Differential transcendence of elliptic hypergeometric functions through Galois theory[1]

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Introduction

Elliptic hypergeometric functions were introduced by Spiridonov in early 2000s; they are analogues/generalizations of classical Euler-Gauss hypergeometric functions, related to elliptic curves.

They find applications in:

- representation theory (connected to math. physics, and conjecturally to reps. of "elliptic quantum groups");
- four-dimensional sypersymmetric quantum field theories;
- exactly solvable models in statistical mechanics;
- entropy of black holes;
- ...

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We have shown that "most" of these special functions do not satisfy any algebraic differential equations with elliptic coefficients.

This is an application of new algorithmic results obtained from *differential Galois theory of difference equations (over elliptic curves)*.

Theta functions

Let $p \in \mathbb{C}^*$ such that $|p| < 1$, and denote $(z; p)_\infty = \prod_{j \geq 0} (1 - zp^j)$.

The (modified Jacobi) *theta function* is the holomorphic function on $\mathbb{C}^*$ defined by

$$\theta(z; p) = (z; p)_\infty (pz^{-1}; p)_\infty.$$

Note that

$$\theta(z_0; p) = 0 \qquad \text{if and only if} \qquad z_0 \in p^{\mathbb{Z}} = \{p^n \mid n \in \mathbb{Z}\},$$

and we have the functional equation

$$\theta(pz; p) = \theta(z^{-1}; p) = -z^{-1}\theta(z; p).$$

Elliptic functions and theta functions

Elliptic functions: meromorphic on $\mathbb{C}^*$ such that $f(pz) = f(z)$.

Same as meromorphic functions on elliptic curve $E = \mathbb{C}^*/p^{\mathbb{Z}}$.

- Given $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ and $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, the map

$$\mathbb{C} \to \mathbb{C}^* : w \mapsto \exp(2\pi i w) = z$$

  induces an isomorphism $\mathbb{C}/\Lambda \simeq \mathbb{C}^*/p^{\mathbb{Z}}$, where $p = \exp(2\pi i \tau)$.

Given $a_1, \dots, a_m, b_1, \dots, b_m, c \in \mathbb{C}^*$ with *balancing condition*

$$\prod_{j=1}^{m} a_j = \prod_{j=1}^{m} b_j, \qquad \text{the function} \qquad f(z) = c\,\frac{\prod_{j=1}^{m} \theta(a_j z; p)}{\prod_{j=1}^{m} \theta(b_j z; p)}$$

is elliptic. *Any* non-zero elliptic function has this form.

Explicit divisor: $\mathrm{div}\big(f(z)\big) = \sum_{j=1}^{m} \big[a_j^{-1}\big]_E - \big[b_j^{-1}\big]_E$.

Elliptic Gamma functions

Now $p, q \in \mathbb{C}^*$ such that $|p|, |q| < 1$ and $p^{\mathbb{Z}} \cap q^{\mathbb{Z}} = \{1\}$.

Denote $(z; p, q)_\infty = \prod_{j,k \geq 0}(1 - zp^j q^k)$.

Ruijsenaars's *elliptic Gamma function* is:

$$\Gamma(z; p, q) = \frac{(pq/z; p, q)_\infty}{(z; p, q)_\infty}.$$

It satisfies the functional equations:

$$\Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q) \quad \& \quad \Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q).$$

- Elliptic analogues of classical Euler Gamma function: $\Gamma(z + 1) = z\Gamma(z)$.
- Classical Gauss hypergeometric functions can be defined in terms of the Euler Gamma function (Barnes integral formula).

Elliptic hypergeometric functions

For $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_8) \in (\mathbb{C}^*)^8$ satisfying the *balancing condition*

$$\prod_{j=1}^{8} \varepsilon_j = p^2 q^2, \tag{1}$$

the *elliptic hypergeometric function* $f_{\boldsymbol{\varepsilon}}(z)$ is defined in terms of elliptic Gamma functions, by

$$V(\underline{t}\ ; p, q) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi i} \int_{\mathbb{T}} \frac{\prod_{j=1}^{8} \Gamma(t_j s^{\pm 1}; p, q)}{\Gamma(s^{\pm 2}; p, q)} \frac{ds}{s},$$

and the reparametrization $(t_1, \ldots, t_8) \mapsto (\varepsilon_1, \ldots, \varepsilon_8, z)$:

[you do not want to see this].

Theorem (Spiridonov)

The elliptic hypergeometric function $f_{\boldsymbol{\varepsilon}}(z)$ satisfies a second-order linear difference equation over $\mathbb{C}(E)$, where $E = \mathbb{C}^*/p^{\mathbb{Z}}$.

Galois theory (philosophy)

$$\text{Group Theory}$$
$$\Downarrow \text{ Galois Theory}$$
$$\text{Relations among Solutions}$$

Computing Galois groups leads directly to computation of relations among the solutions to the corresponding equations.

- "Large" Galois group $\Longleftrightarrow$ "few" relations among solutions.

Base $\sigma\delta$-field of elliptic functions

As before, take $p, q \in \mathbb{C}^*$ such that:

$$|p| < 1, \quad |q| < 1, \quad \text{and} \quad p^{\mathbb{Z}} \cap q^{\mathbb{Z}} = \{1\}.$$

In particular $q \pmod{p^{\mathbb{Z}}}$ is of infinite order in $E = \mathbb{C}^*/p^{\mathbb{Z}}$.

Base field: $\mathbb{C}(E)$, meromorphic functions on E.

Difference operator: Automorphism $\sigma : f(z) \mapsto f(qz)$.

Differential operator: Derivation $\delta = z\frac{d}{dz}$.

$\mathbb{C}(E)$ is a $\sigma\delta$-field: $\sigma \circ \delta = \delta \circ \sigma$.

Differential Galois theory for difference equations

(Hardouin-Singer)

K is a $\sigma\delta$-field and $C = K^\sigma = \{c \in K \mid \sigma(c) = c\}$ is δ-closed.

Consider a linear difference equation:

$$a_n\sigma^n(y) + a_{n-1}\sigma^{n-1}(y) + \cdots + a_1\sigma(y) + a_0 y = 0, \qquad (2)$$

where $a_n, \ldots, a_0 \in K$ and $a_n a_0 \neq 0$.

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There is a $\sigma\delta$-ring R, (essentially) generated as K-algebra by

$$\{\sigma^i\delta^j(y_1), \ldots, \sigma^i\delta^j(y_n) \mid i, j \in \mathbb{Z}_{\geq 0}\}$$

where $y_1, \ldots, y_n \in R$ is a C-basis of solutions of (2).

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The $\sigma\delta$-Galois group is

$$\mathrm{Gal}_{\sigma\delta}(R/K) := \{\gamma \in \mathrm{Aut}_{K\text{-alg}}(R) \mid \gamma \circ \sigma = \sigma \circ \gamma,\ \gamma \circ \delta = \delta \circ \gamma\}.$$

It is a linear differential algebraic group in $\mathrm{GL}_n(C)$.

Linear differential algebraic groups

Definition

A *linear differential algebraic group* is a subgroup of $\mathrm{GL}_n(C)$ defined by polynomial differential equations in the matrix entries.

Examples:

- algebraic groups over C;
- algebraic groups over $C^\delta = \{c \in C \mid \delta(c) = 0\}$;

Let $\mathcal{L} = \sum_{i=0}^{n} c_i \delta^i$ with $c_n, \ldots, c_0 \in C$.

- $\{\alpha \in \mathbb{G}_a(C) \mid \mathcal{L}(\alpha) = 0\}$;
- $\{\alpha \in \mathbb{G}_m(C) \mid \mathcal{L}(\frac{\delta(\alpha)}{\alpha}) = 0\}$.

Theorem (Cassidy)

Every δ-algebraic subgroup of $\mathbb{G}_a(C)$ or $\mathbb{G}_m(C)$ is as above.

Main Result: differential transcendence criteria

[Under mild conditions on the otherwise arbitrary $\sigma\delta$-field K.]

Theorem (A.-Dreyfus-Roques)

Let $f \neq 0$ be a solution of

$$\sigma^2(f) + a\sigma(f) + bf = 0,$$

where $a, b \in K$ and $b \neq 0$. Assume that:

- *There is no $u \in K$ such that $\sigma(u)u + au + b = 0$.*
- *There are no $c_0, \ldots, c_n \in C$ with $c_n \neq 0$ and $h \in K$, such that*

$$c_n\delta^n\left(\frac{\delta b}{b}\right) + \cdots + c_0\frac{\delta b}{b} = \sigma(h) - h.$$

Then f is differentially transcendental over K.

The elliptic hypergeometric equation

Theorem (Spiridonov)

The function $f_\varepsilon(z)$ *satisfies*

$$A(z)(\sigma(y) - y) + A(z^{-1})(\sigma^{-1}(y) - y) + \nu y = 0, \qquad (3)$$

where

$$A(z) = \frac{1}{\theta(z^2; p)\theta(qz^2; p)} \prod_{j=1}^{8} \theta(\varepsilon_j z; p), \quad \nu = \prod_{j=1}^{6} \theta(\varepsilon_j \varepsilon_8 / q; p).$$

- It follows from the *balancing condition* $\prod_{j=1}^{8} \varepsilon_j = p^2 q^2$ that the coefficients $\underline{A(z), A(z^{-1}) \in \mathbb{C}(E)}$.
- Hence, (3) is equivalent to a second-order linear difference equation over the base $\mathbb{C}(E)$ (after applying σ once).
- Because $A(z)$ and $A(z^{-1})$ are given in terms of theta functions, we have complete knowledge of their divisors.

Proving differential transcendence of $f_\varepsilon(z)$

Theorem (A.-Dreyfus-Roques)

If every multiplicative relation among $\varepsilon_1, \ldots, \varepsilon_8 \in \mathbb{C}^*$ is induced from the balancing condition $\prod_{j=1}^{8} \varepsilon_j = p^2 q^2$, then $f_{\boldsymbol{\varepsilon}}(z)$ is differentially transcendental over $\mathbb{C}(E)$.

- Earlier work of Dreyfus-Roques gives criteria to decide (non-)existence of solutions $u \in \mathbb{C}(E)$ of Riccati equation

$$\sigma(u)u + au + b = 0,$$

  depending on the divisors of $a, b \in \mathbb{C}(E)$.

- We prove non-existence of telescoper $0 \neq \mathcal{L} \in C[\delta]$ and certificate $h \in \mathbb{C}(E)$ such that

$$\mathcal{L}\left(\frac{\delta(b)}{b}\right) = \sigma(h) - h$$

  also by analyzing the divisor of $b \in \mathbb{C}(E)$.

Sketch of proof: Main Result (1/2)

We know a priori that one of the following three cases occurs for the $\sigma\delta$-Galois group G.

1. G is conjugate to a group of upper-triangular matrices. This happens if and only if there exists a solution $u \in K$ to the Riccati equation $\sigma(u)u + au + b = 0$.
2. G is conjugate to a subgroup of

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \middle| \alpha, \beta \in C^\times \right\} \bigcup \left\{ \begin{pmatrix} 0 & \gamma \\ \mu & 0 \end{pmatrix} \middle| \gamma, \mu \in C^\times \right\}.$$

3. G contains $\mathrm{SL}_2(C^\delta)$.

No solutions to Riccati equation implies that G is irreducible (i.e., we are not in case 1).

Sketch of proof: Main Result (2/2)

No telescoper/certificate for $\mathcal{L}\left(\frac{\delta(b)}{b}\right) = \sigma(h) - h$ implies that $\det(G) = \mathbb{G}_m(C)$, which in turn implies that G is one of the following groups

- $\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \middle| \alpha, \beta \in C^\times \right\} \bigcup \left\{ \begin{pmatrix} 0 & \gamma \\ \mu & 0 \end{pmatrix} \middle| \gamma, \mu \in C^\times \right\}$;
- $C^\times \cdot \mathrm{SL}_2(C^\delta)$;
- $\mathrm{GL}_2(C)$.

In any of these cases, G is sufficiently large to guarantee that any one solution $f \neq 0$ of the difference equation must be differentially transcendental over K.

Final remarks

- Differential transcendence criteria in Main Result apply uniformly in other cases of interest:
  - Shift: $\sigma(x) = x + 1$;
  - q-dilation: $\sigma(x) = qx$, where $q \in \mathbb{C}^*$ is not a root of unity;
  - Mahler: $\sigma(x) = x^p$, where $p \in \mathbb{Z}_{\geq 2}$.

  Applications include: q-series, deterministic finite automata, walks in the quarter plane, [insert your problem here].

- There exist algorithms to verify first condition (Riccati) in all cases, but require knowledge of divisors in elliptic case.

- For second condition: telescopers hardly ever exist in all cases but elliptic, where fewer general results are known.

- General algorithm to compute the $\sigma\delta$-Galois group only available in shift case (but q-dilation and Mahler will appear soon in joint work with Yi Zhang).