Difference Galois groups of linear differential equations

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(joint with Michael Wibmer)

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hence $G = \mathrm{GL}_1$.

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$\Rightarrow$ $y, \sigma(y)$ and $\sigma^2(y)$ are algebraically dependent over F

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and

$$G(S) = \{c \in S^\times \mid \sigma^2(c)\sigma(c)^{-2}c = 1\}$$

holds for any σ-algebra S over $\mathbb{C}$, so $G(S) \subsetneq \mathrm{GL}_1(S)$ for "sufficient general" S and thus $G \lneq \mathrm{GL}_1$.

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$\rightsquigarrow$ σ-Galois group $G \leq \mathrm{GL}_n$

Fact: G is given by polynomial equations over $\mathbb{C}$ in the matrix entries and their images under σ, $\sigma^2, \ldots$

i.e., G is a (linear) difference algebraic group over $\mathbb{C}$.

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- in particular, if $\mathcal{G}$ is the finite cyclic group of order d over $\mathbb{C}$ we can associate two difference algebraic groups G_1, G_2 to it: $G_1(S) = \mathcal{G}(S) = \{g \in S \mid s^d = 1\}$ and $G_2(S) = \{g \in S \mid s^d = 1 \text{ and } \sigma(s) = s\}$ for all $\mathbb{C}$-σ-algebras S

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- Subgroups of the multiplicative group $\mathrm{GL}_1 = \mathbb{G}_m$:
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- Unitary group: $G(S) = \{g \in \mathrm{GL}_n(S) \mid \sigma(g)^{\mathrm{tr}} g = 1\}$ for all $\mathbb{C}$-σ-algebras S

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- no constant points of finite cyclic groups!

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has the following property:

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$$\delta(y) = \frac{1}{2x} y \tag{3}$$

Solution $y = \sqrt{x}$ with $\sigma(y) = \sqrt{x+1},\ \sigma^2(y) = \sqrt{x+2}, \ldots$

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$$\begin{aligned} G(\mathbb{C}) &= \{\gamma\colon F(y,\sigma(y),\dots) \to F(y,\sigma(y),\dots) F\text{-autom.} \mid \gamma \text{ commutes with } \delta, \sigma\} \\ &= \{\gamma\colon F(y,\sigma(y),\dots) \to F(y,\sigma(y),\dots), y \mapsto c\cdot y \mid c \in \mathbb{C}^\times, c^2 = 1\} \end{aligned}$$

The inverse problem

Inverse Problem: Which difference algebraic groups over $\mathbb{C}$ are difference Galois groups of some differential equation over $\mathbb{C}$?

Examples:

- the subgroup of $\mathbb{G}_m$ defined by the equation $\sigma^2(x)\sigma(x)^{-2}x = 1$.
- constant subgroup of $\mathbb{G}_m$
- $\mathbb{G}_m$, $\mathbb{G}_a$ and finite cyclic groups interpreted as difference algebraic groups
- but <u>no</u> proper, non-trivial subgroup of $\mathbb{G}_a$!
- <u>no</u> constant points of finite cyclic groups!

The additive group

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The σ-Picard-Vessiot extension $E = F(y, \sigma(y), \sigma^2(y), \ldots)$ has σ-Galois group $\mathbb{G}_a$.

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Corollary: Let $\mathcal{G}$ be a unipotent linear algebraic group over $\mathbb{C}$ and let G be its constant subgroup. Then G is not a σ-Galois group over $\mathbb{C}(x)$.

Claim: No non-trivial proper subgroup of $\mathbb{G}_a$ is a σ-Galois group over $\mathbb{C}(x)$

Sketch of proof: Let $E/\mathbb{C}(x)$ be a σ-Picard-Vessiot extension with group $G \lneq \mathbb{G}_a$.

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$$\sum_{l=0}^{n} \sum_{j=1}^{r} \frac{c_l \alpha_j}{x + l + \beta_j}$$

has an antiderivative in $\mathbb{C}(x)$ and is thus zero, but the terms do not cancel unless $a = 0$.

A necessary criterion

Theorem

If G is a σ-Galois group over $\mathbb{C}(x)$ with derivation $\delta = \frac{d}{dx}$ and endomorphism σ given by $\sigma(f(x)) = f(x+1)$, then G is σ-reduced and σ-connected.

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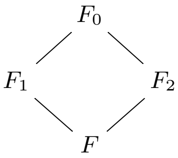
The criterion in the theorem above is far from sufficient!

Theorem (B., Wibmer)

Let $\mathcal{G}$ be a linear algebraic group over $\mathbb{C}$ and interpret it as a difference-algebraic group G over $\mathbb{C}$. Then there exists a σ-Picard-Vessiot extension over $\mathbb{C}(x)$ with σ-Galois group G.

A diamond of fields



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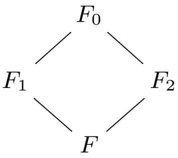

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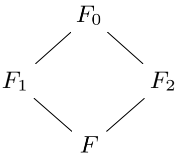

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Algebraic Patching

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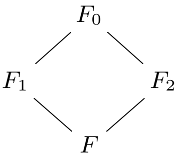

is called a diamond of fields with the factorization property if the following holds:

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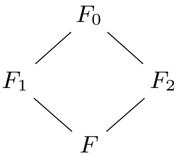

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Fact: Let U_1, U_2 be open, connected proper subsets of the Riemann sphere $\mathcal{X} = \mathbb{P}^1_{\mathbb{C}}$ such that

- $U_0 := U_1 \cap U_2$ is connected and
- $U_1 \cup U_2 = \mathcal{X}$.

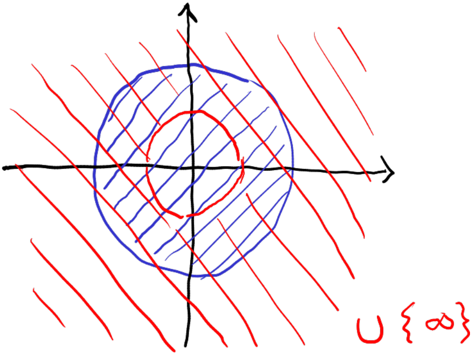
Let F_i be the field of meromorphic functions on U_i. Then (F, F_1, F_2, F_0) is a diamond with the factorization property, where $F = \mathbb{C}(x)$ is the field of meromorphic functions on $\mathcal{X}$.

Example:

$$U_1 = \{x \in \mathbb{P}^1_{\mathbb{C}} \mid |x| < 2\}$$

$$U_2 = \{x \in \mathbb{P}^1_{\mathbb{C}} \mid |x| > 1\}$$

$$U_0 = \{x \in \mathbb{P}^1_{\mathbb{C}} \mid 1 < |x| < 2\}$$



Patching σ-Picard Vessiot extensions

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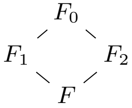

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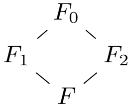

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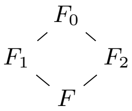

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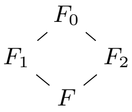

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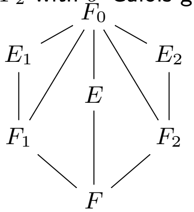

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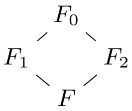

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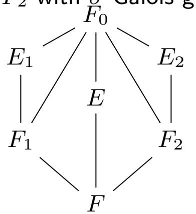

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First step: Choose $n \in \mathbb{N}$ with $G \leq \mathrm{GL}_n$ and show that there exist fundamental solution matrices $Y_1 \in \mathrm{GL}_n(E_1), Y_2 \in \mathrm{GL}_n(E_2)$ (i.e., adjust representations).

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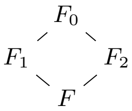

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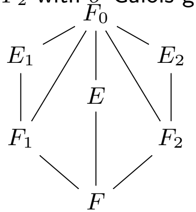

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Second step: Consider $Y_0 = Y_1 Y_2^{-1} \in \mathrm{GL}_n(F_0)$. The factorization property yields matrices $B_1 \in \mathrm{GL}_n(F_1), B_2 \in \mathrm{GL}_n(F_2)$ with $Y_0 = B_1 B_2^{-1}$.

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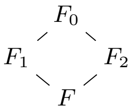

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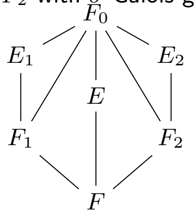

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First step: Choose $n \in \mathbb{N}$ with $G \leq \mathrm{GL}_n$ and show that there exist fundamental solution matrices $Y_1 \in \mathrm{GL}_n(E_1), Y_2 \in \mathrm{GL}_n(E_2)$ (i.e., adjust representations).

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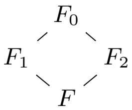
Patching σ-Picard Vessiot extensions



Let $G = \langle H_1, H_2 \rangle$ be a σ-algebraic group with generating (closed) subgroups H_1 and H_2. Consider a diamond with the factorization property (F, F_1, F_2, F_0) such that

- all fields are equipped compatibly with commuting derivations δ and endomorphisms σ such that $F_0^\delta = F^\delta$
- there exist σ-Picard-Vessiot extensions E_1/F_1 and E_2/F_2 with σ-Galois groups isomorphic to H_1 and H_2, resp.and
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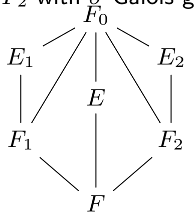

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Third step: Show that the σ-Picard-Vessiot extension $E = F(Y, \sigma(Y), \sigma^2(Y), \dots)$ over F has σ-Galois group G.

Generating subgroups

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Now interpret $\mathcal{G}$ as a difference algebraic group G and similarly $\mathcal{H}_1, \ldots, \mathcal{H}_r$ as difference algebraic groups $H_1, \ldots, H_r$.

Corollary: G is generated as a difference algebraic group by $H_1, \ldots, H_r$.

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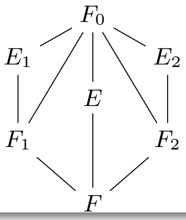

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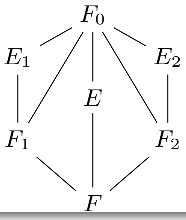

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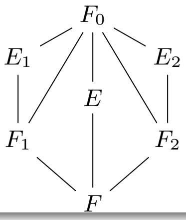

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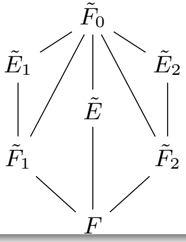

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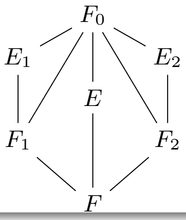

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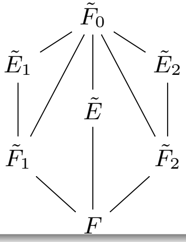

Continue inductively.

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$$U_1 = \{x \in \mathbb{C} \mid \mathrm{Im}(x) > 0\}$$
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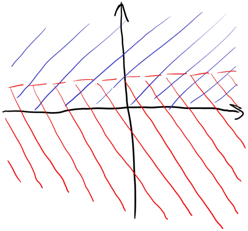

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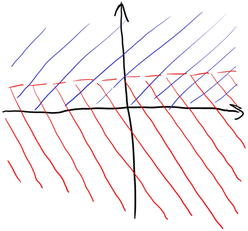

But these sets don't cover $\mathbb{P}^1_{\mathbb{C}}$ and we won't be able to find a connected, σ-invariant $\tilde{U}_2$ in the second step with $U_0 \cup \tilde{U}_2 = \mathbb{P}^1_{\mathbb{C}}$.

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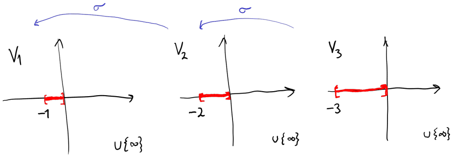

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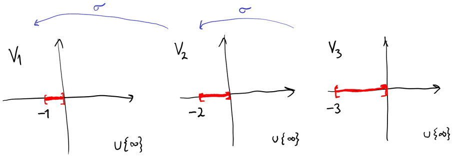

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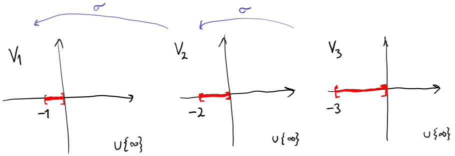

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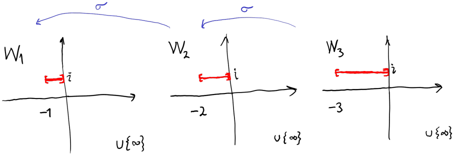

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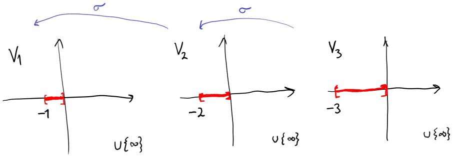

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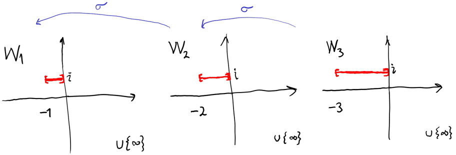

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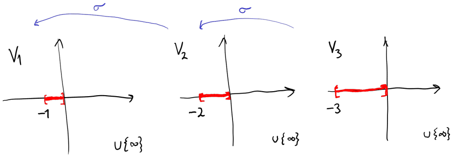

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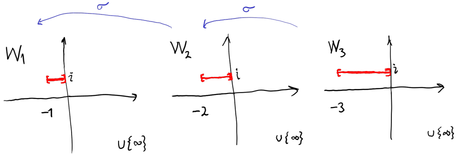

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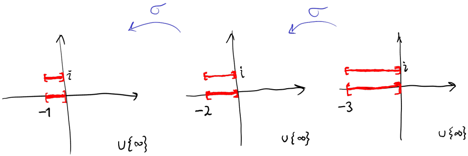

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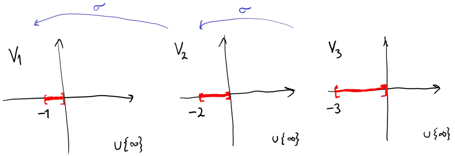

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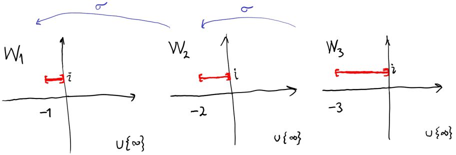

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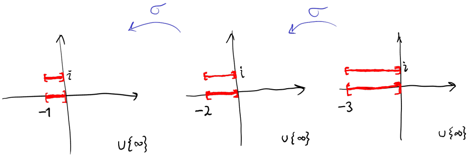

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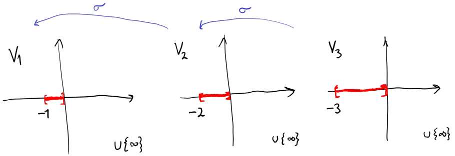

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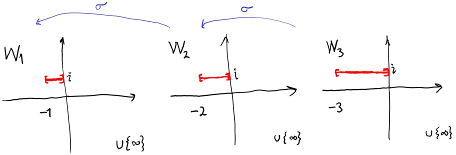

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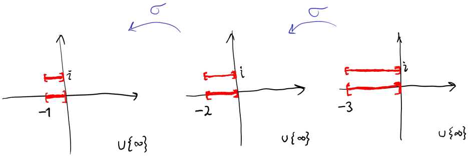

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$\Rightarrow (F, F_1, F_2, F_0)$ is a diamond with the factorization property

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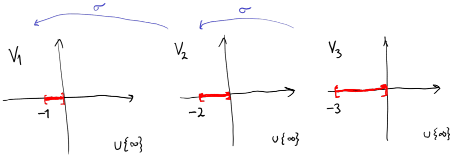

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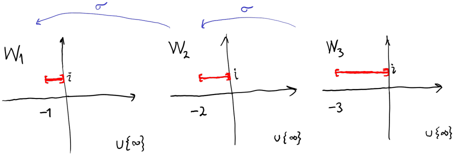
$K_1 \xrightarrow{\sigma} K_2 \xrightarrow{\sigma} K_3 \to \dots$



Define $F_0 = \bigcup_{i\in\mathbb{N}} M_i$

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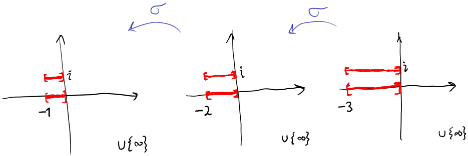

For all $i \in \mathbb{N}$, (F, L_i, K_i, M_i) is a diamond with the factorization property $\Rightarrow$ (F, F_1, F_2, F_0) is a diamond with the factorization property <u>and</u> these fields are equipped compatibly with extensions of σ on $F = \mathbb{C}(x)$.

Suitable diamonds over $F = \mathbb{C}(x)$ in the second induction step

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Define $\tilde{F}_1 = \bigcup_{i \in \mathbb{N}} M_i$

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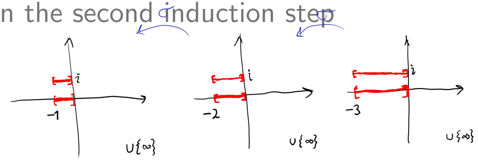

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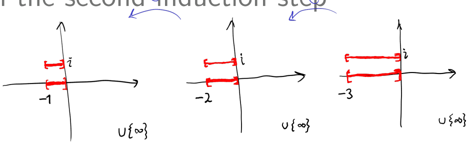

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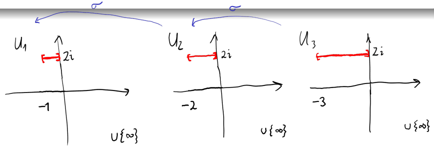

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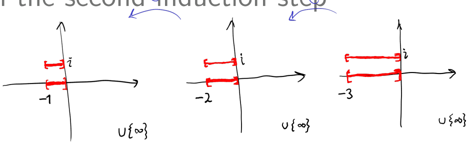

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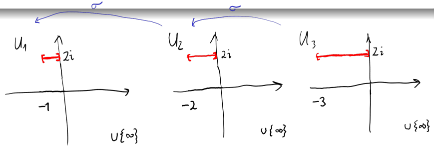

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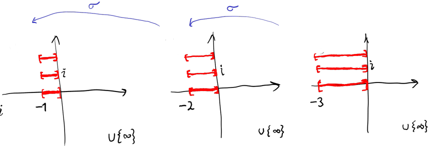

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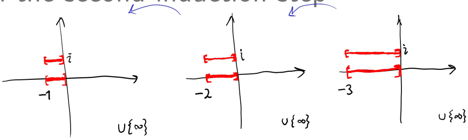

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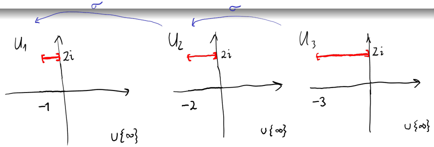

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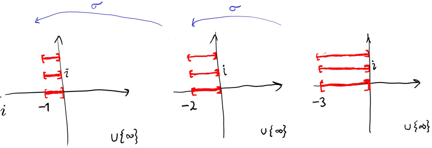

$\Rightarrow (F, \tilde{F}_1, \tilde{F}_2, \tilde{F}_0)$ is a diamond with the factorization property and these fields are equipped compatibly with extensions of σ on $F = \mathbb{C}(x)$.

Building blocks

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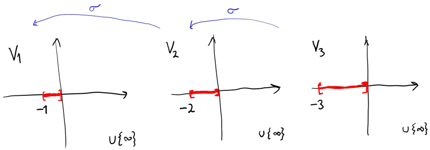

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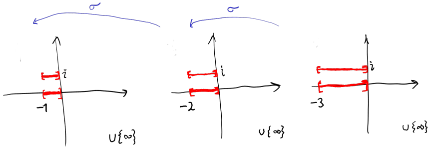

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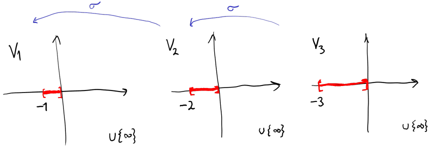

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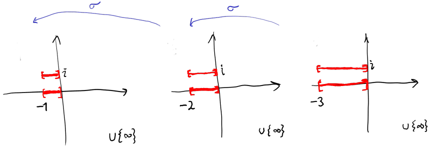

Building blocks: We need to construct a σ-Picard-Vessiot extension E_1/F_1 with σ-Galois group H_1 and $E_1 \subseteq F_0$.

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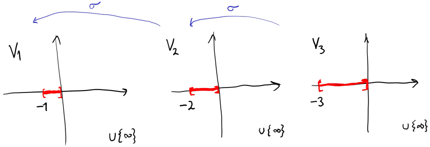

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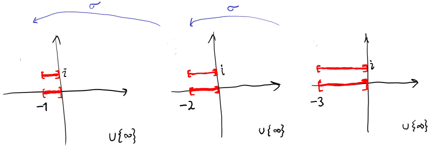

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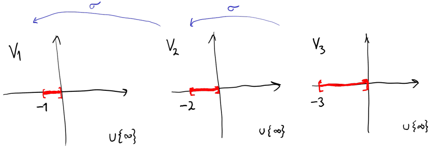

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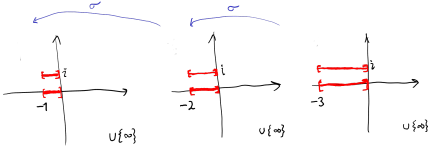

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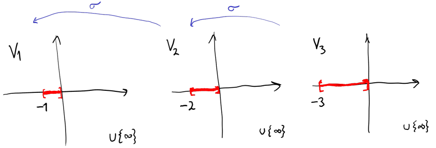

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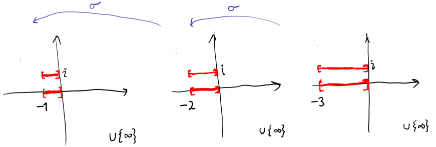

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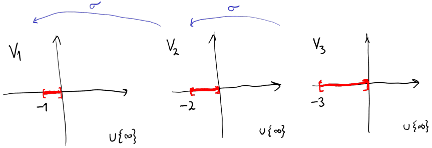

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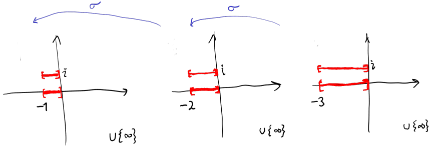

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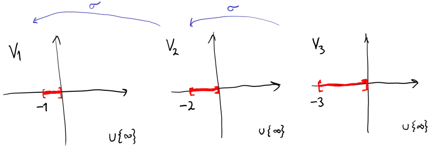

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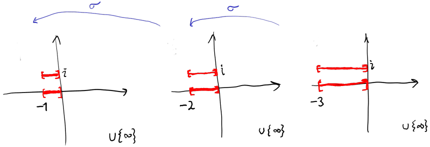

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Third case: H_1 finite cyclic of order d. Define $E_1 = F_1\langle \sqrt[d]{\frac{1}{x-i}+1}\rangle \subseteq F_0$.

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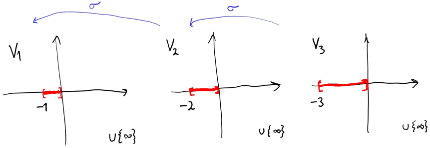

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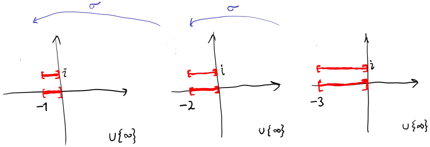

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