Strong minimality for Painleve equations and Fuchsian equations

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Strong minimality in context

- Let X be the solution set of an ODE of order n over differential field K. X is strongly minimal if

  > Whenever $y \in X$ and K_1 is any differential field extending K, then $td(y, y', \dots, y^{(n-1)}/K_1) = 0$ or n.

- Strong minimality is itself a functional transcendence result, but one thing it is very useful for is proving further results.
- Model theorists have developed a detailed classification of strongly minimal sets which can be usefully applied.

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Émile Picard

- **Picard:** What differential equations of the form $y'' = F(y, y', t)$ have the property that the movable singularities of any solution are poles? (*Painlevé property*)
- Movable pole: $\frac{1}{t-c}$. Movable branch point: $\log(t-c)$
- Solutions of linear ODEs have no movable branch points.

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Painlevé equations

Henri Poincaré

- **Poincaré and Fuchs:** Any nonlinear order one ODE with (PP) can be transformed into Riccati equation or a Weierstrass equation.

Painlevé equations

Lazarus Fuchs

- **Poincaré and Fuchs:** Any nonlinear order one ODE with (PP) can be transformed into Riccati equation or a Weierstrass equation.
- **Hadamard** wrote that he hoped someone would extend this work, because it would show that: "... continuing the work of Henri Poincaré was not beyond human capacity."

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Paul Painlevé

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- **Painlevé, Gambier, Fuchs:** Any order two equation whose solutions have the (PP) can be transformed by Möbius transformation to ones coming from one of 50 families of equations.
- Of these, 44 are expressible in terms of previously known special functions (e.g. elliptic functions, solutions of linear ODEs)

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Painlevé equations

General problem: classify the (differential) algebraic relations over $\mathbb{C}(t)$ between solutions of Painlevé equations.

$$P_I: \quad \frac{d^2y}{dt^2} = 6y^2 + t$$

$$P_{II}(\alpha): \quad \frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

$$P_{III}(\alpha,\beta,\gamma,\delta): \quad \frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

$$P_{IV}(\alpha,\beta): \quad \frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}$$

$$P_V(\alpha,\beta,\gamma,\delta): \quad \frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$

$$P_{VI}(\alpha,\beta,\gamma,\delta): \quad \frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

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General problem: classify the (differential) algebraic relations over $\mathbb{C}(t)$ between solutions of Painlevé equations.

- That is, given $y_1, \ldots, y_n$ each a generic solution to some Painlevé equation, what are the possible algebraic relations between $y_1, y_1', \ldots, y_n, y_n'$?
- Strong minimality plays a key roles in our approach to this question.
- Before telling you about what is currently known, and what we currently conjecture, I want to quickly explain a (new) proof of the strong minimality of $P6$ with generic coefficients.

Painlevé six

$$\frac{d^2x}{dt^2} = \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x-t}\right)\left(x'\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{x-t}\right)\left(x'\right) + \frac{x(x-1)(x-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{x^2}+\gamma\frac{t-1}{(x-1)^2}+\delta\frac{t(t-1)}{(x-t)^2}\right)$$

For the elliptic curve given by the projective closure of $y^2 = x(x-1)(x-t)$, the Manin map is given by:

$$\mu(x,y) = -\frac{y}{(x-t)^2} + \left(2t(t-1)\frac{x'}{y}\right)' + \frac{t(t-1)x'}{x-t)y}.$$

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After simplifying, the Kernel of μ is given by

$$\begin{aligned}\frac{d^2x}{dt^2} = \quad & \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\left(\frac{dx}{dt}\right)^2 \\ & -\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{x-t}\right)\left(\frac{dx}{dt}\right) \\ & + \frac{x(x-1)}{2t(t-1)(x-t)}\end{aligned}$$

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This should look familiar, since it is P6 with parameters $(0, 0, 0, \frac{1}{2})$. Hrushovski showed that this equation is strongly minimal (and also more general ones).

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The family P6 has a series of birational transformations acting on it:

$$\begin{array}{ccc} P6 & \xrightarrow{g_*} & P6 \\ \downarrow & & \downarrow \\ \mathbb{C}^4 & \xrightarrow{g} & \mathbb{C}^4 \end{array}$$

where g is any element in the group generated by

$$\begin{aligned}
s_1(v_1, v_2, v_3, v_4) &:= (v_2, v_1, v_3, v_4) \\
s_2(v_1, v_2, v_3, v_4) &:= (v_1, v_3, v_2, v_4) \\
s_3(v_1, v_2, v_3, v_4) &:= (v_1, v_2, v_4, v_3) \\
s_4(v_1, v_2, v_3, v_4) &:= (v_1, v_2, -v_4, -v_3) \\
s_5(v_1, v_2, v_3, v_4) &:= (-v_2 + 1, -v_1 + 1, v_3, v_4)
\end{aligned}$$

Painlevé six

- Suppose that $P6(\alpha, \beta, \gamma, \delta)$ is not strongly minimal for $(\alpha, \beta, \gamma, \delta)$ generic over $\mathbb{Q}$.
- This is equivalent to having (non-algebraic) solutions $d_1, \ldots d_n$ to $P6(\alpha, \beta, \gamma, \delta)$, for some $n \in \mathbb{N}$ such that

$$td_{\mathbb{Q}(\alpha,\beta,\gamma,\delta,t)}(\mathbb{Q}(\alpha, \beta, \gamma, \delta, t)\langle d_1, \ldots, d_n \rangle) = 2n - 1.$$

- So there is a differential polynomial with coefficients in $\mathbb{Q}(\alpha, \beta, \gamma, \delta, t)$ which defines a co-order one differential subvariety $V_{(\alpha,\beta,\gamma,\delta,t)}$ of $P6(\alpha, \beta, \gamma, \delta)^n$.
- But then $tp((\alpha, \beta, \gamma, \delta)/\mathbb{Q}(t))$ implies the the existence of such a variety $V_{(\alpha,\beta,\gamma,\delta,t)}$.

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- Definable subsets of $\mathbb{C}^4$ are given by $\mathbb{Q}$-Zariski-constructible subsets of $\mathbb{A}^4$.
- By compactness, there must be some Zariski open subset U of $\mathbb{A}^4$ over $\mathbb{Q}$ such that for any $(a, b, c, d) \in U$, $V(a, b, c, d, t)$ is a co-order 1 subvariety of $P6(a, b, c, d)^n$.
- This open set U hits the orbit of the Manin Kernel.
- Strong minimality rules out the possibility of such a variety $V(a, b, c, d, t)$.
- Thus $P6(\alpha, \beta, \gamma, \delta)$ for generic $(\alpha, \beta, \gamma, \delta)$ must be strongly minimal.

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Three parts of parts of the classification

General problem: classify the (differential) algebraic relations over $\mathbb{C}(t)$ between solutions of Painlevé equations.

1. Algebraic relations between solutions of a single fixed Painlevé equation. Very complete data for equations with generic coefficients - P6 was completed just last year by Nagloo. For nongeneric coefficients, little is known in general.
2. Algebraic relations between solutions of equations from different families. Very complete data when one of the equations from one of the families has generic coefficients (Freitag). Essentially no other cases known.
3. Algebraic relations between solutions of equations in the same family (different coefficients). Very complete data for one of the equations having generic coefficients. P6 only completed this year (Freitag and Nagloo).

- Progress on one of the three areas usually leads to progress in the others.

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Algebraic relations between generic solutions of Painleve equations

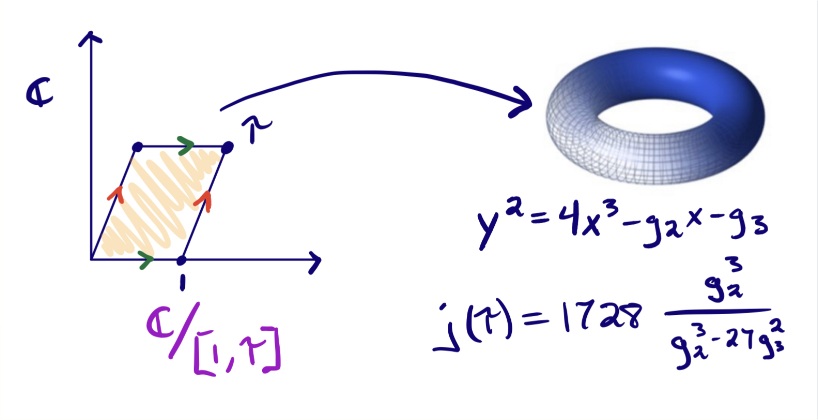
Current conjecture:

The algebraic relations between generic solutions of Painlevé equations are given by

- Backlund transformations.
- Algebraic relations within the orbit of the Manin kernel in P6.
- Isogenies of elliptic curves between fibers in the orbit of the Manin Kernel in P6.

In particular, there are no algebraic relations between pairs of equations from different families.

Part II: around Fuchsian functions



General Problem: The bi-algebraicity problem

- X, Y algebraic varieties[2] over $\mathbb{C}$ and let

$$\phi : X \to Y$$

be a complex analytic map **which is not algebraic.**

- For **most** algebraic subvarieties $X_0 \subset X$, $\phi(X_0)$ is *not* algebraic.
- Pairs of algebraic subvarieties (X_0, Y_0) with $X_0 \subset X$ and $Y_0 \subset Y$ such that $\phi(X_0) = Y_0$ are called **bi-algebraic** for ϕ.
- Casale, F. Nagloo solved this problem for genus zero Fuchsian functions (more on this in a moment).
- Our approach uses strong minimality in an essential way.

[2] More generally an open subset of an algebraic variety.

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Automorphic functions of Fuchsian groups

- Let Γ be a genus zero *Fuchsian group* of the first kind - a discrete, finitely generated subgroup $\Gamma \leq PSL_2(\mathbb{R})$.
- $\mathbb{H}$ is the upper half plane.
- Γ acts on $\mathbb{H}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d}$$

- Take $z \in \mathbb{H}$, consider the orbit Γz.
- Γ is discrete $\rightarrow$ no limit points in $\mathbb{H}$.
- There can be limit points of the orbit in $\mathbb{R} \cup \{\infty\}$.
- **First kind** = every point in $\mathbb{R} \cup \{\infty\}$ is a limit point.
- Assume $\Gamma \backslash \mathbb{H}$ is genus zero.

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Automorphic functions

- The function $j_\Gamma(t)$ satisfies a nonlinear third order differential equation:
$$S_{\frac{d}{dt}}(y) + (y')^2 \cdot R_\Gamma(y) = 0 \qquad (\star)$$
where R_Γ is a rational function and
$$S_{\frac{d}{dt}}(x) = \left(\frac{x''}{x'}\right)' - \frac{1}{2}\left(\frac{x''}{x'}\right)^2$$
is the Schwarzian derivative.
- For instance, the j-function satisfies the differential equation:
$$S_{\frac{d}{dt}}(y) + \frac{y^2 - 1968y + 2654208}{2y^2(y-1728)^2}(y')^2 = 0$$

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Painlevé, irreducibility, and the Japanese school

$$S_{\frac{d}{dt}}(y) + (y')^2 \cdot R_\Gamma(y) = 0 \qquad (\star)$$

Conjecture: (1895) In his famous "Lecons de Stockholm", Painlevé conjectured that over any differential field extension K of $\mathbb{C}(t)$,

$$tr.deg_K(y, y', y'') = 0 \textbf{ or } 3.$$

Progress on Painlevé's conjecture

- **Painlevé** (1902-1903) eventually claimed proofs of the strong minimality various differential equations using results from the thesis of **Drach** (1898).

Drach's thesis (*despite highly favorable reviews by Picard and Darboux*) was flawed.

- Painlevé and Vessiot along with Cartan played a role in pointing out the flaws of Drach's thesis in late 1898.
- Painlevé used the Drach theory to "prove" the strong minimality of P_1 and the automorphic equations.
- He thought that Drach's theory (a kind of nonlinear differential Galois theory) would soon be put on a firm foundation.
- People chose sides - Liouville: the first Painlevé equation is not strongly minimal.

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(Nishioka 1991) When $K = \mathbb{C}(t)$,

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(Casale-F.-Nagloo ≥ 2019*) For any* K,

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- Let Γ be a Fuchsian Group of first kind and not necessarily of genus zero.
- If the compactification C_Γ of the quotient $\Gamma \setminus \mathbb{H}$ is defined over $\mathbb{Q}^{alg}$, then C_Γ is called a **Belyi surface.**

Theorem

(Belyi 1980) Let Γ be a Fuchsian Group of first kind.

1. *If Γ is cocompact, then C_Γ is a Belyi surface if and only if Γ is a finite index subgroup of a cocompact Fuchsian triangle group $\Gamma_{(k,l,m)}$.*
2. *If Γ is not cocompact, then C_Γ is a Belyi surface if and only if one of the following holds*
   (i) *Γ is a finite index subgroup of $\Gamma_{(2,3,\infty)}$; or*
   (ii) *Γ is a finite index subgroup of $\Gamma_{(2,\infty,\infty)}$; or*
   (iii) *Γ is a finite index subgroup of $\Gamma_{(\infty,\infty,\infty)}$.*

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Theorem

(Belyi 1980) Let Γ be a Fuchsian Group of first kind.

1. *If Γ is cocompact, then C_Γ is a Belyi surface if and only if Γ is a finite index subgroup of a cocompact Fuchsian triangle group $\Gamma_{(k,l,m)}$.*
2. *If Γ is not cocompact, then C_Γ is a Belyi surface if and only if one of the following holds*
   - (i) *Γ is a finite index subgroup of $\Gamma_{(2,3,\infty)}$; or*
   - (ii) *Γ is a finite index subgroup of $\Gamma_{(2,\infty,\infty)}$; or*
   - (iii) *Γ is a finite index subgroup of $\Gamma_{(\infty,\infty,\infty)}$.*

Progress on Painlevé's conjecture

- The uniformizing functions of triangle groups fit into the framework of (Casale, F. Nagloo).
- It is not hard to see that if Γ_1 is finite index in Γ_2, then j_{Γ_1} and j_{Γ_2} are interalgebraic over $\mathbb{C}(t)$.
- Strong minimality (and problems like ALW) are not affected by interalgebraicity,

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Another soft target

- $\Delta(\alpha, \beta, \gamma) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^\alpha = (bc)^\beta = (ca)^\gamma = 1 \rangle$. with $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$.
- $\Delta(\alpha, \beta, \gamma)$ is a Fuchsian group.
- The uniformizing functions of triangle groups satisfy Schwarzian differential equations:

$$S_t(y) + (y')^2 R(y) = 0. \qquad (\star)$$

$$R(y) = \frac{1}{2}\left(\frac{1-\beta^{-2}}{y^2} + \frac{1-\gamma^{-2}}{(y-1)^2} + \frac{\beta^{-2}+\gamma^{-2}-\alpha^{-2}-1}{y(y-1)}\right).$$

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Proposition

Let us assume that equation $(\star)$ with $R = R_{\triangle}$ with complex parameters (α, β, γ) is not strongly minimal. One of the following holds:

1. *At least one of the four complex numbers, $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$, $-\alpha^{-1} + \beta^{-1} + \gamma^{-1}$, $\alpha^{-1} - \beta^{-1} + \gamma^{-1}$, $\alpha^{-1} + \beta^{-1} - \gamma^{-1}$ is an odd integer.*
2. *The quantities α^{-1} or $-\alpha^{-1}$, β^{-1} or $-\beta^{-1}$ and γ^{-1} or $-\gamma^{-1}$ take, in an arbitrary order, values given in the following table:*

	$\pm\alpha^{-1}$	$\pm\beta^{-1}$	$\pm\gamma^{-1}$	
1	$\frac{1}{2} + \ell$	$\frac{1}{2} + m$	arbitrary	
2	$\frac{1}{2} + \ell$	$\frac{1}{2} + m$	$\frac{1}{2} + n$	
3	$\frac{2}{3} + \ell$	$\frac{1}{3} + m$	$\frac{1}{4} + n$	$\ell + m + n$ even
4	$\frac{1}{2} + \ell$	$\frac{1}{3} + m$	$\frac{1}{4} + n$	
5	$\frac{2}{3} + \ell$	$\frac{1}{4} + m$	$\frac{1}{4} + n$	$\ell + m + n$ even
6	$\frac{1}{2} + \ell$	$\frac{1}{3} + m$	$\frac{1}{5} + n$	

What's next?

- Many of you know - the other main approach to ALW-type theorems is via o-minimality. Are there settings in which the combination of these methods would be fruitful?
- What about more general transcendence results? Ax-Schanuel?
- What about generalizations to higher dimensional spaces and discrete groups?

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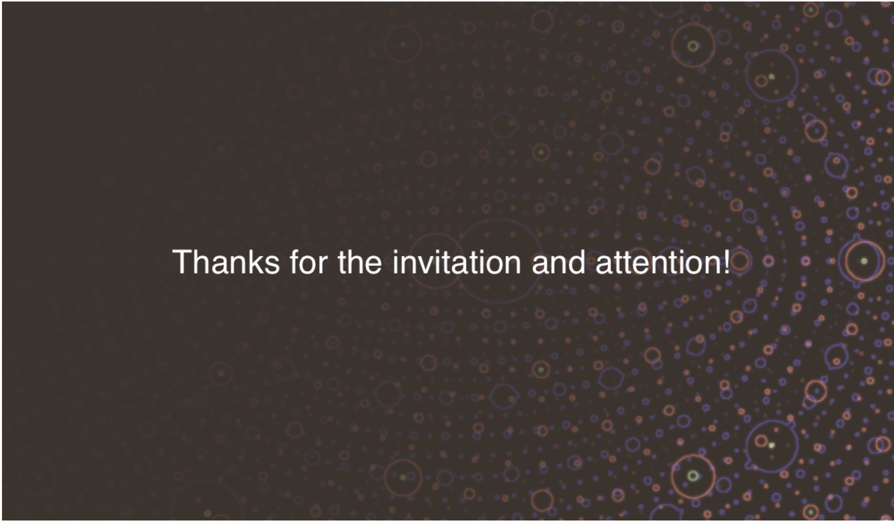
Thanks for the invitation and attention!