1(\sigma_2(y)) - \sigma_1(y) - \sigma_2(y) + y - y^2 z = 0$$

$$\sigma_1(\sigma_2^2(z))\sigma_1(y) + z = 0$$

Partial difference equation

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$$y_{m+1,n+1} - y_{m+1,n} - y_{m,n+1} + y_{m,n} - y_{m,n}^2 z_{m,n} = 0$$
$$z_{m+1,n+2} y_{m+1,n} + z_{m,n} = 0$$

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$$z_{m+1,n+2} y_{m+1,n} + z_{m,n} = 0$$

Undecidability

Given

$$F \subseteq k\left[\sigma_1^{\alpha_1}(\sigma_2^{\beta_1}(y_1)), \ldots, \sigma_1^{\alpha_n}(\sigma_2^{\beta_n}(y_n)) \middle| \, \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \geq 0\right]$$

finite, the problem

"Does F have a solution in $k^{\mathbb{N}^2}$?"

is undecidable.

The heart of the proof: Counterexample and undecidability of $f \in \sqrt{[F]}$

Lemma

$p\colon \mathbb{A}_k^n \to \mathbb{A}_k^n$ piecewise polynomial map, $V \subseteq \mathbb{A}_k^n$ closed. Then $\exists$ $r \geq 1$ and $F \subseteq k\{y_1, \ldots, y_r\}$ finite and $f \in k\{y_1, \ldots, y_n\}$ such that the following are equivalent:

- There exists a sequence $(\mathbf{x}_i)_{i\in\mathbb{N}} = (x_{1,i}, \ldots, x_{n,i})_{n\in\mathbb{N}} \in (k^{\mathbb{N}})^n$ with $\mathbf{x}_0 \in V$, $\mathbf{x}_{i+1} = p(\mathbf{x}_i)$ and $x_{n,i} \neq 0$ for $i \geq 1$.
- $f \notin \sqrt{[F]}$.

Thank you!

Reference:

- G. Pogudin, T. Scanlon and M. Wibmer, Solving difference equations in sequences: Universality and Undecidability, arXiv:1909.03239