# Strong Frobenius structure, rigidity and hypergeometric equations

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INSTITUT CAMILLE JORDAN, LYON

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# AIMS

This talk has two main aims:

- Show that there exists a family $\mathfrak{L}$ of differential operators such that for all $L$ in $\mathfrak{L}$, $L$ has a strong Frobenius structure for almost prime number $p$.

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- Show that there exists a family $\mathfrak{L}$ of differential operators such that for all $L$ in $\mathfrak{L}$, $L$ has a strong Frobenius structure for almost prime number $p$.
- Show the connection between the existence of a strong Frobenius structure for a prime number $p$ and the algebraicity modulo $p$ of the solutions of the corresponding operator.

# Motivation

There is a family $\mathfrak{G}$ of G -functions, $f(t) = \sum_{n\geq 0} a_n t^n \in \mathbb{Q}[[t]]$ , such that for all $f \in \mathfrak{G}$ there exists $\mathcal{S}$ an infinit set of prime numbers such that for every $p \in \mathcal{S}$:

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In particular there is $a_0(t), \ldots, a_c(t) \in \mathbb{F}_p(t)$ such that $f_{|p}$ is zero of

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$$f_1(t) = \sum_{n\geq 0} \frac{1}{16^n} \binom{2n}{n}^2 t^n =_2 F_1(1/2, 1/2; 1, t)$$

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The notion of **strong Frobenius structure** give us a general point of view about the following question:

If $f$ is a $G$-function such that $f \in \mathbb{Z}_{(p)}[[t]]$, is its reduction $f_{|p}$ algebraic over $\mathbb{F}_p(t)$ ? If this is the case, what could we say about its algebraicity degree?

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We say that $A$ and $B$ are **globally equivalents** in $\mathbb{C}(t)$, if there exists $H \in Gl_n(\mathbb{C}(t))$ such that

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Where $E_p$ is the field of analytic elements, that is, the completion of $\mathbb{C}_p(t)$ for the Gauss norm.

# Strong Frobenius structure

Let's set the differential operator
$L := \frac{d}{dt^n} + a_1(t)\frac{d}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{d}{dt} + a_n(t) \in \mathbb{Q}(t)[d/dt]$.

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$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \dots & -a_2(t) & -a_1(t) \end{pmatrix}.$$

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$L$ has a strong Frobenius structure for $p$, of period $h$, if there is an integer $h \geq 1$ , such that $A$ and $p^h t^{p^h-1} A(t^{p^h})$ are $E_p$-**equivalents**.

In other words, $\exists H \in Gl_n(E_p)$ such that

$$\frac{d}{dt}H = AH - H(p^h t^{p^h-1} A(t^{p^h})).$$

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**Frobenius action:** If $L$ has a strong Frobenius structure for $p$ of period h, and $h_{11}, \dots, h_{1n} \in E_p$ is the firt row of $H$ , then for every solution $f$ of $L$, $f$ belongs to a ring containing $E_p$, we have

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Applying the Cayley-Hamilton theorem, we obtain the following

**Theorem (I)**

*Let $f(t) = \sum_{n \geq 0} a(n) t^n \in \mathbb{Z}_{(p)}[[t]]$ be a solution of $L$ endowed of a strong Frobenius structure for $p$ of period h. Then $f_{|p}$ is algebraic over $\mathbb{F}_p(t)$ and*

$$\deg(f_{|p}) \leq p^{n^2 h}$$

*Where $n$ is the order of $L$ .*

# Gauss hypergeometric equations

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**Theorem**

*Let $a, b, c \in \mathbb{Q}$. If $a, b, c-a, c-b \notin \mathbb{Z}$, the operator associated to the Gauss hypergoemetric equation,*

$$\frac{d^2}{dt^2} + \frac{c-(a+b+1)t}{t(1-t)}\frac{d}{dt} - \frac{ab}{t(1-t)}, \qquad (1)$$

*has a strong Frobenius structure for almost all $p$ of period*

$$h \leq \varphi(d(a))\varphi(d(b))\varphi(d(c)),$$

*where $\varphi$ is the Euler phi function and $d$ is the denominator function*

# RIGIDITY

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**Theorem (VM.)**

*Let $L \in \mathbb{Q}(t)[d/dt]$. We suppose that the following conditions are verified.*

1. *The singular points of $L$ are regular .*
2. *The exponents at singular points are rational numbers.*
3. *The monodromy group of $L$ is rigid.*

*Then, the differential operator $L$ has a strong Frobenius structure for almost all prime number $p$.*

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We say that monodromy group of $L$ is rigid if for every $N_1, \ldots, N_r \in Gl_n(\mathbb{C})$ such that $N_1 \cdots N_r = Id_n$ and $N_i$ cojugated to $M_i$ , then there is $U \in Gl_n(\mathbb{C})$ such that

$$UN_iU^{-1} = M_i \quad \forall 1 \leq i \leq r\,.$$

# Generalized hypergeometric equation

The differential operator associated to the generalized hypergeometric equation is

$$\mathcal{H}(\underline{\alpha}, \underline{\beta}) : -t(\delta+\alpha_1)\cdots(\delta+\alpha_n)+(\delta+\beta_1-1)\cdots(\delta+\beta_n-1),$$

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where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{Q}$ and $\alpha_i - \beta_j \notin \mathbb{Z}$.

- The singularities of $\mathcal{H}(\underline{\alpha}, \underline{\beta})$ are regular , wich are $0, 1, \infty$.
- The exponents at infinity are $\alpha_1, \ldots, \alpha_n$, the exponents at $0$ are $1 - \beta_1, \ldots, 1 - \beta_n$, and the exponents at $1$ are $0, 1, \ldots, n-2, -1 + \sum(\beta_i - \alpha_i)$.
- The monodromy group of $\mathcal{H}(\underline{\alpha}, \underline{\beta})$ is rigid. (Levelt).

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where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{Q}$ and $\alpha_i - \beta_j \notin \mathbb{Z}$.

Let $\mathcal{S} := \{p \in \mathcal{P} \setminus \{2\} \text{ such that } |\alpha_i|_p, |\beta_j|_p = 1\}$.

**Theorem (II)**

*$\mathcal{H}(\underline{\alpha}, \underline{\beta})$ has a strong Frobenius structure for $p \in \mathcal{S}$ of period*

$$h \leq \prod_{i=1}^{n} \varphi(d(\alpha_i)) \prod_{j=1}^{n} \varphi(d(\beta_j)).$$

# Algebraicity modulo $p$

We come back to the serie

$$f_2(t) :={}_2F_1(1/2,1/2;2/3,t)=\sum_{k=0}^{\infty}\left(\frac{(1/2)_k^2}{(2/3)_k k!}\right)t^n\in\mathbb{Q}[[t]].$$

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For $p \equiv 1 \mod 3$, $f_2(t) \in \mathbb{Z}_{(p)}[[t]]$ and for these primer numbers the period $h = 1$, the **theorem I** implies that $f_{2|p}$ is algebraic and

$$\deg(f_{2|p}) \leq p^{2^2}.$$

Let us consider

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For $p \neq 3$, $f_3(t) \in \mathbb{Z}_{(p)}[[t]]$. Applying **theorem I** we have that for $p \geq 7$, $f_{3|p}$ is algebraic and

$$\deg(f_{3|p}) \leq p^{3^2 \times 6}.$$

**Theorem (VM.)**

*Let $L \in \mathbb{Q}(t)[d/dt]$. We suppose that the following conditions are verified.*

1. *The singular points of $L$ are regular .*
2. *The exponents at singular points are rational numbers.*
3. *The monodromy group of $L$ is rigid.*

*Then, the differential operator $L$ has a strong Frobenius structure for almost all prime numbers $p$.*

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Since the exponents are rational numbers, there is $h \geq 1$ such that $\forall p \in \mathcal{S}$, $p^h\alpha \equiv \alpha \mod \mathbb{Z}$, for every exponent $\alpha$ and $|\gamma_i^{p^h} - \gamma_i|_p < 1$. $\forall 1 \leq i \leq r-1$.

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$$G = -\frac{1}{t+1}\left(\sum_{j=1}^{r} N_j\right) + \sum_{j=1}^{r-1} \frac{1}{t-\gamma_j} N_j \in M_n(\mathbb{C}_p(t)),$$

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where $N_j \in M_n(\mathbb{C}_p)$ and the eigenvalues of $N_j$ are the exponents of $L$ at $\gamma_j$ multplied by $p^h$ and $\sum_{j=1}^{r} N_j \in M_n(\mathbb{Z})$ is a diagonal matrix.

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In particular the eigenvalues of $N_j's$ are $p^h \alpha_{i,j} \in \mathbb{Q}$. Where $\alpha_{i,j}$ is an exponent of L at $\gamma_j$ .

The main points in the proof are the following:

1. Let $\kappa : \mathbb{C}_p \to \mathbb{C}$ be a isomorphisme of fields.

$$G^{\kappa} = -\frac{1}{t+1}\left(\sum_{j=1}^{r} N_j^{\kappa}\right) + \sum_{j=1}^{r-1} \frac{1}{t-\kappa(\gamma_j)} N_j^{\kappa} \in M_n(\mathbb{C}(t)) .$$

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2. Finally the rigidity hypothesis implies that $A$ and $G^{\kappa}$ are $\mathbb{C}(t)$-equivalents. So that, $A$ and $G$ are $E_p$-equivalents, which implies that $A$ and $B$ are $E_p$-equivalents.

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Since $p^h \alpha_{i,j} \equiv \alpha_{i,j} \mod \mathbb{Z}$ for $p \in \mathcal{S}$, the matrices $exp(2\pi i C_j)$ and $exp(2\pi i p^h C_j)$ are conjugated.

# Monodromy of $G^\kappa$

$$G^\kappa = -\frac{1}{t+1}\left(\sum_{j=1}^{r} N_j^\kappa\right) + \sum_{j=1}^{r-1}\frac{1}{t-\kappa(\gamma_j)}N_j^\kappa \in M_n(\mathbb{C}(t)) .$$

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So that, the eigenvalues of $L_j$ are $p^h \alpha_{i,j} + m$, where $p^h \alpha_{i,j}$ is an eigenvalue of $N_j^\kappa$. We conclude that, $exp(2\pi i L_j)$ and $\exp(2\pi i p^h C_j)$ are conjugated.

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Consenquently, $A$ and $G$ are $E_p$**-equivalents**, but $G$ is $E_p$**-equivalent** to $B$ , then by transitivity we have that $A$ and $B$ are $E_p$**-aquivalents.** □