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Generalization of an Integrability Theorem of Darboux and the Stable Configuration Condition.

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joint work with

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- M. Benfield, H. K. Jenssen, and I. A. Kogan.
  A generalization of an integrability theorem of Darboux,
  Journal of Geometric Analysis, Vol. 29 No. 4, (2019), 3470-3493.
  `https://doi.org/10.1007/s12220-018-00119-6`

- H. K. Jenssen, and I. A. Kogan.
  A mixed boundary value problem for $u_{xy} = f(x, y, u, u_x, u_y)$, (2019), 26 pp.
  in press in The Journal of Differential Equations,
  online: `https://doi.org/10.1016/j.jde.2019.11.063`

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Integrability theorems for PDEs: – theorems about local existence and the "size" of the solution set for an overdetermined system of PDEs.

The "size" of the solution set is the number of arbitrary functions and constants the general solution depends on.

A more subtle question is about type of data that can be prescribed to guarantee the uniqueness of the solution.

Examples of the integrability theorems for PDEs:

- Cartan-Kähler theorem – a theorem determining the size of the solution set of general systems of PDEs and EDSs, but requires analyticity of the equations and the data.

- Darboux [Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. (1910)] – quite specialized, provides explicit prescription of data that guarantees uniqueness, requires only C^1-regularity of the equations and the data.

- PDE version of the Frobenius integrability theorem is a particular case of the Darboux theorem.

We formulate and prove a generalization of the Darboux theorem.

Darboux Théorème III

Chapitre I, Livre III, Leçons sur les systèmes orthogonaux et les coordonnées curvilignes. (1910).

Consider a system of PDEs:

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha\big(x, u(x)\big), \quad i \in I_\alpha \subseteq \{1, \ldots, n\},$$

where

- $x = (x_1, \ldots, x_n)$ are independent variables;
- $u = (u_1, \ldots, u_m)$ are unknown functions;
- $I_\alpha \subseteq \{1, \ldots, n\}$ determines the set of partial derivatives $\partial_{x_i} u_\alpha$ prescribed by the system for the unknown function u_α.
- $f_i^\alpha(x, u)$ are given C^1-functions on some open subset $\Omega \times \Upsilon \subset \mathbb{R}^n \times \mathbb{R}^m$.

Example of Darboux-type system:

- system:

$$\partial_x u = f(x, y, u, v)$$
$$\partial_x v = g(x, y, u, v)$$
$$\partial_y v = h(x, y, u, v)$$

- two unknown functions u and v of (x, y).
- $I_1 = \{1\}$ and $I_2 = \{1, 2\}$.
- f, g, h are given C^1-functions of (x, y, u, v) on an open subset of $\mathbb{R}^2 \times \mathbb{R}^2$.

Returning to the Darboux theorem

the system:

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha\big(x, u(x)\big), \quad i \in I_\alpha \subseteq \{1, \ldots, n\},$$

with the data prescribed near a point $\bar{x} \in \mathbb{R}^n$ by:

$$u_\alpha|_{\Xi_\alpha} = \phi_\alpha, \quad \alpha = 1, \ldots, m,$$

where

- $\Xi_\alpha \subset \{x \,|\, x_i = \bar{x}_i, \text{ for all } i \in I_\alpha\}$
- ϕ_α is a given C^1-function on Ξ_α
- f_i^α are given C^1-functions on open $\Omega \times \Upsilon$, where $\bar{x} \in \Omega \subset \mathbb{R}^n$ and $\phi(\bar{x}) \in \Upsilon \subset \mathbb{R}^m$.

Under appropriate integrability conditions

has a unique local C^1-solution near $\bar{x}$.

Example (data): for the system

$$\partial_x u = f(x, y, u, v)$$
$$\partial_x v = g(x, y, u, v)$$
$$\partial_y v = h(x, y, u, v)$$

we prescribe data near $(\bar{x}, \bar{y})$:

$$u(\bar{x}, y) = \phi(y), \quad v(\bar{x}, \bar{y}) = \psi,$$

where $\phi(y)$ is an arbitrary C^1-function of one variable, ψ is a constant.

Example (integrability conditions):

$$
\begin{aligned}
u_x &= f(x, y, u, v) \\
v_x &= g(x, y, u, v) \\
v_y &= h(x, y, u, v)
\end{aligned}
$$

Equality of partials $v_{xy} = v_{yx}$ imposes a condition on f, g, h:

$$g_y + g_u\, u_y + g_v\, v_y = h_x + h_u\, u_x + h_v\, v_x$$

$\Downarrow$ substitute $u_x, u_y, v_x,$ and v_y from the system ...

Example (integrability conditions):

$$u_x = f(x, y, u, v)$$
$$v_x = g(x, y, v) \qquad \boxed{g_u = 0}$$
$$v_y = h(x, y, u, v)$$

Equality of partials $v_{xy} = v_{yx}$ imposes a condition on f, g, h:

$$g_y + g_u\, u_y + g_v\, v_y = h_x + h_u\, u_x + h_v\, v_x$$

$\Downarrow$ substitute $u_x, v_x,$ and v_y from the system :

$$\boxed{g_y + g_v\, h = h_x + h_u\, f + h_v\, g}$$

The Darboux theorem implies that a system:

$$\begin{aligned} u_x &= f(x, y, u, v) \\ v_x &= g(x, y, v) \\ v_y &= h(x, y, u, v) \end{aligned} \qquad \boxed{g_u = 0}$$

with the data

$$u(\bar{x}, y) = \phi(y), \quad v(\bar{x}, \bar{y}) = \psi,$$

where $\phi(y)$ is an arbitrary C^1-function of one variable, ψ is a constant and f, g, h are C^1-functions such that the equality

$$\boxed{g_y + g_v\, h = h_x + h_u\, f + h_v\, g}$$

is identically satisfied in a neighborhood of a point $(\bar{x}, \bar{y}, \phi(\bar{y}), \psi) \in \mathbb{R}^2 \times \mathbb{R}^2$,

has a unique C^1-solution near $(\bar{x}, \bar{y})$.

The Darboux theorem (Théorème III)

A system:

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha\Big(x, u(x)\Big), \quad i \in I_\alpha \subseteq \{1, \ldots, n\},$$

with the data prescribed near a point $\bar{x} \in \mathbb{R}^n$ by:

$$u_\alpha|_{\Xi_\alpha} = \phi_\alpha, \quad \alpha = 1, \ldots, m, \quad \text{where}$$

- $\Xi_\alpha \subset \{x \,|\, x_i = \bar{x}_i,$ for all $i \in I_\alpha\}$ and $\cap_\alpha \Xi_\alpha = \bar{x}$.
- ϕ_α is an arbitrary C^1-function on Ξ_α
- $f_i^\alpha(x, u)$ are C^1-functions on an open ngbhd. of $(\bar{x}, \phi(\bar{x})) \in \mathbb{R}^{n+m}$, s.t.:

$\forall \alpha$ and $\forall i, j \in I_\alpha$, such that $i \neq j$:

1. $\forall \beta \in \{1, \ldots, m\}$, if $i \notin I_\beta$ then $\partial_{u_\beta} f_j^\alpha \equiv 0$
2. $\partial_{x_i} f_j^\alpha + \sum_{\beta : i \in I_\beta} \left(\partial_{u_\beta} f_j^\alpha\right) f_i^\beta \equiv \partial_{x_j} f_i^\alpha + \sum_{\beta : j \in I_\beta} \partial_{u_\beta} f_i^\alpha \, f_j^\beta$

has a unique local C^1-solution near $\bar{x}$.

Particular cases

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha\big(x, u(x)\big), \quad i \in I_\alpha \subseteq \{1, \ldots, n\},$$

- if, for all α, $|I_\alpha| = 1$ then the system is determined.
  (Darboux's Théorème I)

- if, for all α, $|I_\alpha| = n$ then the system is Frobenius
  (Darboux's Théorème II)

Outline of Darboux's proof

$$\partial_{x_i} u_\alpha(x) = f_i^\alpha\big(x, u(x)\big), \quad i \in I_\alpha \subseteq \{1, \ldots, n\},$$

- Darboux's Théorème I ($|I_\alpha| = 1$ for all α) is proved via Picard iterations.
- Darboux's Théorème III ($|I_\alpha|$ is arbitrary)

  Darboux wrote out a proof only for $n = 2$ and $n = 3$:

  "Pour établier cette importante proposition, sans employer un trop grand luxe de notations, nous nous bornerons au cas de deux et de trois variables indépendantes, qui suffira d'ailleurs pour les applications que nous avons en vue".

  – for $n = 2$, the proof uses Théorème I.

  – for $n = 3$, Darboux identifies sub-systems that can be treated by Théorème I or by $n = 2$ case. These sub-systems are solved in a "right" order so that the solution of one sub-system provides initial data to the next.

This suggests a proof by induction.

Extending Darboux's argument to an inductive proof for an arbitrary number of independent variables turned out to be non-trivial:

Benfield, Jenssen, and IK, ”On two theorems of Darboux” (2017) preprint, 27 pp

`http://www.math.ncsu.edu/~iakogan/papersPDF/BJK-dar.pdf`

This inductive argument does not work for a generalization we were interested in.

We realized that a direct proof of a more general version of the theorem can be given.

Generalization of Théorème III

Our theorem generalizes Darboux's in two ways:

(i) Instead of partial derivatives, the directional derivatives of the unknown functions are prescribed along C^1-vector fields comprising a local frame $\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ near $\bar{x}$:

$$\mathbf{r}_i(u_\alpha)\Big|_x = f_i^\alpha(x, u(x)) \qquad \text{for each } i \in I_\alpha \subseteq \{1, \ldots, n\},.$$

(I_α may vary with α.)

(ii) The prescribed data ϕ_α for unknown u_α may be given along an arbitrary $(n - |I_\alpha|)$-dimensional manifold through the point $\bar{x}$, transversal to the vector fields $\{\mathbf{r}_i \,|\, i \in I_\alpha\}$.

Example of a generalized Darboux system:

- system:

$$\mathbf{r}_1(u) = f(x,y,u,v)$$
$$\mathbf{r}_1(v) = g(x,y,u,v)$$
$$\mathbf{r}_2(v) = h(x,y,u,v)$$

- $\mathbf{r}_1 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $\mathbf{r}_2 = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

- non-commutative frame on $\mathbb{R}^2 \backslash \{(x,y)|xy = 1\}$:

$$[\mathbf{r}_1, \mathbf{r}_2] = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = c^1_{12}\mathbf{r}_1 + c^2_{12}\mathbf{r}_2,$$

  where $c^1_{12} = \frac{x+y^2}{1-xy}$ and $c^2_{12} = -\frac{y+x^2}{1-xy}$

- $\Xi_1 = \{(\mathrm{arctan}(y), y)| -1 < x < 1\}$ and $\Xi_2 = (0,0)$

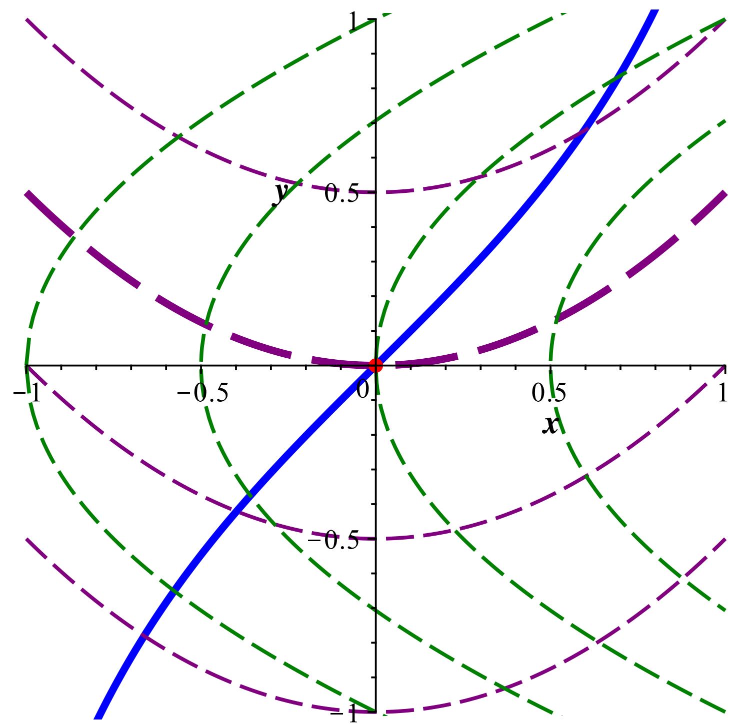
- $u|_{\Xi_1} = u(\mathrm{arctan}(y), y) = \phi(y)$ and $v|_{\Xi_2} = v(0,0) = \psi$.

- Does a solution in a neighborhood of $(0,0)$ exist? Unique?

$$\mathbf{r}_1(u) = f(x,y,u,v)$$
$$\mathbf{r}_1(v) = g(x,y,u,v)$$
$$\mathbf{r}_2(v) = h(x,y,u,v)$$

$\mathbf{r}_1 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ and $\mathbf{r}_2 = y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$

$$u|_{\Xi_1} = u(\mathrm{arctan}(y), y) = \phi(y)$$
$$v|_{\Xi_2} = v(0,0) = \psi$$



Integrability conditions in the case of non-commuting derivations:

The derivatives, prescribed by the system must be consistent with the structure equations of the frame:

$$[\mathbf{r}_i, \mathbf{r}_j] = \sum_{k=1}^{n} c_{ij}^k \mathbf{r}_k.$$

In other words, we substitute the derivatives $\mathbf{r}_j(u_\alpha)$ prescribed by the system into

$$\mathbf{r}_i\Big(\mathbf{r}_j(u_\alpha)\Big) - \mathbf{r}_j\Big(\mathbf{r}_i(u_\alpha)\Big) = \sum_{k=1}^{n} c_{ij}^k \mathbf{r}_k(u_\alpha) \qquad (*)$$

and require that:

1. No unprescribed derivatives of u_α are present in (*).

2. Equality (*) holds as an identity near $(\bar{x}, \phi(\bar{x})) \in \mathbb{R}^n \times \mathbb{R}^m$.

Integrability conditions in details:

$$\mathbf{r}_i\Big(\mathbf{r}_j(u_\alpha)\Big) - \mathbf{r}_j\Big(\mathbf{r}_i(u_\alpha)\Big) = \sum_{k=1}^{n} c_{ij}^k\,\mathbf{r}_k(u_\alpha) \qquad (*)$$

$\forall\alpha$ and $\forall i, j \in I_\alpha$, such that $i \neq j$:

1. No unprescribed derivatives of u_α are present in (*):
   - $\forall\beta \in \{1, \ldots, m\}$, if $i \notin I_\beta$ then $\partial_{u_\beta} f_j^\alpha \equiv 0$
   - if $k \notin I_\alpha$ then $c_{ij}^k \equiv 0$
2. Equality (*) holds as an identity near $(\bar{x}, \bar{u}) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$D_x f_j^\alpha(x,u) \cdot \mathbf{r}_i\Big|_x + \sum_{\beta : i \in I_\beta} \partial_{u_\beta} f_j^\alpha(x,u) f_i^\beta(x,u)$$

$$-D_x f_i^\alpha(x,u) \cdot \mathbf{r}_j\Big|_x - \sum_{\beta : j \in I_\beta} \partial_{u_\beta} f_i^\alpha(x,u) f_j^\beta(x,u) \equiv \sum_{k \in I_\alpha} c_{ij}^k(x) f_k^\alpha(x,u).$$

Geometric Stable Configuration Condition (SCC) is the condition on the relative position of the frame vector fields and the data manifolds.

- SCC is sufficient for proving the existence via Picard iteration scheme as described above.
- SCC is necessary and sufficient for the uniqueness.
- SCC is automatically satisfied in the original Darboux setting.
- SCC plays role even for determined systems with standard partial derivatives.

SCC-examples for determined systems

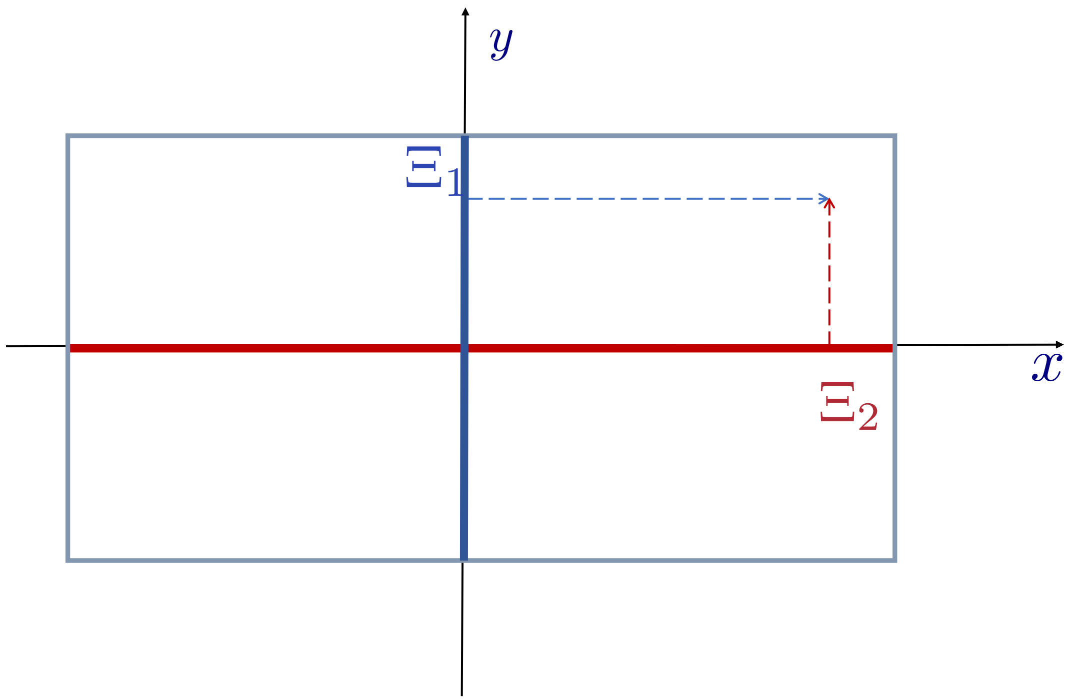
$$\partial_x u = f(x, y, u, v)$$
$$\partial_y v = h(x, y, u, v).$$

Data manifolds are one dimensional:

- Ξ_1 is transversal to ∂_x (never has a horizontal tangent) and so can be parametrized by y.
- Ξ_2 is transversal to ∂_y (never has a vertical tangent) and so can be parametrized by x.
- $\Xi_1 \cap \Xi_2 = (0, 0)$

$$u|_{\Xi_1} = \phi(y) \text{ and } v|_{\Xi_2} = \psi(x).$$

The original Darboux setting :

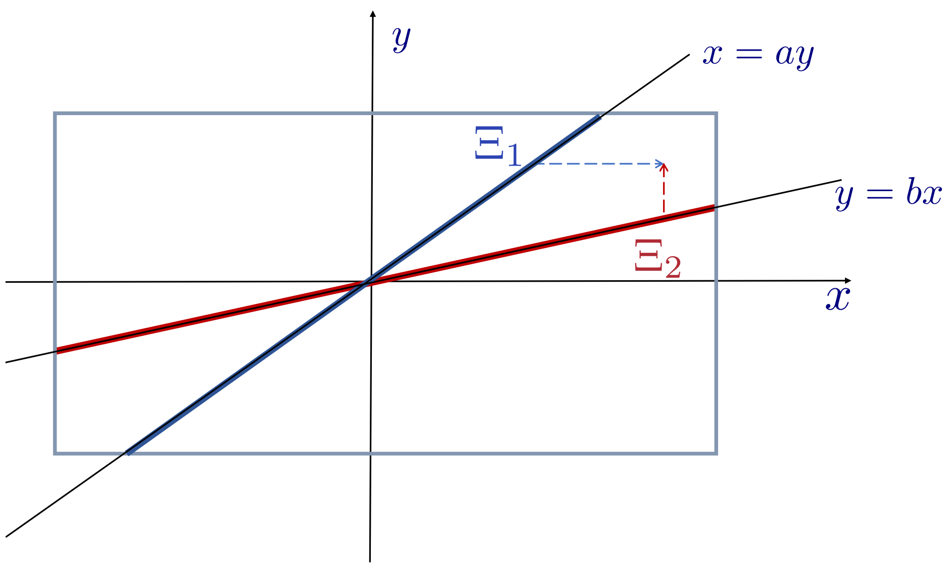


$$u|_{\Xi_1} = u(0, y) = \phi(y) \text{ and } v|_{\Xi_2} = v(x, 0) = \psi(x).$$

More general data manifolds:

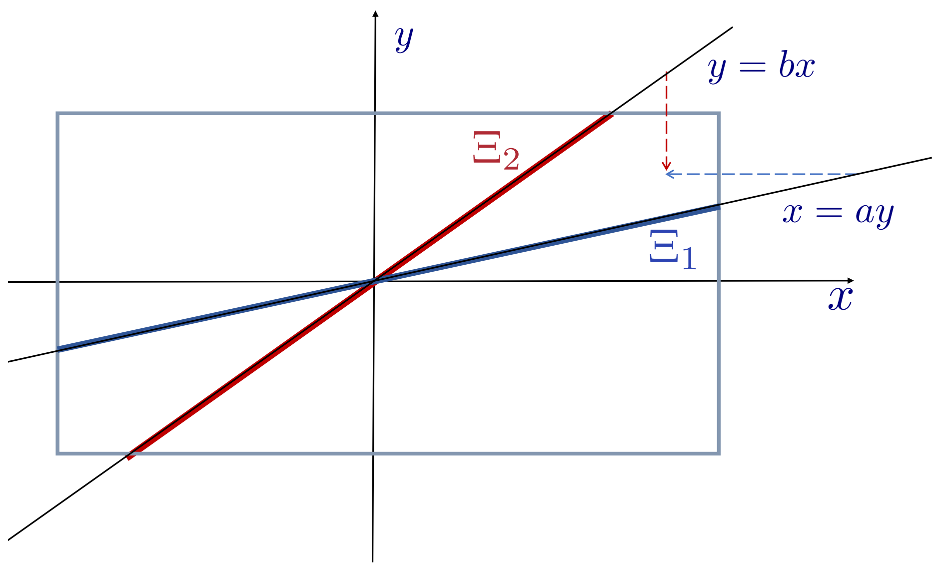
$$\partial_x u = f(x, y, u, v)$$
$$\partial_y v = h(x, y, u, v).$$

$$u|_{\Xi_1} = u\,(ay, y) = \phi(y), \quad a > 0$$
$$v|_{\Xi_2} = v(x, bx) = \psi(x), \quad b > 0.$$



$ab < 1$

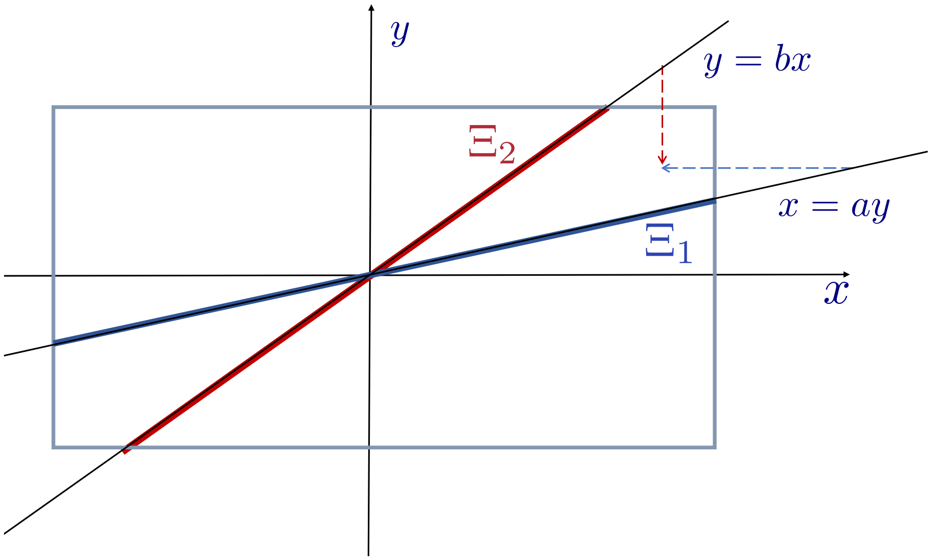
Stable configuration



$ab > 1$

Non-stable configuration

Issues with the non-stable case even for simple examples



$$\partial_x u = f(x, y, u, v)$$
$$\partial_y v = h(x, y, u, v).$$

$$u|_{\Xi_1} = u\,(ay, y) = \phi(y), \qquad a > 0$$
$$v|_{\Xi_2} = v(x, bx) = \psi(x), \qquad b > 0,$$
$$ab > 1$$

- Proving local existence by Picard iterations (integral curves "run out" of the domain of the definition of the iterates).

- Strong non-uniqueness example: $u_x = v, \quad v_y = u \ \ (u_{xy} = u)$.

$$u|_{\Xi_1} = 0 \text{ and } v|_{\Xi_2} = 0.$$

$v \equiv u \equiv 0$ is a solution and $\exists$ a C^1-solution, such that u has strictly positive values everywhere in an open wedge between Ξ_1 and Ξ_2!

Jenssen and IK. *A mixed boundary value problem for* $u_{xy} = f(x, y, u, u_x, u_y)$. (2019)

The Generalized Darboux theorem

A system:

$$\mathbf{r}_i(u_\alpha)\Big|_x = f_i^\alpha\big(x, u(x)\big), \quad i \in I_\alpha \subseteq \{1, \ldots, n\},$$

with the data prescribed near a fixed point $\bar{x} \in \mathbb{R}^n$ by:

$$u_\alpha|_{\Xi_\alpha} = \phi_\alpha, \quad \alpha = 1, \ldots, m, \quad \text{where}$$

- $\mathbf{r}_1, \ldots, \mathbf{r}_n$ is a local C^1-frame on an open $\Omega \supset \bar{x}$, with uniformly bounded structure coefficients c_{ij}^k.
- $\Xi_\alpha \subset \mathbb{R}^n$ is an $(n - |I_\alpha|)$-dimensional manifold through $\bar{x}$, transversal to $\{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$,
- $\mathbf{r}$'s and Ξ's satisfy SCC,
- ϕ_α is an arbitrary C^1-function on Ξ_α,
- $f_i^\alpha(x, u)$ are C^1-functions on an open $\Omega \times \Upsilon \supset (\bar{x}, \phi(\bar{x}))$, satisfying the

  the integrability conditions stated above.

has a unique local C^1-solution near $\bar{x}$.

Proof outline

1. Use Picard-type argument to prove existence and uniqueness of the solution $\tilde{u}$ of the restricted system, which

   - has the same equations and data as the original system

   - each equation is required to hold only for x on a certain, in general lower dimensional, submanifold of $\mathbb{R}^n$, containing $\bar{x}$.

   Integrability conditions are not used for this part!

2. Prove that $\tilde{u}$ is, in fact, a solution of the original system by:
   - introducing functions:

$$A_i^\alpha(x) = \mathbf{r}_i(\tilde{u}_\alpha)|_x - f_i^\alpha(x, \tilde{u}(x)), \qquad 1 \leq \alpha \leq m,\ i \in I_\alpha$$

   - using integrability conditions of the original system to show that functions $A_i^\alpha(x)$ satisfy a linear homogeneous system of equations of the "restricted type" covered by part 1.
   - observing that $A_i^\alpha(x) \equiv 0$ is a unique solution of such system.

More details on the "restricted system"

1. Let $W_i^t(x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ denote the flow of $\mathbf{r}_i$:

$$\frac{d}{dt} W_i^t(x) = \mathbf{r}_i \Big|_{W_i^t(x)}.$$

2. For each α, choose an increasing order on the set of indices $I_\alpha = \{i_1, \ldots, i_{p(\alpha)}\}$ and define a map ρ from an appropriate open neighborhood of $(0, \bar{x})$ in $\mathbb{R}^p \times \Xi_\alpha$ to a neighborhood Ω of $\bar{x}$, by

$$\rho(t_1, t_2, \ldots, t_p, \xi) := W_{i_p}^{t_p} \cdots W_{i_2}^{t_2} W_{i_1}^{t_1} \xi.$$

3. By shrinking the domain of ρ, we can insure that $\Xi_\alpha^n = Im(\rho)$ is an open neighborhood of $\bar{x}$ with local C^1-coordinates $(\xi_1, \ldots, \xi_{n-p}, t_{i_1}, \ldots t_{i_p})$ and hence

$$\Xi_\alpha^k = \{x \in \Xi_\alpha^i \,|\, t_{i_{k+1}} = 0, \ldots, t_{i_p} = 0\}$$

are C^1-submanifolds of $\mathbb{R}^n$.

4. For the restricted system, we require that for each $\alpha = 1, \ldots, m$, and each $i_k \in I_\alpha$:

$$\mathbf{r}_{i_k}(u_\alpha)\Big|_x = f^\alpha_{i_k}(x, u(x)) \qquad \text{for } x \in \Xi^k_\alpha.$$

5. If SCC conditions are satisfied , Picard-type argument implies that the fixed point $\tilde{u}$ of a contractive map:

$$\begin{aligned}\Phi[u]_\alpha(x) = \phi_\alpha(\xi) &+ \int_0^{t_1} f^\alpha_{i_1}\Big(W^b_{i_1}\xi,\, u(W^b_{i_1}\xi)\Big)\, db \\ &+ \int_0^{t_2} f^\alpha_{i_2}\Big(W^b_{i_2}W^{t_1}_{i_1}\xi,\, u(W^b_{i_2}W^{t_1}_{i_1}\xi)\Big)\, db \\ &\vdots \\ &+ \int_0^{t_p} f^\alpha_{i_p}\Big(W^b_{i_p}W^{t_{p-1}}_{i_{p-1}}\cdots W^{t_1}_{i_1}\xi,\, u(W^b_{i_p}W^{t_{p-1}}_{i_{p-1}}\cdots W^{t_1}_{i_1}\xi)\Big) db\end{aligned}$$

is the unique solution of the restricted system with the data

$$u_\alpha|_{\Xi_\alpha} = \phi_\alpha.$$

Motivation: Geometric study of systems of hyperbolic conservation laws

$$U_t + F(U)_x = 0.$$

A sub-problem (the Jacobian Problem):

Given a local frame $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ on $\Omega \subset \mathbb{R}^n$, find all maps $F\colon \Omega \to \mathbb{R}^n$ such that $\mathcal{R}$ is the set of eigenvectors of the Jacobian matrix DF.

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1. Jenssen, H. K., and I.K., Conservation laws with prescribed eigencurves. *J. of Hyperbolic Differential Equations (JHDE)* Vol. 7, No. 2., (2010), 211– 254.

2. Benfield M., Jenssen, H. K., and I.K., Jacobians with prescribed eigenvectors.*Journal of Differential Geometry and its Applications.* Vol. 65, (2019), 108–146.

Example: The Euler system for 1-dim. compressible flow

- Euler system in thermodynamic variables

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ S_t &= 0\,. \end{aligned}$$

  $v = \frac{1}{\rho}$ is volume per unit mass, u is velocity, S is entropy per unit mass, $\boxed{p(v,S) > 0}$ is pressure as a given function, s.t $\boxed{p_v < 0}$.

- $\boxed{U_t + F(U)_x = 0}$, where $U = [v, u, S]^T$ and $F(U) = [-u, p(v,S), 0]^T$.

- eigenvectors of $[D_U F]$ are:
  $\mathbf{r}_1 = [\,1,\ \sqrt{-p_v},\ 0\,]^T$, $\mathbf{r}_2 = [\,-p_S,\ 0,\ p_v\,]^T$, $\mathbf{r}_3 = [\,1,\ -\sqrt{-p_v},\ 0\,]^T$

- eigenvalues of $[D_u f]$ are $\lambda^1 = -\sqrt{-p_v}\,,\qquad \lambda^2 \equiv 0\,,\qquad \lambda^3 = \sqrt{-p_v}$.

Example: The Jacobian problem for the Euler frame.

Given:

- (v, u, S) are coordinate functions in $\mathbb{R}^3$.
- $\boxed{p(v, S) > 0}$, s.t $\boxed{\text{-}p_v < 0}$
- vector fields $\mathbf{r}_1 = \begin{bmatrix} 1 \\ \sqrt{-p_V} \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -p_S \\ 0 \\ p_V \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 1 \\ -\sqrt{-p_V} \\ 0 \end{bmatrix}$

Find: the set $\mathcal{F}(\mathcal{R})$ of all maps $F : \mathbb{R}^3 \to \mathbb{R}^3$, such that $\mathcal{R} = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a set of eigenvector-fields of the Jacobian matrix $[D_U F]$.

Answer:

- If $\left(\frac{p_S}{p_v}\right)_v \neq 0$

$$F = c\begin{bmatrix} -u \\ p(v,S) \\ 0 \end{bmatrix} + \bar{\lambda}\begin{bmatrix} v \\ u \\ S \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c\begin{bmatrix} -u \\ p(v,S) \\ 0 \end{bmatrix} + \text{trivial flux.}$$

eigenvalues: $\lambda^1 = -c\sqrt{-p_v} + \bar{\lambda}\,, \qquad \lambda^2 \equiv \bar{\lambda}\,, \qquad \lambda^3 = c\sqrt{-p_v} + \bar{\lambda}.$

- If $\left(\frac{p_S}{p_v}\right)_v \equiv 0$, then $\mathcal{F}(\mathcal{R})$ depends on 3 arbitrary functions of one variable.

More examples for the Jacobian problem in $\mathbb{R}^3$ (coordinates (u, v, w))

(1) • $\mathbf{r}_1 = \begin{bmatrix} 0 \\ 1 \\ u \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} w \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} u \\ 0 \\ -w \end{bmatrix}$

(integral curves: lines, parabolas, circles)

$$F(U) = \bar{\lambda} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad \bar{\lambda}, a_1, a_2, a_3 \in \mathbb{R},$$

- we call such fluxes trivial fluxes: $\mathcal{F}(\mathcal{R}) = \mathcal{F}^{\text{triv}}$.

- $\lambda_1 = \lambda_2 = \lambda_3 = \bar{\lambda}$.

(2) • $\mathbf{r}_1 = \begin{bmatrix} v \\ u \\ 1 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -v \\ u \\ 0 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ on Ω, where $u\,v \neq 0$.

("hyperbolic spiral":

$$u = \bar{u} \cosh t + \bar{v} \sinh t, v = \bar{u} \sinh t + \bar{v} \cosh t, w = \bar{w} + t,$$

circles, lines)

• $\mathcal{F}(\mathcal{R})/\mathcal{F}^{\text{triv}}$ is a 1-dimensional space

$$F(U) = c\left[v^3, \, u^3, \, \frac{3}{4}(u^2 + v^2)\right]^T + \text{a trivial flux}, \quad c \in \mathbb{R}$$

$$\lambda^1 = 3\,c\,u\,v + \bar{\lambda}, \quad \lambda^2 = -3\,c\,u\,v + \bar{\lambda}, \quad \lambda^3 = \bar{\lambda}.$$

(3) (the coordinate frame)

- $$\mathbf{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- $$f = \left[\phi^1(u),\, \phi^2(v),\, \phi^3(w)\right]^T, \quad \phi^i : \mathbb{R} \to \mathbb{R} \text{ arbitrary}$$

  $\mathcal{F}(\mathcal{R})$ is a ∞-dimensional space

- $$\lambda^1 = \left(\phi^1\right)'(u), \quad \lambda^2 = \left(\phi^2\right)'(v), \quad \lambda^3 = \left(\phi^3\right)'(w).$$

Thank you!

Additional slides

Applications

We encountered systems of the generalized Darboux type in our study of hyperbolic conservative systems:

$$U_t + F(U)_x = 0.$$

1. Jenssen, H. K., Kogan, I. A., Conservation laws with prescribed eigencurves. *J. of Hyperbolic Differential Equations (JHDE)* Vol. 7, No. 2., (2010) pp. 211– 254.

2. Jenssen, H. K., Kogan, I. A., Extensions for systems of conservation laws *Communications in PDE's*, No. 37, (2012) , pp. 1096 – 1140.

At that time, we had to impose analyticity assumptions and use Cartan-Kähler theorem.

Conservation laws with prescribed eigencurves

Jacobian problem: Given a local frame $\mathcal{R} = \{\mathbf{r}_1, \ldots, \mathbf{r}_n\}$ on $\Omega \subset \mathbb{R}^n$, find all maps $F\colon\ \Omega \to \mathbb{R}^n$ such that $\mathcal{R}$ is the set of eigenvectors of the Jacobian matrix Df.

This leads to the λ-system, on the eigenvalues-to-be:

$$
\begin{aligned}
\mathbf{r}_i(\lambda^j) &= \Gamma^j_{ji}(\lambda^i - \lambda^j) && \text{for } i, j \in \{1, \ldots, n\},\ i \neq j,\\
(\lambda^i - \lambda^k)\Gamma^k_{ji} &= (\lambda^j - \lambda^k)\Gamma^k_{ij} && \text{for } i, j, k \in \{1, \ldots, n\},\\
& && i < j,\ i \neq k,\ j \neq k.
\end{aligned}
$$

For $n > 2$ the size of the solution set depends on the properties of $\mathcal{R}$.

In a case when:
$\Gamma^1_{23} \neq 0$, but $c^1_{23} = 0$, $\quad \Gamma^2_{31} = 0$, $\quad \Gamma^3_{21} = 0$, $\quad \Gamma^3_{31} = \Gamma^2_{21}$,

the λ-system reduces to $\lambda^2 = \lambda^3$ and a generalized Darboux-type system:

$$
\begin{aligned}
\mathbf{r}_2(\lambda^1) &= \Gamma^1_{12}\,(\lambda^2 - \lambda^1), & \mathbf{r}_1(\lambda^2) &= \Gamma^2_{21}\,(\lambda^1 - \lambda^2)\\
\mathbf{r}_3(\lambda^1) &= \Gamma^1_{13}\,(\lambda^2 - \lambda^1), & \mathbf{r}_2(\lambda^2) &= 0\\
& & \mathbf{r}_3(\lambda^2) &= 0
\end{aligned}
$$

Example: The Euler system for 1-dim. compressible flow

- Euler system in thermodynamic variables

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p_x &= 0 \\ S_t &= 0\,. \end{aligned}$$

  $v = \frac{1}{\rho}$ is volume per unit mass, u is velocity, S is entropy per unit mass, $\boxed{p(v,S) > 0}$ is pressure as a given function, s.t $\boxed{p_v < 0}$.

- $\boxed{U_t + F(U)_x = 0}$, where $U = [v, u, S]^T$ and $F(U) = [-u, p(v,S), 0]^T$.

- eigenvectors of $[D_U F]$ are:
  $\mathbf{r}_1 = [\,1,\ \sqrt{-p_v},\ 0\,]^T$, $\mathbf{r}_2 = [\,-p_S,\ 0,\ p_v\,]^T$, $\mathbf{r}_3 = [\,1,\ -\sqrt{-p_v},\ 0\,]^T$

- eigenvalues of $[D_u f]$ are $\lambda^1 = -\sqrt{-p_v}\,, \qquad \lambda^2 \equiv 0\,, \qquad \lambda^3 = \sqrt{-p_v}$.

Example: The Jacobian problem for the Euler frame.

Given:

- (v, u, S) are coordinate functions in $\mathbb{R}^3$.

- $\boxed{p(v, S) > 0}$, s.t $\boxed{\text{-}p_v < 0}$

- vector fields $\mathbf{r}_1 = \begin{bmatrix} 1 \\ \sqrt{-p_V} \\ 0 \end{bmatrix}, \quad \mathbf{r}_2 = \begin{bmatrix} -p_S \\ 0 \\ p_V \end{bmatrix}, \quad \mathbf{r}_3 = \begin{bmatrix} 1 \\ -\sqrt{-p_V} \\ 0 \end{bmatrix}$

Find: the set $\mathcal{F}(\mathcal{R})$ of <u>all</u> maps $F : \mathbb{R}^3 \to \mathbb{R}^3$, such that $\mathcal{R} = \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ is a set of eigenvector-fields of the Jacobian matrix $[D_U F]$.

Answer:

- If $\left(\frac{p_S}{p_v}\right)_v \neq 0$

$$F = c\begin{bmatrix} -U \\ p(v,S) \\ 0 \end{bmatrix} + \bar{\lambda}\begin{bmatrix} v \\ u \\ S \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = c\begin{bmatrix} -u \\ p(v,S) \\ 0 \end{bmatrix} + \text{trivial flux.}$$

eigenvalues: $\lambda^1 = -c\sqrt{-p_v} + \bar{\lambda}\,, \qquad \lambda^2 \equiv \bar{\lambda}\,, \qquad \lambda^3 = c\sqrt{-p_v} + \bar{\lambda}.$

- If $\left(\frac{p_S}{p_v}\right)_v \equiv 0$, then $\mathcal{F}(\mathcal{R})$ depends on 3 arbitrary functions of one variable.

Extensions of systems of conservation laws

Hessian problem: Given a local frame $\mathcal{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_n\}$ on $\Omega \subset \mathbb{R}^n$, find all functions $\eta \colon \Omega \to \mathbb{R}$, such that $\mathcal{R}$ is orthogonal with respect to the inner product defined by the Hessian matrix $D^2\eta$.

This leads to the β-system, on the "lengths" of vectors $\mathbf{r}_i$ with respect to $D^2\eta$:

$$r_i(\beta^j) = \beta^j\,(\Gamma^j_{ij} + c^j_{ij}) - \beta^i\,\Gamma^i_{jj} \qquad \text{for } i \neq j,$$

$$\beta^k\,c^k_{ij} + \beta^j\Gamma^j_{ik} - \beta^i\,\Gamma^i_{jk} = 0 \qquad \text{for } i < j,\ i \neq k,\ j \neq k,$$

For $n > 2$ the size of the solution set depends on the properties of $\mathcal{R}$.

In a case when: $c^1_{23} \neq 0$, but $c^3_{12} = \Gamma^3_{12} = \Gamma^3_{21} \equiv 0$, $c^2_{13} = \Gamma^2_{13} = \Gamma^2_{31} \equiv 0$, and $\Gamma^2_{11} = \Gamma^3_{11} \equiv 0$,

the β-system reduces to $\beta^1 = 0$ and a generalized Darboux-type system:

$$\mathbf{r}_1(\beta^2) = \beta^2\,(\Gamma^2_{12} + c^2_{12}), \qquad \mathbf{r}_1(\beta^3) = \beta^3\,(\Gamma^3_{13} + c^3_{13}),$$

$$\mathbf{r}_3(\beta^2) = \beta^2\,(\Gamma^2_{32} + c^2_{32}) - \beta^3\,\Gamma^3_{22}, \qquad \mathbf{r}_2(\beta^3) = \beta^3\,(\Gamma^3_{23} + c^3_{23}) - \beta^2\,\Gamma^2_{33}.$$