A (truly) universal differential equation

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Joint work with Olivier Bournez and Daniel Graça

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INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

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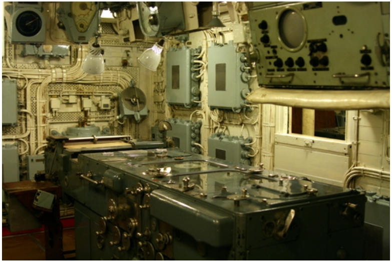
What is a computer?

What is a computer ?

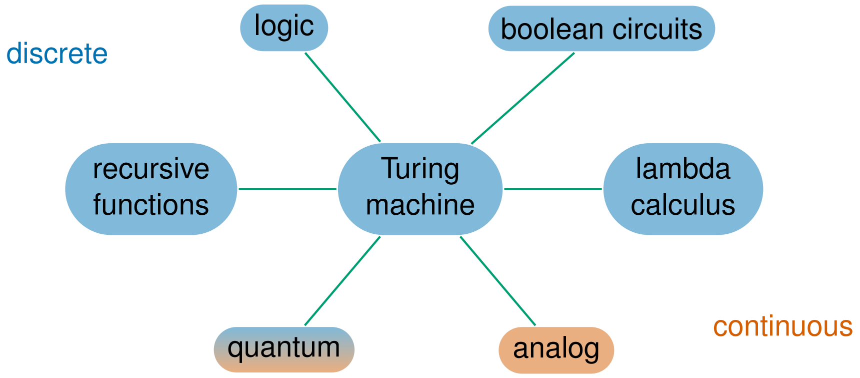
What is a computer ?

VS



Church Thesis

Computability

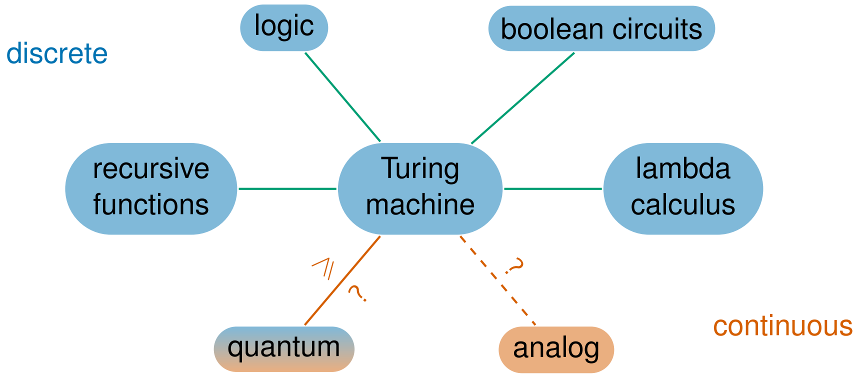


Church Thesis

All **reasonable** models of computation are equivalent.

Church Thesis

Complexity



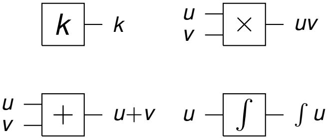
Effective Church Thesis

All **reasonable** models of computation are equivalent for complexity.

Polynomial Differential Equations

Differential Analyzer

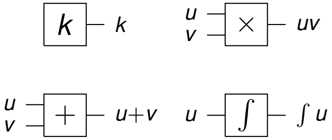
Polynomial Differential Equations



General Purpose Analog Computer, Shannon 1936

Differential Analyzer

Polynomial Differential Equations



General Purpose Analog Computer, Shannon 1936

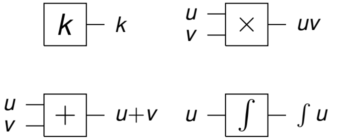
Differential Analyzer

Polynomial differential equations :

$$\begin{cases} y(0) = y_0 \\ y'(t) = p(y(t)) \end{cases}$$

- Rich class
- Stable $(+, \times, \circ, /, \text{ED})$
- No closed-form solution

Polynomial Differential Equations



General Purpose Analog Computer, Shannon 1936

Differential Analyzer

Newton mechanics

Reaction networks :

- chemical
- enzymatic

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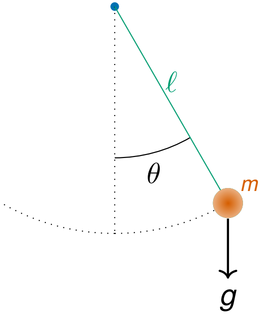
Example of dynamical system



$$\ddot{\theta} + \frac{g}{\ell} \sin(\theta) = 0$$

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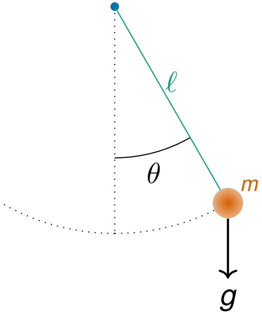

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$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{l} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

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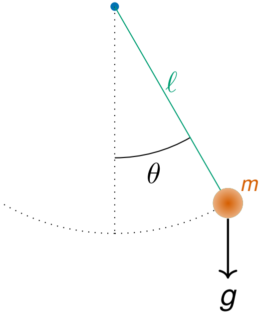

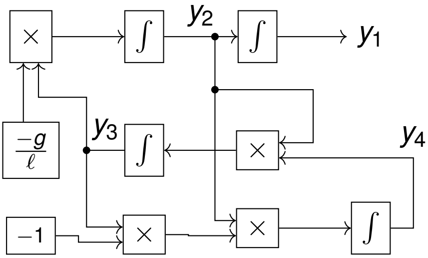

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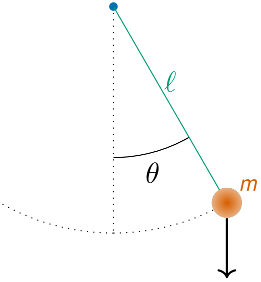

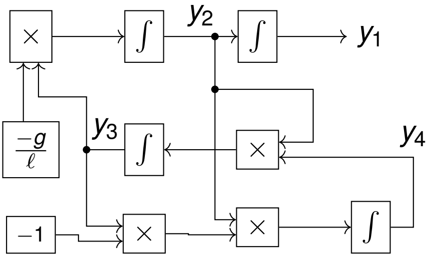

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Historical remark : the word "analog"

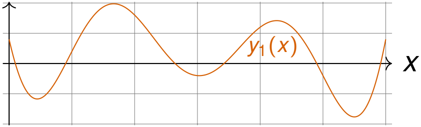
The pendulum and the circuit have the same equation. One can study one using the other by analogy.

Computing with differential equations

Generable functions

$$\begin{cases} y(0) = y_0 \\ y'(x) = p(y(x)) \end{cases} \qquad x \in \mathbb{R}$$

$f(x) = y_1(x)$



Shannon's notion

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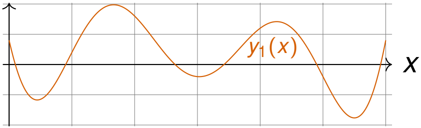


Shannon's notion

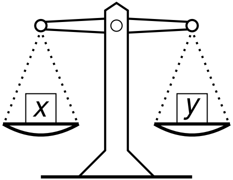
$\sin, \cos, \exp, \log, \ldots$

Considered "weak" : not Γ and ζ

Only analytic functions

Does a balance scale compute a function ?

Inputs : $x, y \in [0, +\infty)$



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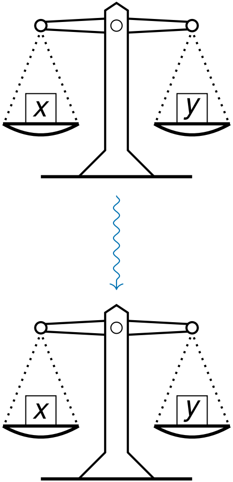

$x = y$

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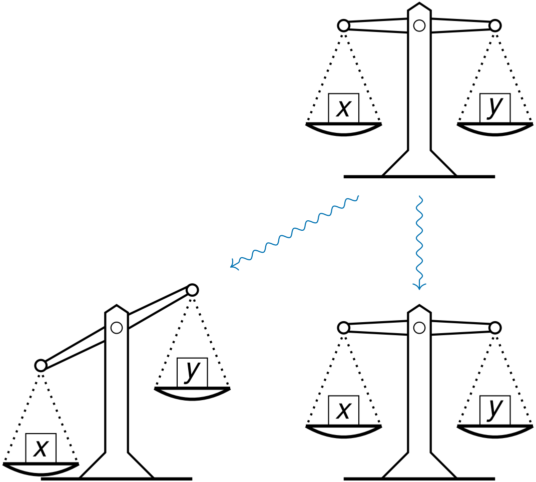

$x > y$ $\quad$ $x = y$

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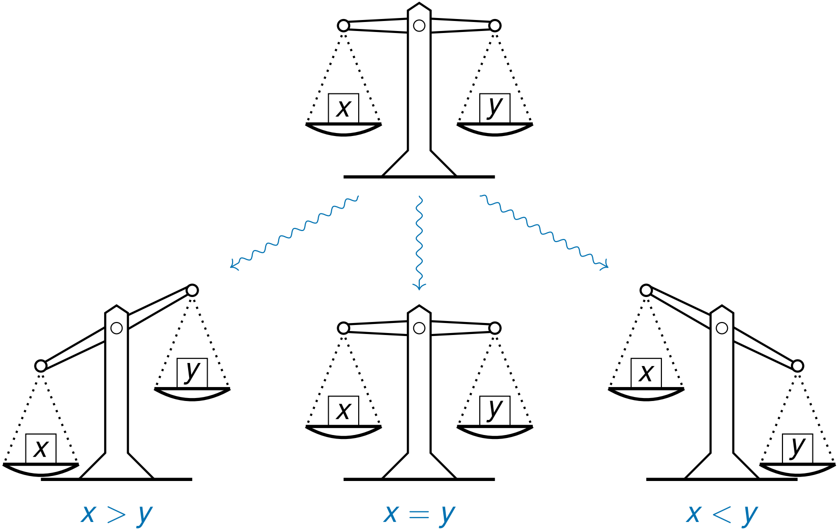

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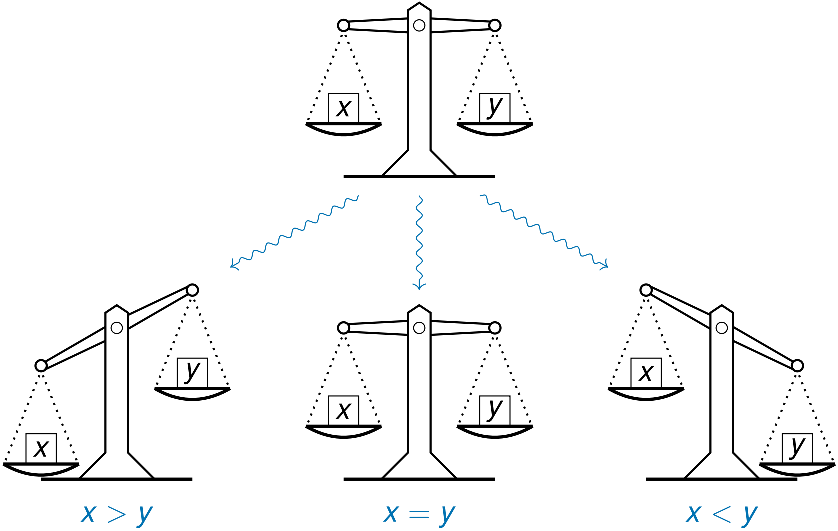
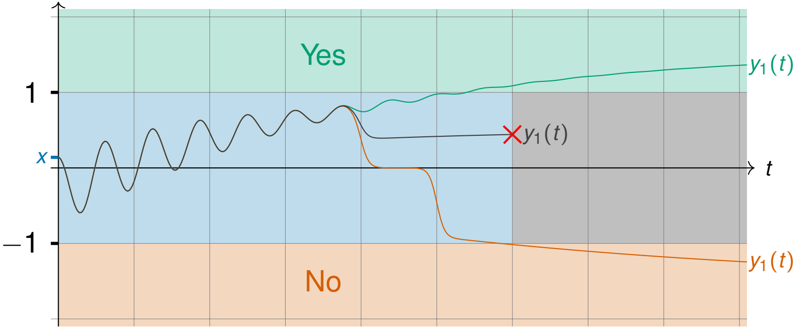


Output : $\text{sign}(x - y)$?

More formally



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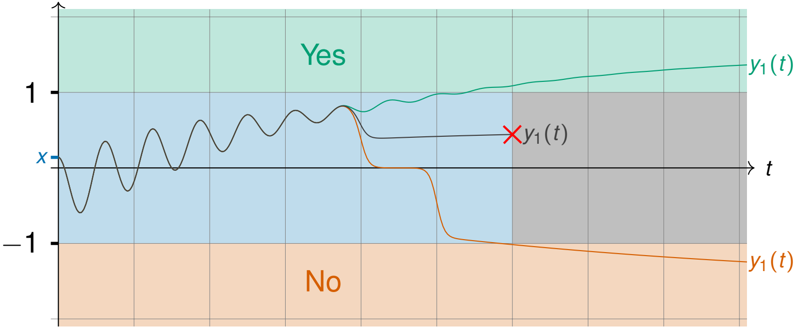
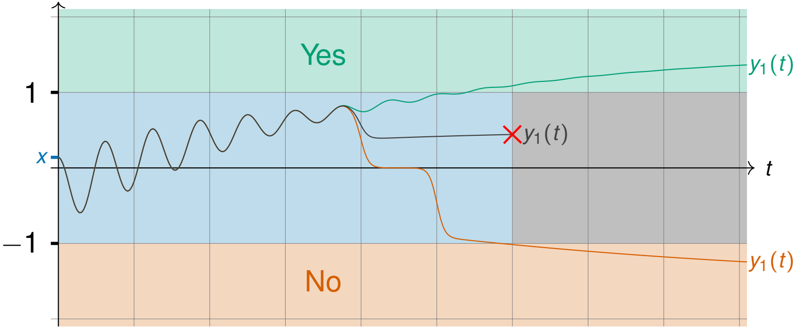


Theorem (Bournez et al, 2010)

This is equivalent to a Turing machine.

More formally



Theorem (Bournez et al, 2010)

This is equivalent to a Turing machine.

- analog computability theory
- purely continuous characterization of classical computability

Computing with differential equations (cont.)

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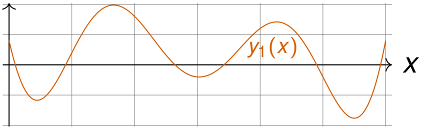


Shannon's notion

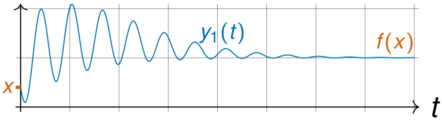
$\sin, \cos, \exp, \log, \ldots$

Considered "weak" : not Γ and ζ
Only analytic functions

Computable

$$\begin{cases} y(0) = q(x) & x \in \mathbb{R} \\ y'(t) = p(y(t)) & t \in \mathbb{R}_+ \end{cases}$$

$$f(x) = \lim_{t\to\infty} y_1(t)$$



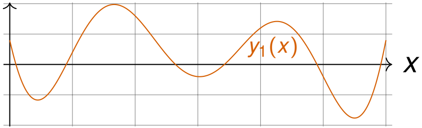
Modern notion

$\sin, \cos, \exp, \log, \Gamma, \zeta, \ldots$

Turing powerful
[Bournez et al., 2007]

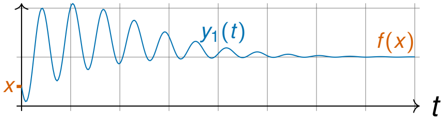
Universal differential equations

Generable functions



subclass of analytic functions

Computable functions



any computable function

Universal differential equations

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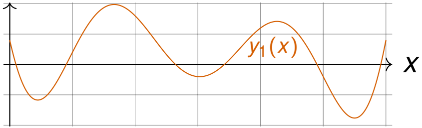

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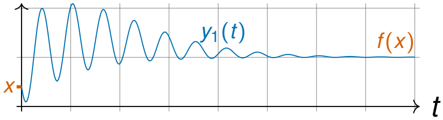

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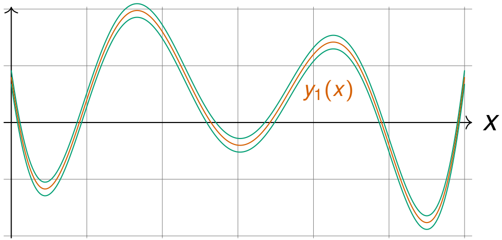
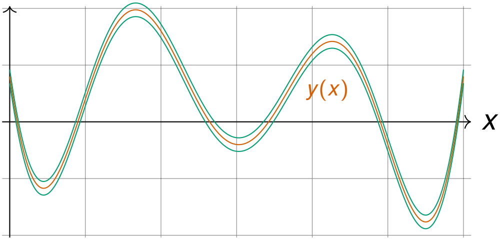


Universal differential algebraic equation (DAE)



Theorem (Rubel, 1981)

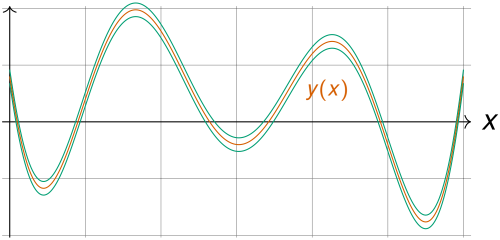
For any continuous functions f and ε, there exists $y : \mathbb{R} \to \mathbb{R}$ solution to

$$3y'^4 y'' y''''^2 - 4y'^4 y'''^2 y'''' + 6y'^3 y''^2 y''' y'''' + 24y'^2 y''^4 y'''' - 12y'^3 y'' y'''^3 - 29y'^2 y''^3 y'''^2 + 12y''^7 = 0$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

Universal differential algebraic equation (DAE)



Theorem (Rubel, 1981)

*There exists a **fixed** polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to*

$$p(y, y', \ldots, y^{(4)}) = 0$$

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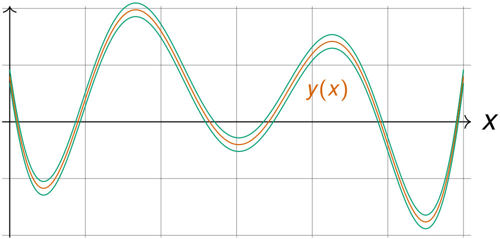

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Problem : this is «weak» result.

The problem with Rubel's DAE

The solution y is not unique, **even with added initial conditions** :

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel's proof to work !

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In fact, this is fundamental for Rubel's proof to work !

- Rubel's statement : this DAE is universal
- More realistic interpretation : this DAE allows almost anything

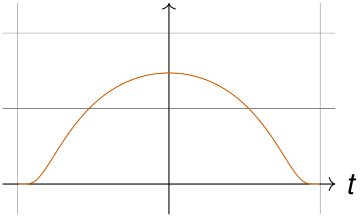
Open Problem (Rubel, 1981)

Is there a universal ODE $y' = p(y)$?
Note : explicit polynomial ODE $\Rightarrow$ unique solution

Rubel's proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

  It satisfies $\quad (1-t^2)^2 f''(t) + 2tf'(t) = 0.$



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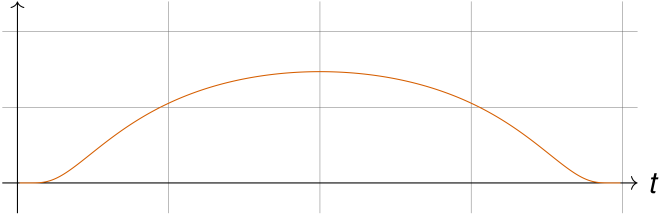
- For any $a, b, c \in \mathbb{R}$, $y(t) = cf(at+b)$ satisfies

$$\begin{aligned} 3y'^4 y'' y''''^2 \quad & -4y'^4 y''^2 y'''' + 6y'^3 y''^2 y''' y'''' + 24 y'^2 y''^4 y'''' \\ & -12 y'^3 y'' y'''^3 - 29 y'^2 y''^3 y'''^2 + 12 y''^7 = 0 \end{aligned}$$

Translation and rescaling :



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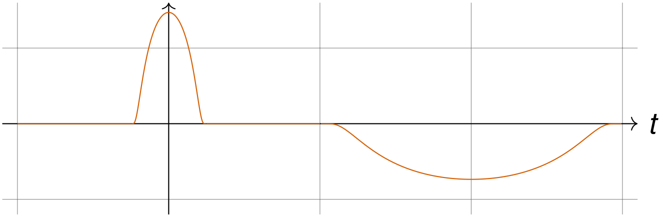
- Can glue together arbitrary many such pieces



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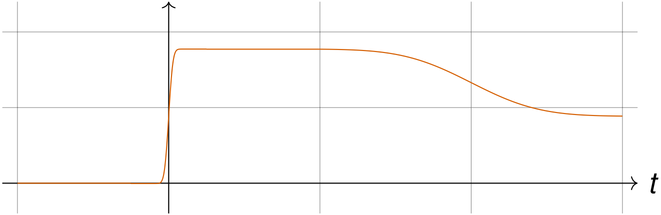
- Can glue together arbitrary many such pieces
- Can arrange so that $\int f$ is solution : piecewise pseudo-linear



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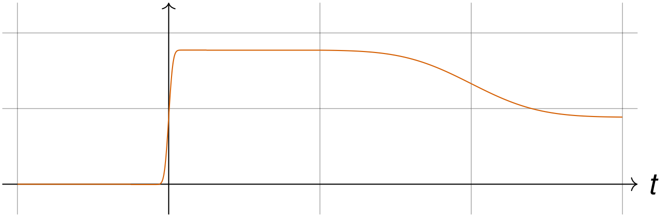
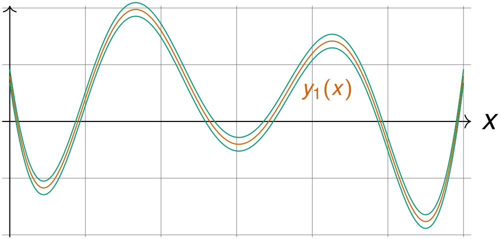


Conclusion : Rubel's equation allows any piecewise pseudo-linear functions, and those are **dense in** C^0

Universal initial value problem (IVP)



Theorem

*There exists a **fixed** (vector of) polynomial p such that for any continuous functions f and ε, there exists $\alpha \in \mathbb{R}^d$ such that*

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

*has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,*

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

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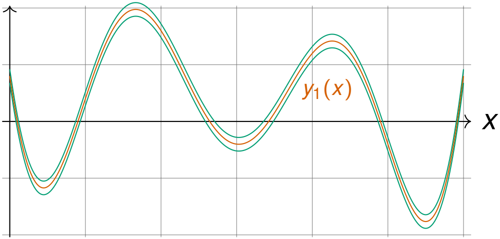

Notes :

- **system** of ODEs,
- y is analytic,
- we need $d \approx 300$.

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has a ***unique solution*** $y : \mathbb{R} \to \mathbb{R}^d$ *and* $\forall t \in \mathbb{R}$*,*

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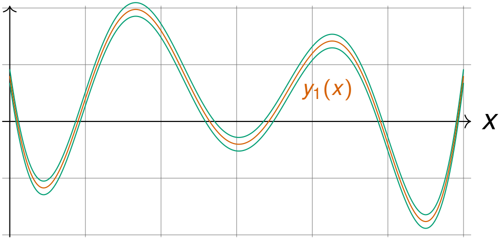

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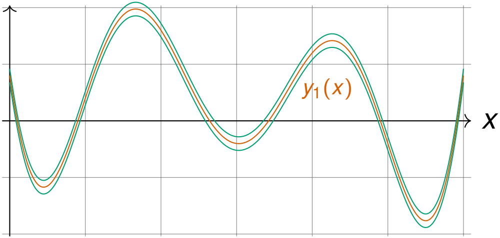
Remark : α is usually transcendental, but computable from f and ε

Universal DAE revisited



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$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

*has a **unique analytic solution** and this solution satisfies such that*

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

A brief stop

Before I can explain the proof, you need to know more of polynomial ODEs and what I mean by programming with ODEs.

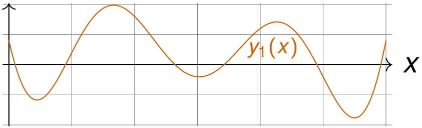
Generable functions : a summary

Definition

$f : \mathbb{R} \to \mathbb{R}$ is generable if $\exists\ d, p$ and y_0 such that the solution y to

$$y(0) = y_0, \qquad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.



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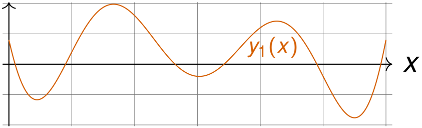

Nice theory for the class of total and univariate generable functions :

- analytic
- contains polynomials, sin, cos, tanh, exp
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)

$$y' = f(y)$$

- solutions to polynomial ODEs form a **very large class**

Why is this useful ?

Writing polynomial ODEs by hand is hard.

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Using generable functions, we can build complicated **multivariate partial functions** using other operations, and we know they are solutions to polynomial ODEs **by construction**.

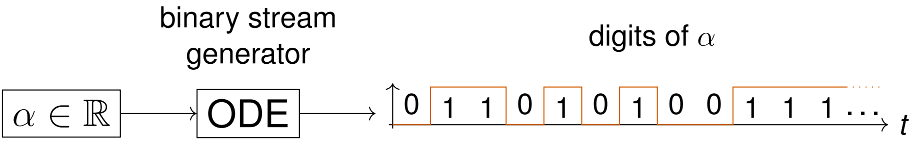
Example : almost rounding function

There exists a generable function round such that for any $n \in \mathbb{Z}$, $x \in \mathbb{R}$, $\lambda > 2$ and $\mu \geqslant 0$:

- if $x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right]$ then $|\operatorname{round}(x, \mu, \lambda) - n| \leqslant \frac{1}{2}$,
- if $x \in \left[n - \frac{1}{2} + \frac{1}{\lambda}, n + \frac{1}{2} - \frac{1}{\lambda}\right]$ then $|\operatorname{round}(x, \mu, \lambda) - n| \leqslant e^{-\mu}$.

See proof

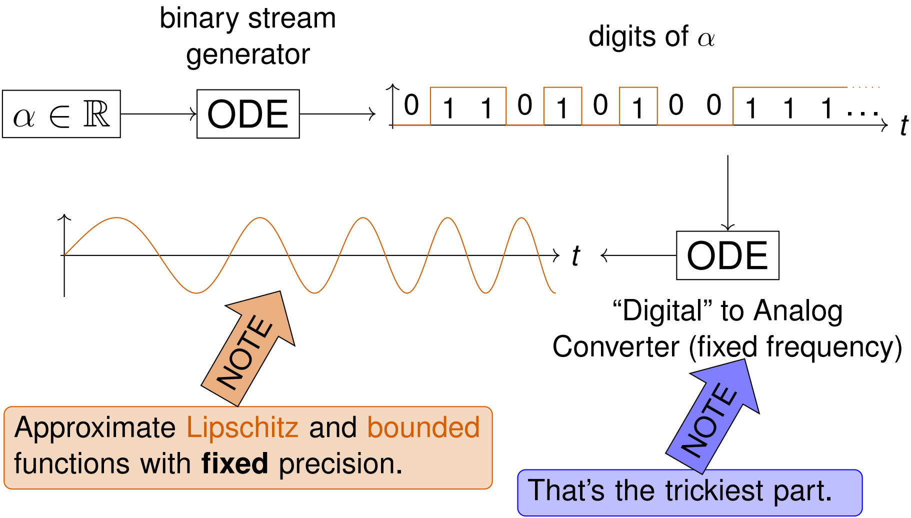
A simplified proof



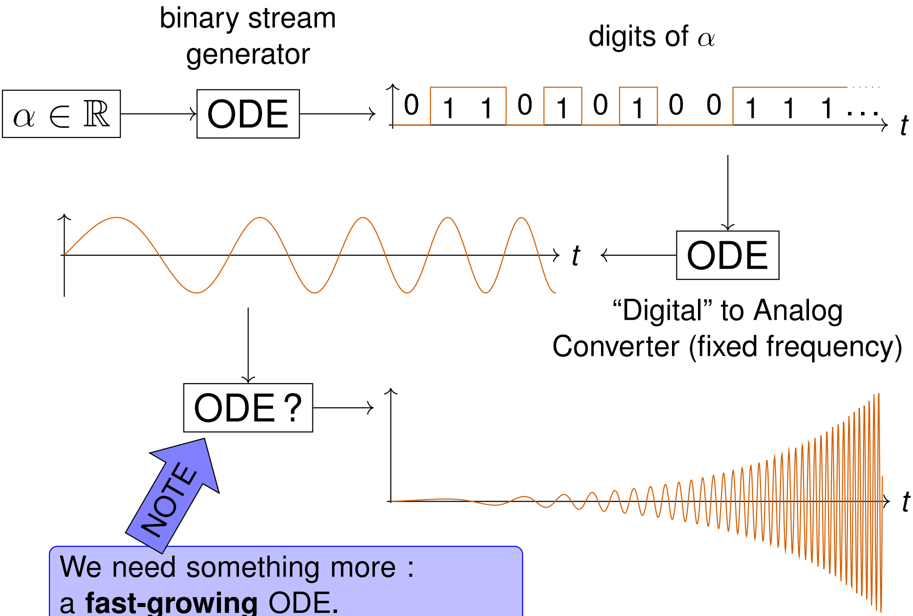
NOTE

This is the ideal curve, the real one is an approximation of it.

A simplified proof

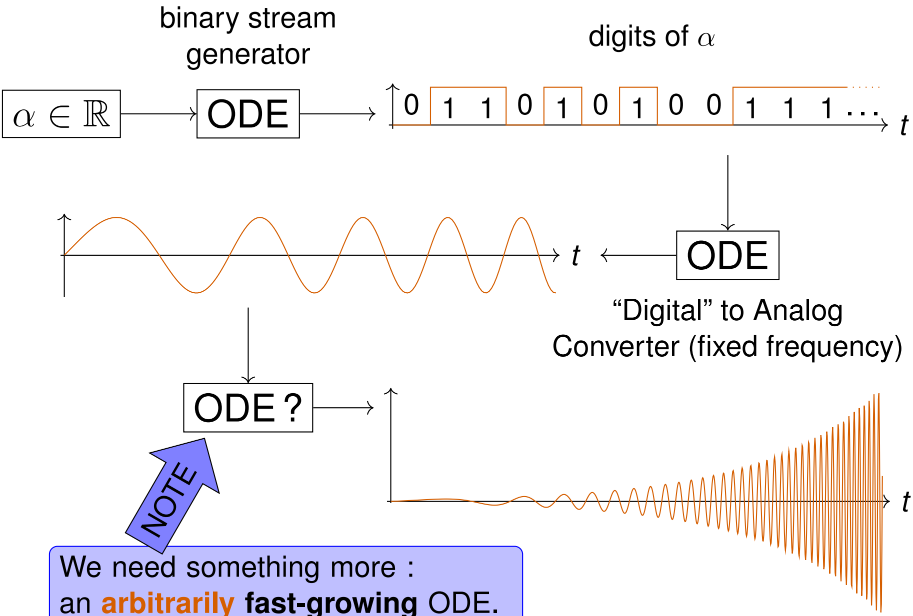


A simplified proof



We need something more :
a **fast-growing** ODE.

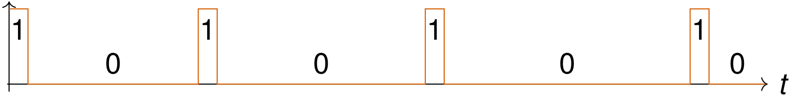
A simplified proof



We need something more :
an **arbitrarily** **fast-growing** ODE.

A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$



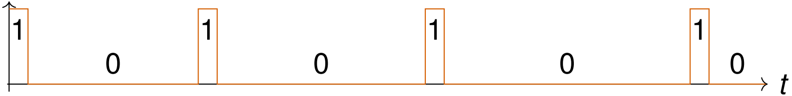
$$f(\alpha, \mu, \lambda, t) = \frac{1}{2} + \frac{1}{2}\tanh(\mu \sin(2\alpha\pi 4^{\text{round}(t-1/4,\lambda)} + 4\pi/3))$$

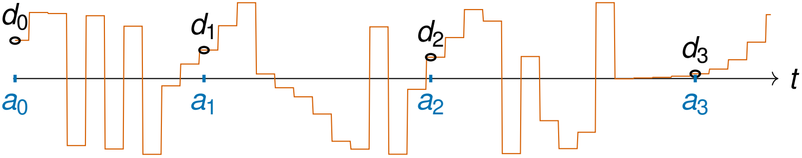
It's horrible, but generable

round is the mysterious rounding function...

A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$



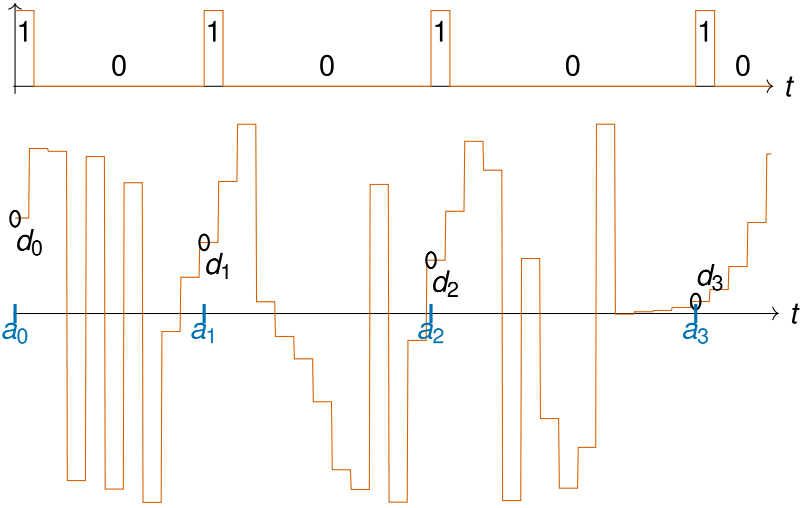


dyadic stream generator : $d_i = m_i 2^{-d_i}$, $a_i = 9i + \sum_{j<i} d_j$

$$f(\alpha, \gamma, t) = \sin(2\alpha\pi 2^{\text{round}(t-1/4,\gamma)}))$$

round is the mysterious rounding function...

A less simplified proof



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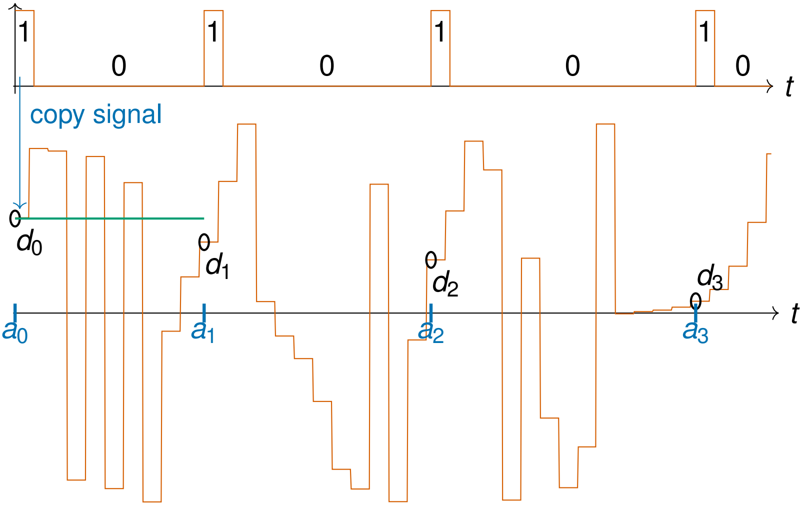

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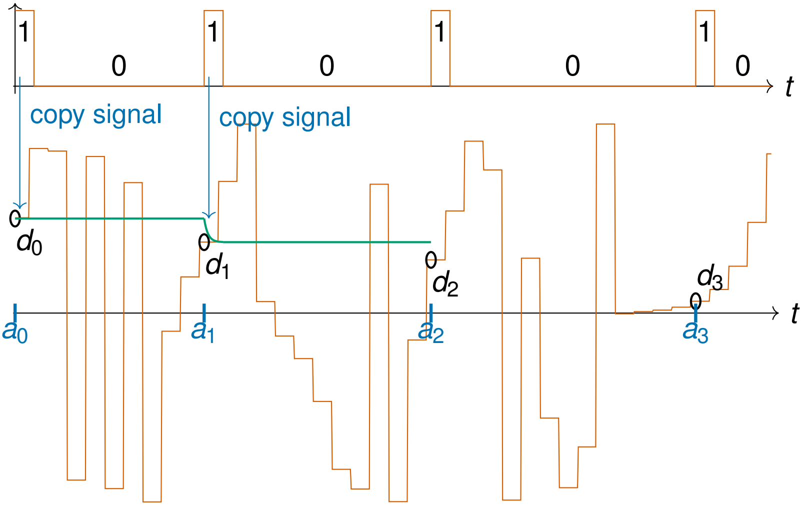

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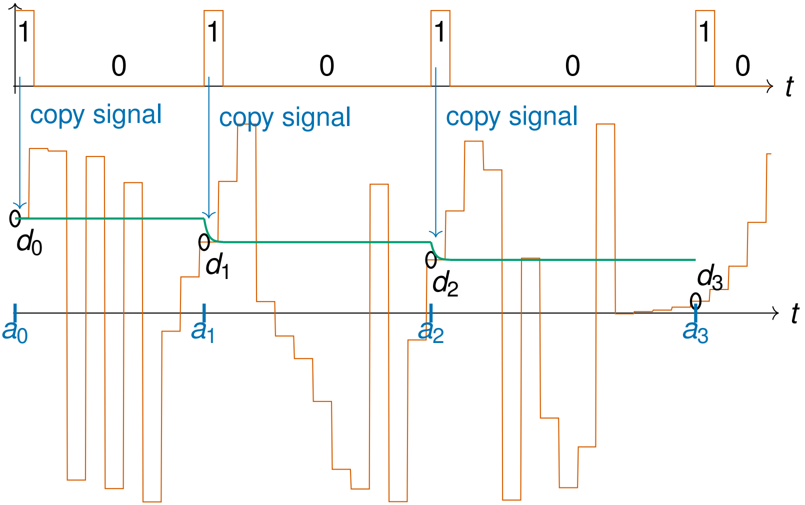

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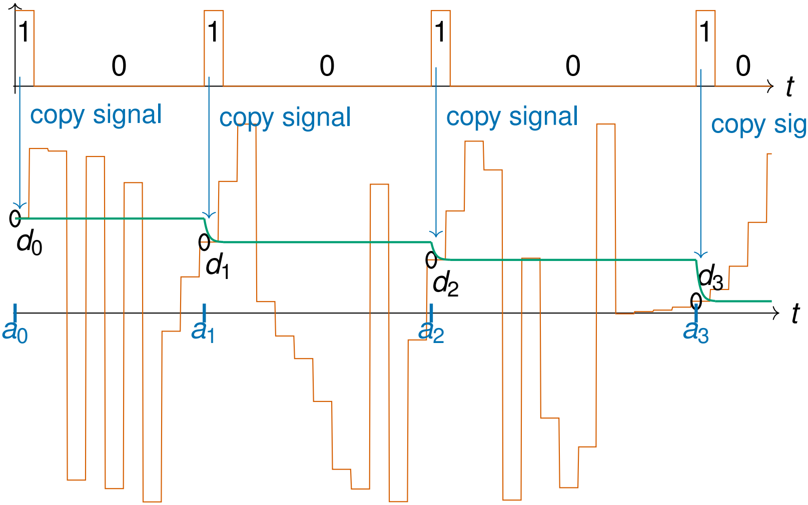

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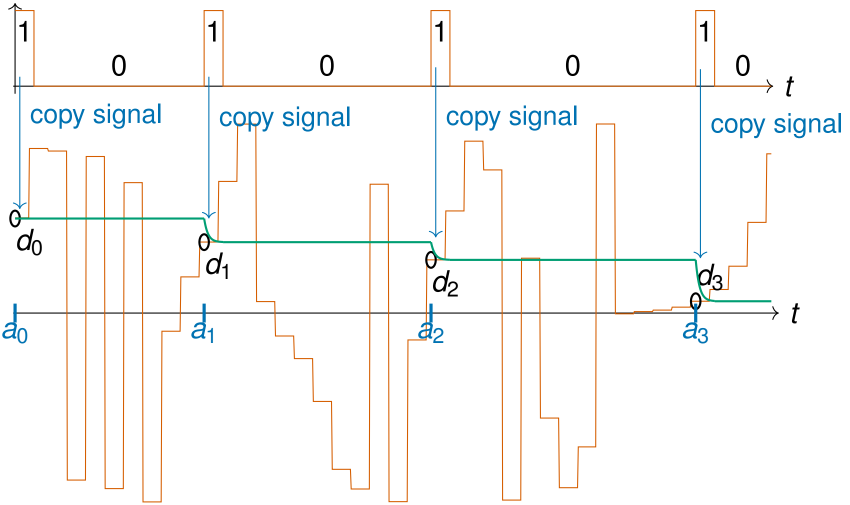
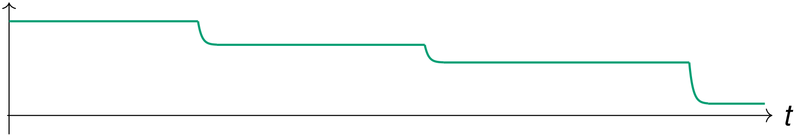


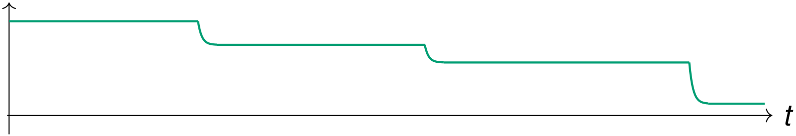
This copy operation is the “non-trivial” part.

A less simplified proof



We can do **almost piecewise constant functions...**

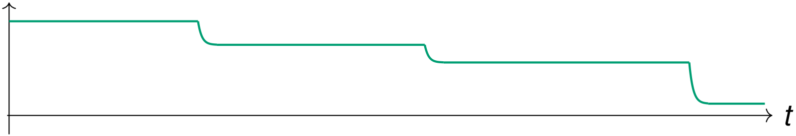
A less simplified proof



We can do **almost piecewise constant functions...**

- ...that are bounded by 1...
- ...and have super slow changing frequency.

A less simplified proof



We can do **almost piecewise constant functions...**

- ...that are bounded by 1...
- ...and have super slow changing frequency.

How do we go to arbitrarily large and growing functions ? Can a polynomial ODE even have arbitrary growth ?

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$y_1' = y_1 \qquad \leadsto \qquad y_1(t) = \exp(t)$$

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$y_1' = y_1 \quad \rightsquigarrow \quad y_1(t) = \exp(t)$$
$$y_2' = y_1 y_2 \quad \rightsquigarrow \quad y_1(t) = \exp(\exp(t))$$

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$\begin{array}{lcl} y_1' = y_1 & \rightsquigarrow & y_1(t) = \exp(t) \\ y_2' = y_1 y_2 & \rightsquigarrow & y_1(t) = \exp(\exp(t)) \\ \dots & & \dots \\ y_n' = y_1 \cdots y_n & \rightsquigarrow & y_n(t) = \exp(\cdots \exp(t) \cdots) := e_n(t) \end{array}$$

An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$:

$$
\begin{array}{lcl}
y_1' = y_1 & \rightsquigarrow & y_1(t) = \exp(t) \\
y_2' = y_1 y_2 & \rightsquigarrow & y_1(t) = \exp(\exp(t)) \\
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y_n' = y_1 \cdots y_n & \rightsquigarrow & y_n(t) = \exp(\cdots \exp(t) \cdots) := e_n(t)
\end{array}
$$

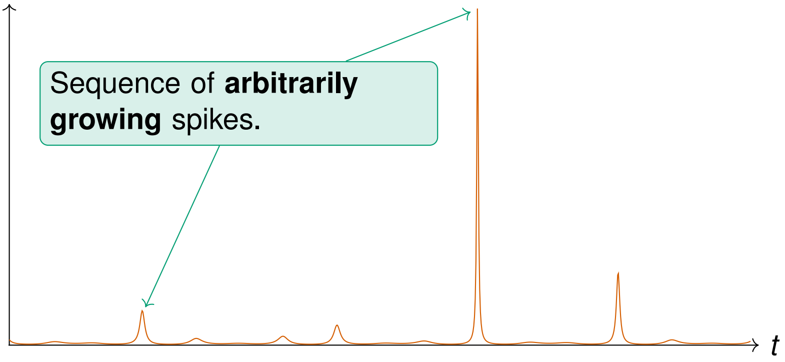
Conjecture (Emil Borel, 1899)

With n variables, cannot do better than $\mathcal{O}_t(e_n(At^k))$.

An old question on growth

Counter-example (Vijayaraghavan, 1932)

$$\frac{1}{2 - \cos(t) - \cos(\alpha t)}$$



An old question on growth

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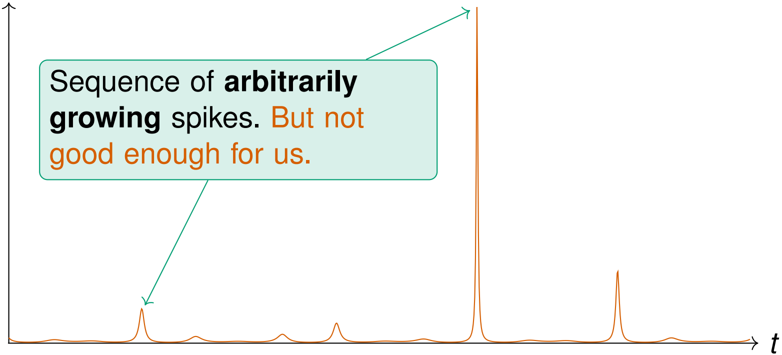

An old question on growth

Theorem

There exists a polynomial $p : \mathbb{R}^d \to \mathbb{R}^d$ such that for any continuous function $f : \mathbb{R}_+ \to \mathbb{R}$, we can find $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

satisfies

$$y_1(t) \geqslant f(t), \qquad \forall t \geqslant 0.$$

An old question on growth

Theorem

There exists a polynomial $p : \mathbb{R}^d \to \mathbb{R}^d$ *such that for any continuous function* $f : \mathbb{R}_+ \to \mathbb{R}$*, we can find* $\alpha \in \mathbb{R}^d$ *such that*

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satisfies

$$y_1(t) \geqslant f(t), \qquad \forall t \geqslant 0.$$

Note : both results require α to be **transcendental.** Conjecture still open for **rational** (or algebraic) coefficients.

Proof gem : iteration with differential equations

Assume f is generable, can we **iterate** f with an ODE ?
That is, build a generable y such that $y(x, n) \approx f^{[n]}(x)$ for all $n \in \mathbb{N}$

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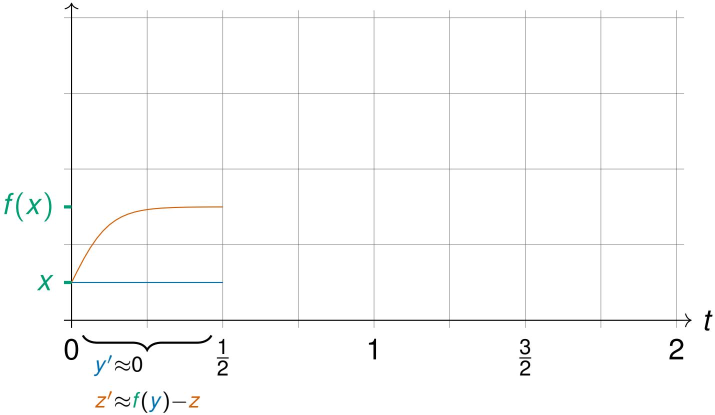

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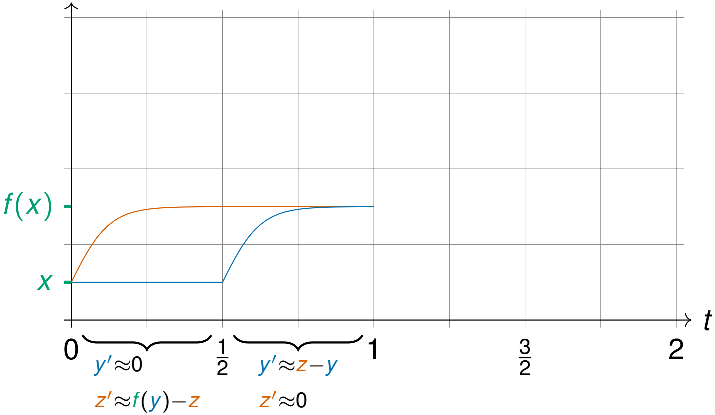

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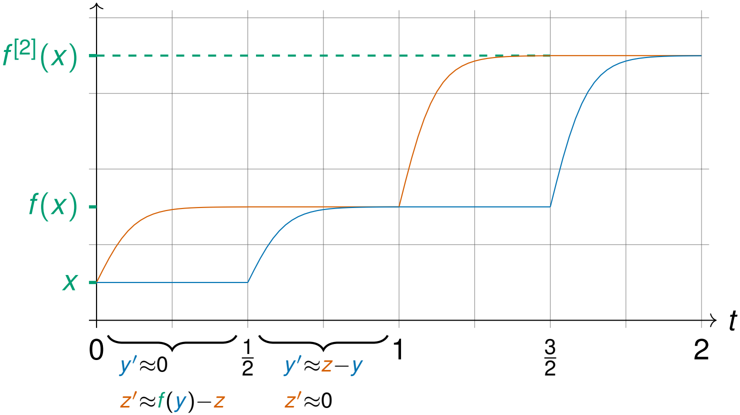
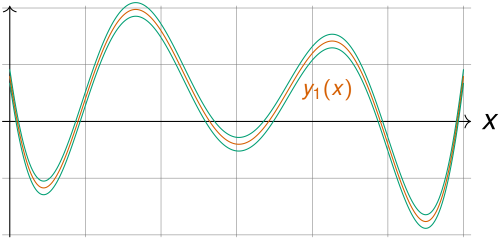


Universal initial value problem (IVP)



Notes :

- **system** of ODEs,
- y is analytic,
- we need $d \approx 300$.

Theorem

There exists a ***fixed*** *(vector of) polynomial p such that for any continuous functions f and* ε*, there exists* $\alpha \in \mathbb{R}^d$ *such that*

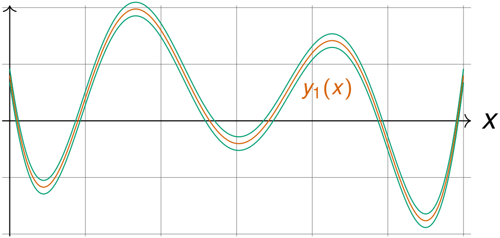
$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a ***unique solution*** $y : \mathbb{R} \to \mathbb{R}^d$ *and* $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Remark : α is usually transcendental, but computable from f and ε

Universal DAE revisited



Theorem

*There exists a **fixed** polynomial p and $k \in \mathbb{N}$ such that for any continuous functions f and ε, there exists $\alpha_0, \ldots, \alpha_k \in \mathbb{R}$ such that*

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

*has a **unique analytic solution** and this solution satisfies such that*

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

Backup slides

Generable functions (total, univariate)

Definition

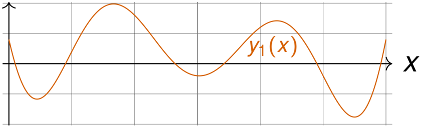
$f : \mathbb{R} \to \mathbb{R}$ is generable if there exists d, p and y_0 such that the solution y to

$$y(0) = y_0, \qquad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

Types

- $d \in \mathbb{N}$: dimension
- $p \in \mathbb{R}^d[\mathbb{R}^n]$: polynomial vector
- $y_0 \in \mathbb{R}^d, y : \mathbb{R} \to \mathbb{R}^d$



Note : existence and unicity of y by Cauchy-Lipschitz theorem.

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Example : $f(x) = x$ ▶ identity

$$y(0) = 0, \qquad y' = 1 \quad \rightsquigarrow \quad y(x) = x$$

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Example : $f(x)=x^2$ ▶ squaring

$$\begin{array}{lll} y_1(0)=0, & y_1'=2y_2 & \rightsquigarrow & y_1(x)=x^2 \\ y_2(0)=0, & y_2'=1 & \rightsquigarrow & y_2(x)=x \end{array}$$

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Example : $f(x)=x^n$ ► n^{th} power

$$\begin{array}{lll}
y_1(0)=0, & y_1'=ny_2 & \rightsquigarrow\; y_1(x)=x^n\\
y_2(0)=0, & y_2'=(n-1)y_3 & \rightsquigarrow\; y_2(x)=x^{n-1}\\
\dots & \dots & \qquad\dots\\
y_n(0)=0, & y_n=1 & \rightsquigarrow\; y_n(x)=x
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Example : $f(x) = \exp(x)$ ► exponential

$$y(0) = 1, \qquad y' = y \quad \rightsquigarrow \quad y(x) = \exp(x)$$

Generable functions (total, univariate)

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Example : $f(x)=\sin(x)$ or $f(x)=\cos(x)$ ► sine/cosine

$$\begin{aligned} y_1(0)&=0, & y_1'&=y_2 &\rightsquigarrow\quad y_1(x)&=\sin(x)\\ y_2(0)&=1, & y_2'&=-y_1 &\rightsquigarrow\quad y_2(x)&=\cos(x)\end{aligned}$$

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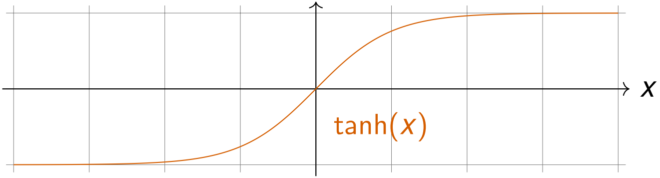
Example : $f(x) = \tanh(x)$ ▶ hyperbolic tangent

$$y(0) = 0, \qquad y' = 1 - y^2 \quad \rightsquigarrow \quad y(x) = \tanh(x)$$



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Example : $f(x) = \frac{1}{1+x^2}$ ▶ rational function

$$f'(x) = \frac{-2x}{(1+x^2)^2} = -2xf(x)^2$$

$$\begin{aligned} y_1(0) &= 1, \quad & y_1' &= -2y_2y_1^2 \quad \rightsquigarrow \quad & y_1(x) &= \frac{1}{1+x^2} \\ y_2(0) &= 0, \quad & y_2' &= 1 \quad \rightsquigarrow \quad & y_2(x) &= x \end{aligned}$$

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Example : $f=g\pm h$ ► sum/difference

$$(f\pm g)'=f'\pm g'$$

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Example : $f = gh$ ▶ product

$$(gh)' = g'h + gh'$$

Generable functions (total, univariate)

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Example : $f = \frac{1}{g}$ ▶ inverse

$$f' = \frac{-g'}{g^2} = -g' f^2$$

Generable functions (total, univariate)

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Example : $f=\int g$ ► integral

$$f'=g$$

Generable functions (total, univariate)

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Example : $f = g'$ ► derivative

$$f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'$$

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Example : $f = g \circ h$ ▶ composition

$$(z \circ h)' = (z' \circ h)h' = p(z \circ h)h'$$

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Example : $f' = \tanh \circ f$ ► Non-polynomial differential equation

$$f'' = (\tanh' \circ f)f' = (1 - (\tanh \circ f)^2)f'$$

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Example : $f(0) = f_0, f' = g \circ f$ ► Initial Value Problem (IVP)

$$f' = g'' = (p(z))' = \nabla p(z) \cdot z'$$

Generable functions : a first summary

Nice theory for the class of total and univariate generable functions :

- analytic
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Limitations :

- total functions
- univariate

Generable functions (generalization)

Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is generable if X is open **connected** and $\exists d, p, x_0, y_0, y$ such that

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and $f(x) = y_1(x)$ for all $x \in X$.

$J_y(x) =$ Jacobian matrix of y at x

Types

- $n \in \mathbb{N}$: input dimension
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- $y_0 \in \mathbb{K}^d$, $y : X \to \mathbb{R}^d$

Notes :

- Partial differential equation !
- Unicity of solution y...
- ... **but not existence** (ie you have to show it exists)

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Example : $f(x_1, x_2) = x_1 x_2^2 \quad (n = 2, d = 3)$ ▶ monomial

$$y(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} y_3^2 & 3y_2y_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightsquigarrow \quad y(x) = \begin{pmatrix} x_1 x_2^2 \\ x_1 \\ x_2 \end{pmatrix}$$

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$$\begin{array}{llll} y_1(0,0) = 0, & \partial_{x_1} y_1 = y_3^2, & \partial_{x_2} y_1 = 3 y_2 y_3 & \rightsquigarrow \quad y_1(x) = x_1 x_2^2 \\ y_2(0,0) = 0, & \partial_{x_1} y_2 = 1, & \partial_{x_2} y_2 = 0 & \rightsquigarrow \quad y_2(x) = x_1 \\ y_3(0,0) = 0, & \partial_{x_1} y_3 = 0, & \partial_{x_2} y_3 = 1 & \rightsquigarrow \quad y_3(x) = x_2 \end{array}$$

This is tedious !

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Last example : $f(x) = \frac{1}{x}$ for $x \in (0, \infty)$ ▶ inverse function

$$y(1) = 1, \qquad \partial_x y = -y^2 \quad \rightsquigarrow \quad y(x) = \frac{1}{x}$$

Generable functions : summary

Nice theory for the class of multivariate generable functions (over connected domains) :

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Riemann Γ and ζ are not generable.

From discrete to real computability

Computable Analysis : lift Turing computability to real numbers

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Definition

$x \in \mathbb{R}$ is computable iff $\exists$ a computable $f : \mathbb{N} \to \mathbb{Q}$ such that :

$$|x - f(n)| \leqslant 10^{-n} \qquad n \in \mathbb{N}$$

Examples : rational numbers, π, e, ...

n	**f**(**n**)	$\lvert\pi - \mathbf{f}(\mathbf{n})\rvert$
0	3	$0.14 \leqslant 10^{-0}$
1	3.1	$0.04 \leqslant 10^{-1}$
2	3.14	$0.001 \leqslant 10^{-2}$
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Beware :there exists uncomputable real numbers !

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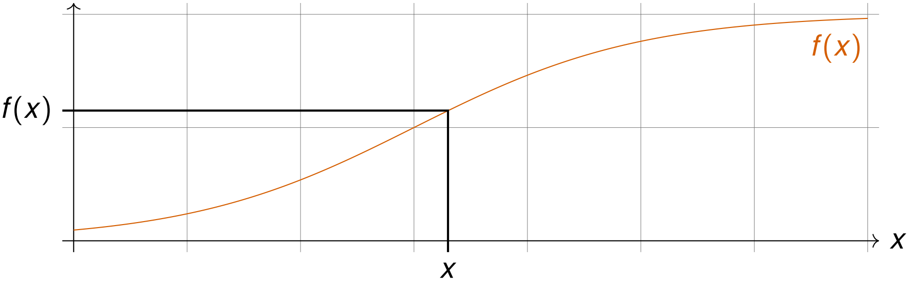

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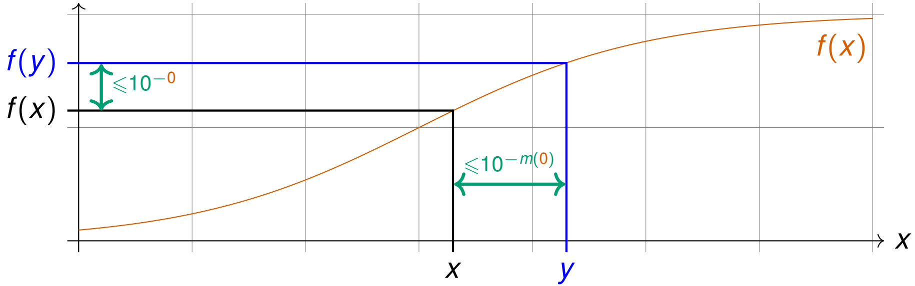

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m : modulus of continuity

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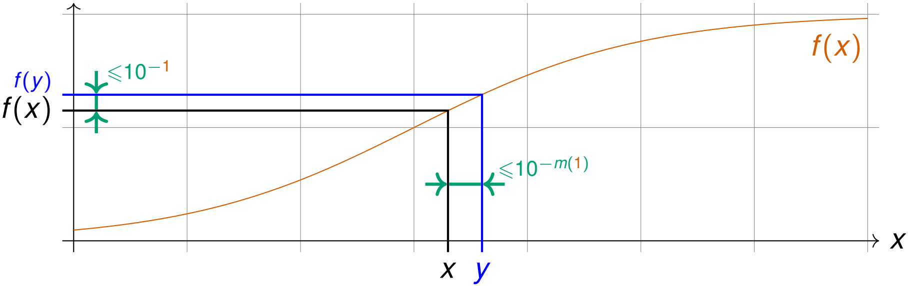

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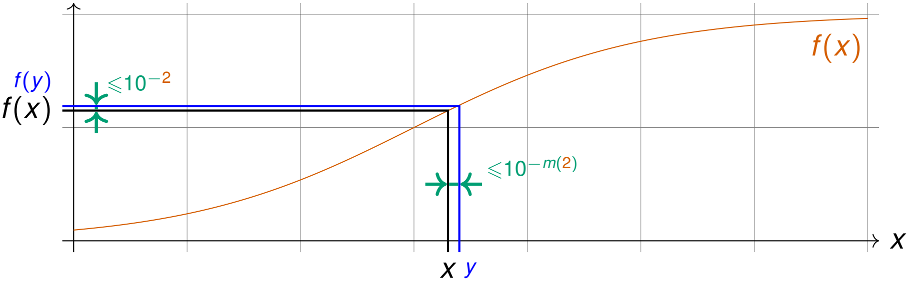

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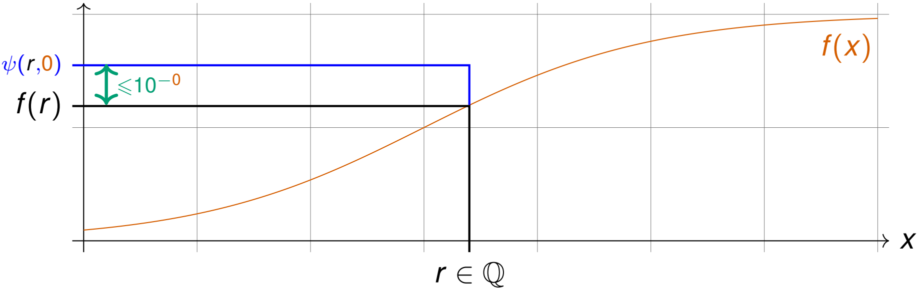

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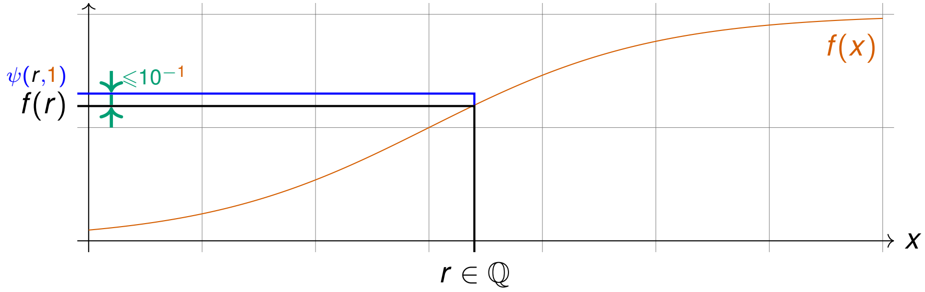

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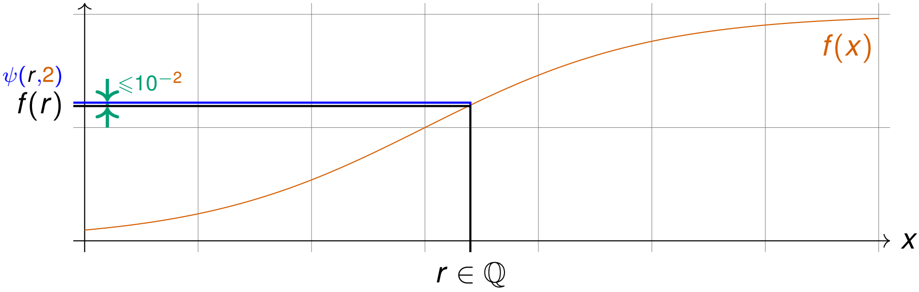

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Polytime complexity

Add “polynomial time computable” everywhere.

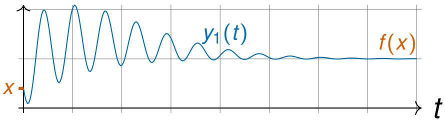
Equivalence with computable analysis

Definition (Bournez et al, 2007)

f **computable by GPAC** if $\exists p$ polynomial such that $\forall x \in [a, b]$

$$y(0) = (x, 0, \ldots, 0) \qquad y'(t) = p(y(t))$$

satisfies $|f(x) - y_1(t)| \leqslant y_2(t)$ et $y_2(t) \xrightarrow[t\to\infty]{} 0$.



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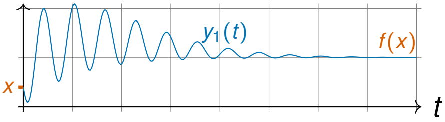

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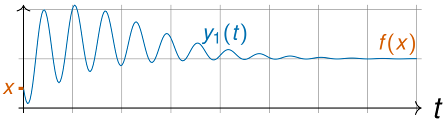

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1. In Computable Analysis, a standard model over reals built from Turing machines.

Almost-rounding function

"Perfect round" :

Back to presentation

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$$\mathrm{nz}(x) = x + \text{some variation on } \tanh$$

Almost-rounding function : gory details

Formally :

Back to presentation

$$\mathrm{rnd}(x,\mu,\lambda) = x - \frac{1}{\pi}\arctan(\mathrm{cltan}(\pi x,\mu,\lambda))$$

$$\mathrm{cltan}(\theta,\mu,\lambda) = \frac{\sin(\theta)}{\sqrt{\mathrm{nz}(\cos^2\theta,\mu+16\lambda^3,4\lambda^2)}}\,\mathrm{sg}(\cos\theta,\mu+3\lambda,2\lambda)$$

$$\mathrm{nz}(x,\mu,\lambda) = x + \frac{2}{\lambda}\,\mathrm{ip}_1\left(1-x+\frac{3}{4\lambda},\mu+1,4\lambda\right)$$

$$\mathrm{ip}_1(x,\mu,\lambda) = \frac{1+\mathrm{sg}(x-1,\mu,\lambda)}{2}$$

$$\mathrm{sg}(x,\mu,\lambda) = \tanh(x\mu\lambda)$$

All generable functions !