Volterra Integral Operators and Generalized Reynolds Algebras

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Abstract

- We study algebraic structures underlying Volterra integral operators, in particular the operator identities satisfied by such operators.
- While the operator satisfies the Rota-Baxter identity when the kernel of the operator only depends on the phantom (dummy) variable, we show that when the kernel is more generally separable, the operator satisfies a generalized Reynolds identity which, in its original form, can be tracked back to the famous study of Reynolds in fluid mechanics in the late 19th century.
- Furthermore a generalized differential operator arises which combined with the generalized Reynolds operator provides an algebraic context to study Volterra operators and equations.
- Notions of operated algebra and matching Rota-Baxter algebras are applied to give a general context of integral equations and to explore the linearity of integral equations.

Volterra integral equations

- Let $C(V)$ denote the algebra of continuous functions on an Euclidean space. For a given $K(x,t) \in C(\mathbb{R}^2)$ and $a \in \mathbb{R}$, the integral operator

$$P_{K,a} : C(\mathbb{R}) \to C(\mathbb{R}),$$

$$P_{K,a}(f)(x) = \int_a^x K(x,t) f(t)\, dt,$$

  is called a Volterra (integral) operator with kernel $K(x,t)$.
- If the upper limit x is fixed, then the operator is called a Fredholm operator.
- An equation in which all integral operators are Volterra operators is called a Volterra integral equations.
- A kernel $K(x,t)$ is called separable if it can be decomposed as $K(x,t) = k(x)h(t)$ for some functions k and h in $C(\mathbb{R})$. If $k(x)$ is a constant, that is $K(x,t)$ is a function of the dummy variable t only, then K is called phantom.

Examples

- The Volterra's population model for a species in a closed system describes the population $u(t)$ of a species when exposed to both crowding and toxicity; it can be written as an integral equation of the form

$$u(t) = u_0 + a \int_0^t u(x)\, dx - b \int_0^t u(x)^2\, dx - c \int_0^t u(x) \int_0^x u(y)\, dy\, dx,$$

  where a, b, and c are the birth rate, crowding coefficient, and toxicity coefficient, respectively, and u_0 is the initial population.
- The Thomas-Fermi equation describes the potential $y(x)$ of an atom in terms of the radius x, and can be written as an integral equation of the form

$$y(x) = 1 + Bx + \int_0^x \int_0^t s^{-1/2} y(s)^{3/2}\, ds\, dt,$$

  where B is a known parameter.

Matching Rota-Baxter algebras

- For any $\lambda \in \mathbf{k}(=\mathbb{R})$, a Rota-Baxter algebra of weight λ is a $\mathbf{k}$-algebra A together with a linear operator $P : A \to A$ satisfying

$$P(f)P(g) = P(fP(g)) + P(P(f)g) + \lambda P(fg), \quad \forall f, g \in A.$$

- Fix Ω a non-empty set and let $\lambda_\Omega := (\lambda_\omega)_{\omega\in\Omega}$ be a family from $\mathbf{k}$ parameterized by Ω. An Ω-matching Rota-Baxter algebra of weight λ_Ω, or simply a matching Rota-Baxter algebra is a $\mathbf{k}$-algebra A together with a family of linear operators $P_\Omega := \{P_\omega \,|\, \omega \in \Omega\}$, where $P_\omega : A \to A$ for all $\omega \in \Omega$, satisfying

$$P_\alpha(f)P_\beta(g) = P_\alpha(fP_\beta(g)) + P_\beta(P_\alpha(f)g) + \lambda_\beta(P_\alpha(fg)), \forall f, g \in A,\ \alpha, \beta \in \Omega.$$

If $\lambda_\omega = 0, \forall \omega \in \Omega$, the matching Rota-Baxter algebra is said to have weight 0.

Volterra operators with phantom kernels

- A kernel $K(x,t)$ is called phantom if it only dependent on t.
- For any $K, H \in C(\mathbb{R})$, $P_K(\bullet) := \int_a^x K(t) \bullet dt$ and P_H satisfy the matching Rota-Baxter identity

$$P_K(f)P_H(g) = P_K(fP_H(g)) + P_H(P_K(f)g), \forall f, g \in C(\mathbb{R}).$$

- For a family $K_\omega, \omega \in \Omega$ of phantom kernels, $C(\mathbb{R})$ with the Volterra operators $P_{K_\omega, a}$ is a matching Rota-Baxter algebra.

Volterra operators with general kernels

- If the kernel of a Volterra operator is not phantom, then the operator is almost never a Rota-Baxter operator.
- Example Let $K(x,t) = x$ and $f = g = 1$, then

$$P_K(f)(x)\, P_K(g)(x) = \left(\int_a^x x dt\right)^2 = x^2(x-a)^2.$$

$$P_K(P_K(f)g)(x) + P_K(fP_K(g))(x) = 2x\left(\frac{x^3 - a^3}{3} - a\frac{x^2 - a^2}{2}\right).$$

Thus $P_K(f)P_K(g) \neq P_K(P_K(f)g) + P_K(fP_K(g))$.
- Theorem Suppose $K(x,t) = k(x)h(t)$ with $k \in C^1(\mathbb{R})$ and $h \in C(\mathbb{R})$, and suppose k is not constant on any open interval of $\mathbb{R}$. Consider the Volterra operator $P_K : C(\mathbb{R}) \to C(\mathbb{R})$

$$P_K(f)(x) := \int_a^x k(x)h(t)f(t)dt.$$

For given $f, g \in C(\mathbb{R})$, the action of P_K on f, g satisfies the Rota-Baxter identity if and only if both sides of the equation is zero. In particular, $P_K(f)P_K(g) = 0$.

Generalized Reynolds operators

- A Reynolds algebra is a **k**-algebra A together with a linear operator $R : A \to A$ satisfying
$$R(f)R(g) = R(fR(g)) + R(R(f)g) - R(R(f)R(g)), \quad \forall f, g \in A.$$
Then R is called a Reynolds operator.
- Let $K(x,t) = k(x)h(t)$ with $k \in C^1(\mathbb{R})$ and $h \in C(\mathbb{R})$ both zero free. Define
$$D_K : C^1(\mathbb{R}) \to C(\mathbb{R}), D_K(f) := \frac{1}{h(x)} \left(\frac{f(x)}{k(x)}\right)'.$$
Then D_K satisfies the operator identity
$$D_K(fg) = D_K(f)g + fD_K(g) - D_K(1)fg, \forall f, g \in C^1(\mathbb{R}).$$
Further the Volterra operator
$$P_K : C(\mathbb{R}) \to C^1(\mathbb{R}), P_K(f) := k(x)\int_a^x h(t)f(t)dt$$
satisfies the operator identity
$$P_K(f)P_K(g) = P_K(P_K(f)g) + P_K(fP_K(g)) - P_K(D_K(1)P_K(f)P_K(g))$$

Special cases and examples

- Since $D_K(1) = -\frac{k'(x)}{h(x)k(x)^2}$, $D_K(1) = 0$ if and only if $k(x)$ is a nonzero constant. Thus when K is separable, the Volterra operator P_K is a Rota-Baxter operator (of weight zero) if and only if K is phantom.
- For $K(x,t) = e^{-x+t} = e^t/e^x$, we have $D_K(1) = 1$. So the operator $P_K : C((1,\infty)) \to C^1((1,\infty))$ satisfies the identity

$$P_K(f)P_K(g) = P_K(P_K(f)g) + P_K(fP_K(g)) - P_K(P_K(f)P_K(g)).$$

  This is the defining identity of the Reynolds operator arising from turbulence theory in fluid mechanics and interested to R. Birkhoff and G.-C. Rota many years ago.
- Next: set up an algebraic framework for integral equations

What to expect for integral algebra

- In general and loose language, an *algebraic* integral equation is the annihilation of an "integral" algebraic expression $\Phi(Y, P_\Omega, A)$ consisting of several ingredients and restrictions:
- a set Y of variables or unknown functions to be solved from the integral equation;
- a set $P_\Omega = \{P_\omega \,|\, \omega \in \Omega\}$ of integral operators in various forms, set apart by
  1. the lower or upper limits (each being fixed or variable, and in the later case, independent variables or intermediate variables).
  2. the kernels for the integral operators, as functions in both the dummy variables of the integrals and the independent variables for the integral equation;

  Special cases are the Volterra operators and the Fredholm operators;
- a set A of "free term" or coefficient functions which can appear both inside and outside of the integrals. Some of them can be constant, treated as parameters;
- These ingredients can be put together only by the algebraic operations together with the action of the integral operators.

Motivations for algebraic integral equations I

- To motivate our general framework to interpret integral equations, let us first recall how we formulate *algebraic* equations. An algebraic equation consists several ingredients:
- a set X of variables;
- a set of “free term” elements from a prefixed ring or an algebra A.
- The ingredients can be put together by only the algebraic operations.
- The algebraic framework for algebraic equation is the polynomial algebra, as a realization of the free (commutative) algebra.

Motivations for algebraic integral equations II

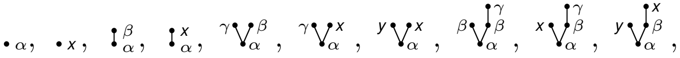
- As another motivation which is more closely related to our situation, let us consider differential equations, consisting of ingredients and conditions:
- a set X of unknown functions;
- a set of differential operators;
- a set of "free term" or coefficient functions from a differential algebra (A, d_0);
- The ingredients can be put together by only the algebraic operations, together with the differential operators.
- Note that here A carries its own derivation d_0. Elements in A are called coefficient functions because they are the coefficients in the differential equation.
- We know that a differential equation with coefficients in a differential algebra (A, d_0) is an element in the differential polynomial algebra $A\{X\}$, as a realization of the relative free (commutative) differential algebra on (A, d_0).

Operated algebras

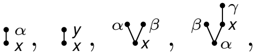
- From this viewpoint, an integral equation should be an element of a suitably defined "integral polynomial algebra". In view of the freeness of the polynomial algebra $\mathbf{k}[X]$ and the differential polynomial algebra $A\{X\}$, this sought-after "integral polynomial algebra" should be the free commutative "integral" algebra on a set X of unknown functions.
- Thus we should find a suitable notion of integral algebra which, in contrast to differential algebras, can have different definitions depending on the nature of the integral operators. To be versatile, we begin by putting no conditions on the integral operators except its linearity, in which case, such an algebra is called an operated algebra, which can be traced back to Kurosh who called it an Ω-algebra.
- Free operated algebras were constructed in terms of bracketed words, rooted trees and Motzkin paths in [Gop,GGZy], with constant coefficient.
- Here we extend the construction to free operated algebras with coefficients in a base operated algebra, similar to the differential polynomial algebra on a differential ring.

Free operated algebras by rooted trees

- Let X and Ω be disjoint sets. Let $\mathcal{F}(X,\Omega)$ denote the set of vertex decorated rooted trees (resp. forests) with the property that elements of only Ω can decorate the internal vertices, namely vertices which are not leafs. The unique vertex of the tree $\bullet$ is regarded as a leaf vertex. In other words, elements of X can only be used to decorate the leaf vertices. Of course, some of the leaf vertices can also be decorated by elements from Ω.
- For example,



$x, y, \in X, \alpha, \beta, \gamma \in \Omega$, are in $\mathcal{F}(X,\Omega)$ whereas, the following are not in $\mathcal{F}(X,\Omega)$:


$x, y \in X, \alpha, \beta, \gamma \in \Omega.$

Free relative matching Rota-Baxter algebras I

- Let Ω be a nonempty set and (A, α_Ω) an Ω-operated matching Rota-Baxter algebra (MRBA). An (A, α_Ω)-MRBA is an Ω-operated MRBA (R, P_Ω) together with a homomorphism $(A, \alpha_\Omega) \longrightarrow (R, P_\Omega)$ of MRBAs.
- Let V be a $\mathbf{k}$-module, e.g., $V = \mathbf{k}X$. We use the notations

$$\mathfrak{A} := S_A(V) := A \otimes S(V), \quad \mathfrak{A}^+ := S_A^+(V) := A \otimes S^+(V).$$

So $\mathfrak{A} = A \oplus \mathfrak{A}^+$, that is, $S_A(V) = A \oplus S_A^+(V)$.
- For a $\mathbf{k}$-module U, denote the *colored* tensor power

$$U^{\otimes_\Omega n} := \mathbf{k}\{1 \otimes_{\omega_1} u_1 \otimes_{\omega_2} u_2 \otimes_{\omega_3} \cdots \otimes_{\omega_n} u_n \,|\, u_i \in U, \omega_i \in \Omega, 2 \leq i \leq n\}.$$

- Let

$$\mathrm{Sh}_\Omega(\mathfrak{A}^+) := \oplus_{n\geq 0}(\mathfrak{A}^+)^{\otimes_\Omega n}$$

denote the Ω-shuffle product algebra on $\mathfrak{A}^+$, with the convention that $\mathfrak{A}^{\otimes 0} = \mathbf{k}$. Then take the tensor product algebra

$$Ш_\Omega^{\mathrm{rel}}(A, V) := \mathfrak{A} \otimes \mathrm{Sh}(\mathfrak{A}^+) := \oplus_{n\geq 0}\mathfrak{A} \otimes_\Omega (\mathfrak{A}^+)^{\otimes_\Omega n}$$

Free relative matching Rota-Baxter algebras II

- We note the direct sum
$$\mathfrak{A}\otimes_\Omega(\mathfrak{A}^+)^{\otimes_\Omega n}=(A\otimes_\Omega(\mathfrak{A}^+)^{\otimes_\Omega n})\oplus(\mathfrak{A}^+)^{\otimes_\Omega(n+1)}.$$
- So a pure tensor $\mathfrak{u}$ in $\mathfrak{A}\otimes_\Omega(\mathfrak{A}^+)^{\otimes_\Omega n}$ is of the form $\mathfrak{u}=u_0\otimes_{\omega_1}\cdots\otimes_{\omega_n}u_n$ where u_0 is either in A or in $\mathfrak{A}^+$.
- We accordingly define $P_\omega(\mathfrak{u}):=P_{\Omega,A,V}(\mathfrak{u})$
$$:=\begin{cases}\alpha_\omega(u_0)\otimes_{\omega_1}u_1\otimes_{\omega_2}\cdots\otimes_{\omega_n}u_n \\ -1\otimes_\omega\alpha_{\omega_1}(u_0)u_1\otimes_{\omega_2}u_2\otimes_{\omega_3}\cdots\otimes_{\omega_n}u_n, & u_0\in A,\\ 1\otimes_\omega u_0\otimes_{\omega_1}u_1\otimes_{\omega_2}\cdots\otimes_{\omega_n}u_n, & u_0\in\mathfrak{A}^+.\end{cases}$$
- Theorem Let (A,α_Ω) be a matching Rota-Baxter algebra and let V be a $\mathbf{k}$-module. The free (A,α_Ω)-MRBA on V of weight 0 is $Ш_\Omega^{\mathrm{rel}}(A,V)$.

Operator linearity of phantom Volterra equations

- An integral equation is called operator linear if it does not contain any products of Volterra integral operators. It is called linear if it is operator linear and the unknown functions appear linearly in the equation.
- Theorem Any Volterra integral equation with phantom kernels is equivalent to one that is operator linear. Further, each of the (nested) integrals acts on variable functions.
- The Thomas-Fermi equation

$$y(x) = 1 + Bx + \int_0^x \int_0^t s^{-1/2} y(s)^{3/2}\, ds\, dt$$

is already operator linear. But there is no variable function between the outer and inner integrals. Integration by parts gives

$$\int_0^x \int_0^t s^{-1/2} y(s)^{3/2} ds\, dt = x \int_0^x s^{-1/2} y(s)^{3/2} ds - \int_0^x t^{1/2} y(t)^{3/2} dt.$$

Free generalized Reynolds algebras

- In order to give an algebraic framework for Volterra equations with separable kernels, we construct free generalized Reynolds algebras, with possible multiple operators.
- The difficulty lies with the cyclic property of the Reynolds identity:

$$R(f)R(g) = R(fR(g)) + R(R(f)g) - R(\underline{R(f)R(g)}).$$

  So there is no finite closed formula for this rewriting system.
- So we introduced complete operated and Reynolds algebras, and constructed free objects in the corresponding categories.
- Then can hope to prove a pro-operator linear property of Volterra equations with separable kernels. (work in progress).

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Thanks you!