A topos-theoretic view of difference algebra

Ivan Tomašić

Queen Mary University of London

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Outline

Ritt-style difference algebra

The topos of difference sets

Difference homological algebra

Difference algebraic geometry

Ritt-style difference algebra

Ritt 1930s

Difference algebra is the study of rings and modules with added endomorphisms.

Difference categories à la Ritt

Let $\mathscr{C}$ be a category. Define its difference category

$$\boldsymbol{\sigma}\text{-}\mathscr{C}$$

- objects are pairs

$$(X, \sigma_X),$$

  where $X \in \mathscr{C}$, $\sigma_X \in \mathscr{C}(X, X)$;
- a morphism $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is a commutative diagram in $\mathscr{C}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

  i.e., an $f \in \mathscr{C}(X, Y)$ such that

$$f \circ \sigma_X = \sigma_Y \circ f.$$

Examples

We will consider:

- $\boldsymbol{\sigma}$-$\mathbf{Set}$;
- $\boldsymbol{\sigma}$-$\mathbf{Gr}$;
- $\boldsymbol{\sigma}$-$\mathbf{Ab}$;
- $\boldsymbol{\sigma}$-$\mathbf{Rng}$.

Given $R \in \boldsymbol{\sigma}$-$\mathbf{Rng}$, consider

- R-$\mathbf{Mod}$, the category of difference R-modules.

The topos of difference sets

Let $\underline{\boldsymbol{\sigma}}$ be the category associated with the monoid $(\mathbb{N}, +)$:

- single object o;
- $\mathrm{Hom}(o, o) \simeq \mathbb{N}$.

Note

$$\boldsymbol{\sigma}\text{-}\mathbf{Set} \simeq [\underline{\boldsymbol{\sigma}}, \mathbf{Set}]$$

is a Grothendieck topos (as the presheaf category on $\underline{\boldsymbol{\sigma}}^{\mathrm{op}} \simeq \underline{\boldsymbol{\sigma}}$).
The literature also calls it the classifying topos of $\mathbb{N}$, written

$$\mathbf{B}\mathbb{N}.$$

The topos of difference sets

The canonical 'global sections' geometric morphism

$$\gamma : \boldsymbol{\sigma}\textbf{-Set} \to \textbf{Set}$$

has direct image functor

$$\gamma_* = \mathrm{Fix}\,.$$

The topos $\boldsymbol{\sigma}\text{-}\mathrm{Set}$ has a 'point' (geometric morphism)

$$\textbf{Set} \to \boldsymbol{\sigma}\textbf{-Set}$$

whose inverse image functor is the forgetful functor

$$\lfloor\,\rfloor : \boldsymbol{\sigma}\textbf{-Set} \to \textbf{Set}.$$

Topos theory view of difference algebra

Old adage of topos theory:

> *A topos can serve as a universe for developing mathematics.*

Motto

Difference algebra is the study of algebraic objects internal in the topos $\boldsymbol{\sigma}$-$\mathbf{Set}$.

We study ordinary algebra in the internal logic of $\boldsymbol{\sigma}$-$\mathbf{Set}$, and then externalise to observe what actually happened.

Topos theory view of difference algebra

Indeed,

$$\boldsymbol{\sigma}\text{-}\mathbf{Gr} \simeq \mathbf{Gr}(\boldsymbol{\sigma}\text{-}\mathbf{Set})$$
$$\boldsymbol{\sigma}\text{-}\mathbf{Ab} \simeq \mathbf{Ab}(\boldsymbol{\sigma}\text{-}\mathbf{Set})$$
$$\boldsymbol{\sigma}\text{-}\mathbf{Rng} \simeq \mathbf{Rng}(\boldsymbol{\sigma}\text{-}\mathbf{Set}).$$

For $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$,

$$R\text{-}\mathbf{Mod} \simeq \mathbf{Mod}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, R)$$

is the category of modules in a ringed topos.

Internal homs for difference sets

Consider $N_+ = (\mathbb{N}, i \mapsto i+1) \in \boldsymbol{\sigma}\text{-}\mathbf{Set}$.

Internal homs for $\boldsymbol{\sigma}$-Set

- $$[X,Y] = \boldsymbol{\sigma}\text{-}\mathbf{Set}(N_+ \times X, Y)$$
  $$\simeq \{(f_i) \in \mathbf{Set}(X,Y)^{\mathbb{N}} : f_{i+1} \circ \sigma_X = \sigma_Y \circ f_i\}.$$

  $$\begin{array}{ccc} X & \xrightarrow{f_0} & Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow{f_1} & Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow{f_2} & Y \\ \vdots & & \vdots \end{array}$$

- shift $s : [X,Y] \to [X,Y]$, $s(f_0, f_1, \ldots) = (f_1, f_2, \ldots)$.

Cartesian closed structure

For $X, Y, Z \in \boldsymbol{\sigma}\text{-}\mathbf{Set}$, we have a natural bijection

$$\boldsymbol{\sigma}\text{-}\mathbf{Set}(X \times Y, Z) \simeq \boldsymbol{\sigma}\text{-}\mathbf{Set}(X, [Y, Z]).$$

Internal homs vs homs

Note

$$\mathrm{Fix}[X, Y] = \boldsymbol{\sigma}\text{-}\mathbf{Set}(X, Y),$$

so the Ritt-style difference algebra only sees the tip of an iceberg.

Difference modules are monoidal closed

Let $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$.

Internal homs for R-$\mathbf{Mod}$

Given $A, B \in R\text{-}\mathbf{Mod}$,

$$f = (f_i) \in [A, B]_R \in R\text{-}\mathbf{Mod}$$

is a 'ladder' with

$$f_i \in \lfloor R \rfloor\text{-}\mathbf{Mod}(\lfloor A \rfloor, \lfloor B \rfloor).$$

Hom-tensor duality for difference modules

$$\mathrm{Hom}_R(A \otimes B, C) \simeq \mathrm{Hom}_R(A, [B, C]_R).$$

Difference homological algebra

Let $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$.

Fact (in any Grothendieck topos)

$R\text{-}\mathbf{Mod} = \mathbf{Mod}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, R)$ is abelian with enough injectives and enough internal injectives.

Cohomology of difference modules is an instance of topos cohomology; we have

- $\mathrm{Ext}^i_R(M, N)$;
- $H^i(\boldsymbol{\sigma}\text{-}\mathbf{Set}, M) = R^i\gamma_*(M) = \mathrm{Ext}^i_R(R, N)$.

Ext of difference modules

Let $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$, and $F, F' \in R\text{-}\mathbf{Mod}$, with F étale. Then

$$\mathrm{Ext}^i_R(F, F') = \begin{cases} R\text{-}\mathbf{Mod}(F, F'), & i = 0, \\ [F, F']_s, & i = 1, \\ 0, & i > 1, \end{cases}$$

In particular,

$$R^i \,\mathrm{Fix}(M) = \begin{cases} \mathrm{Fix}(M) = M^\sigma & i = 0, \\ M/\mathrm{Im}(\sigma - \mathrm{id}) = M_\sigma, & i = 1, \\ 0, & i > 1. \end{cases}$$

Difference algebraic geometry

Motto

Difference algebraic geometry is algebraic geometry over the base topos $\boldsymbol{\sigma}\text{-}\mathbf{Set}$.

Hakim spectra and relative schemes

Hakim-Cole Zariski spectrum

For a ringed topos $(\mathscr{E}, A)$, $\mathrm{Spec.Zar}(\mathscr{E}, A)$ is the locally ringed topos equipped with a morphism of ringed topoi

$$\mathrm{Spec.Zar}(\mathscr{E}, A) \xrightarrow{\pi_{\mathrm{Zar}}} (\mathscr{E}, A)$$

which solves a certain 2-universal problem.

For any 'scheme topology' τ (Zariski, étale, fppf etc), using Hakim's techniques, we can define the τ-spectrum

$$\mathrm{Spec}\,.\tau(\mathscr{E}, A) \xrightarrow{\pi_\tau} (\mathscr{E}, A).$$

Difference schemes

Definition

The affine difference scheme associated to a difference ring A is the locally ringed topos

$$(X, \mathscr{O}_X) = \mathrm{Spec.Zar}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, A)$$

Its τ-spectrum is

$$(X_\tau, \mathscr{O}_\tau) = \mathrm{Spec}\,.\tau(X, \mathscr{O}_X)$$

Difference spectra in detail

Let

$$S_\tau \hookrightarrow \mathbf{Sch}_{/\mathrm{Spec}(\lfloor A \rfloor)}$$

be the classical τ-site of $\mathrm{Spec}(\lfloor A \rfloor)$, so that

$$\lfloor X \rfloor_\tau = \mathrm{Sh}(S_\tau)$$

is the classical τ-topos of $\mathrm{Spec}(\lfloor A \rfloor)$.
Let

$$\varsigma : S_\tau \to S_\tau$$

be the base change functor along $\mathrm{Spec}(\sigma_A)$.
The data

$$\mathbb{S}_\tau = (S_\tau, \varsigma)$$

defines an internal site in $\boldsymbol{\sigma}\text{-}\mathbf{Set}$, and the τ-spectrum is the topos of internal sheaves

$$X_\tau = \mathrm{Sh}_{\boldsymbol{\sigma}\text{-}\mathbf{Set}}(\mathbb{S}_\tau).$$

Difference spectra vs classical spectra

We have

$$X_\tau \simeq \lfloor X \rfloor_\tau^{\varsigma},$$

the category of ς-equivariant sheaves in $\lfloor X \rfloor_\tau$, i.e., those $F \in \lfloor X \rfloor_\tau$ equipped with a morphism

$$F \to F \circ \varsigma.$$

There is a natural geometric morphism

$$X_\tau \xrightarrow{\pi_\tau} \boldsymbol{\sigma}\text{-}\mathbf{Set}.$$

We can recover the difference ring as

$$\pi_{\mathrm{Zar}*} \mathscr{O}_X \simeq A.$$

Cohomology of difference schemes

We have geometric morphisms

$$\begin{array}{ccc} X_\tau & \xrightarrow{\pi_\tau} & \boldsymbol{\sigma}\text{-}\mathbf{Set} \\ & {\scriptstyle \gamma_\tau} \searrow & \downarrow {\scriptstyle \gamma} \\ & & \mathbf{Set} \end{array}$$

If M is an $\mathscr{O}_\tau$-module or an abelian group in X_τ, we define the τ-cohomology and the relative τ-cohomology groups as the topos cohomology groups

$$H^i(X_\tau, M) = R^i\gamma_{\tau,*}(M), \quad \text{and} \quad H^i(X_\tau/\boldsymbol{\sigma}\text{-}\mathbf{Set}, M) = R^i\pi_{\tau,*}(M),$$

the latter being difference abelian groups.

Comparison theorems

Grothendieck-Leray spectral sequence comparing the relative and absolute cohomology degenerates into exact sequences

$$0 \to H^{n-1}(X_\tau/\boldsymbol{\sigma}\text{-}\mathbf{Set}, N)_\sigma \to H^n(X_\tau, N) \to H^n(X_\tau/\boldsymbol{\sigma}\text{-}\mathbf{Set}, N)^\sigma \to 0.$$

Comparison to the cohomology of the underlying scheme:

$$H^n(X_\tau/\boldsymbol{\sigma}\text{-}\mathbf{Set}, N) \simeq H^n(\lfloor X \rfloor_\tau, \lfloor N \rfloor).$$

Classifying torsors

General nonsense: if G is an abelian group in X_τ (most interesting for $\tau = \mathrm{fppf}$), then

$$\mathrm{Tors}^1(X_\tau, G) \simeq H^1(X_\tau, G).$$

Connections to Bachmayr-Wibmer, Chałupnik-Kowalski.
We define the difference groupoid

$$\underline{\mathrm{Tors}}(X_\tau, G)$$

of generalised G-torsors, and they are classified by the relative cohomology

$$\underline{\mathrm{Tors}}^1(X_\tau, G) \simeq H^1(X_\tau/\boldsymbol{\sigma}\textbf{-Set}, G).$$

Cohomology of difference quasi-coherent sheaves

Let $A \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$, $M \in A\text{-}\mathbf{Mod}$, let $(X, \mathscr{O}_X) = \mathrm{Spec.Zar}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, A)$ and

$$\tilde{M} = M \otimes_A \mathscr{O}_X.$$

Then, for $i > 0$,

$$H^i(X/\boldsymbol{\sigma}\text{-}\mathbf{Set}, \tilde{M}) = 0.$$

Hilbert's Theorem 90 for difference schemes

For a difference scheme X,

$$\mathrm{Pic}(X) \simeq H^1(X, \mathscr{O}_X^\times) \simeq H^1(X_{\text{ét}}, \mathbb{G}_m) \simeq H(X_{\text{fppf}}, \mathbb{G}_m).$$

Étale cohomology of difference schemes

Kummer theory

If n is invertible in a difference scheme X, we have a short exact sequence in $X_{\text{ét}}$

$$1 \to \mu_n \to \mathbb{G}_n \xrightarrow{(\,)^n} \mathbb{G}_n \to 1,$$

and we can study the resulting long exact cohomology sequence.

Artin-Schreier theory

If X is a difference scheme of characteristic $p > 0$, we have an exact sequence in $X_{\text{ét}}$

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathscr{O}_{\text{ét}} \xrightarrow{F-\text{id}} \mathscr{O}_{\text{ét}} \to 0,$$

where $F - \text{id}$ is associated to a homomorphism of additive groups $f \mapsto f^p - f$.

Étale cohomology of a difference field

We obtain

$$H^1(k_{\text{ét}}, \mathbb{G}_m) \simeq \mathrm{Pic}(k) \simeq (k^\times)_\sigma,$$

and

$$H^2(k_{\text{ét}}, \mathbb{G}_m) \simeq \mathrm{Br}(k) \simeq \mathrm{Br}(\lfloor k \rfloor)^\sigma.$$

Kummer theory gives

$$0 \to ((k^\times)^\sigma)_n \to H^1(k, \mu_n) \to {}_n((k^\times)_\sigma) \to 0,$$

and

$$0 \to ({}_n(k^\times))_\sigma \to H^1(k, \mu_n) \to ((k^\times)_n)^\sigma \to 0.$$

Étale fundamental groupoid of a difference scheme

Difficulties:

1. a difference scheme X can be topologically totally disconnected, yet indecomposable;
2. X may not have geometric points;
3. the base topos $\boldsymbol{\sigma}\text{-}\mathbf{Set}$ is not Boolean.

We use:

- Janelidze's categorical Galois theory to obtain a difference profinite Galois groupoid, a difference version of Magid's separable Galois theory of commutative rings;
- Bunge-Moerdijk pro-$(\boldsymbol{\sigma}\text{-}\mathbf{Set})$-localic fundamental groupoid associated to the geometric morphism

$$X_{\text{ét}} \to \boldsymbol{\sigma}\text{-}\mathbf{Set}.$$

Difference-differential algebra

Keigher: differential algebra in a topos $\mathscr{E}$

we have the category of differential rings

$$\mathbf{DRng}(\mathscr{E}).$$

Difference-differential rings

Note

$$\delta\text{-}\sigma\text{-}\mathbf{Rng} \simeq \mathbf{DRng}(\boldsymbol{\sigma}\text{-}\mathbf{Set}).$$

Work in progress by Antonino, see his poster.

Why pursue this programme?

1. Ariadne's thread.

2. Generalisations: we work over the base topos

$$\mathbf{B}\mathbb{N},$$

but one can replace $\mathbb{N}$ by an arbitrary monoid, group, category etc to obtain the corresponding 'equivariant' geometry.

Studying Elephant