A DIFFERENTIAL APPROACH TO THE AX-SCHANUEL, I

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Abstract. In this paper, we prove several Ax-Schanuel type results for uniformizers of geometric structures. In particular, we give a proof of the full Ax-Schanuel Theorem with derivatives for uniformizers of any Fuchsian group of the first kind and any genus. Our techniques combine tools from differential geometry, differential algebra and the model theory of differentially closed fields. The proof is very similar in spirit to Ax's proof of the theorem in the case of the exponential function.

1. Introduction

In this paper we use techniques from differential geometry, differential algebra and the model theory of differentially closed fields to prove several results of Ax-Schanuel type. Our main result is stated in the context of a G-principal bundle $\pi : P \to Y$ with Y a complex algebraic variety and G an algebraic group.

Theorem A. *Let ∇ be a G-principal connection on $P \to Y$ with Galois group $Gal(\nabla) = G$. Let V be an algebraic subvariety of P and $\mathcal{L}$ an horizontal leaf. If $\dim V < \dim(V \cap \mathcal{L}) + \dim G$ and $\dim(V \cap \mathcal{L}) > 0$ then the projection of $V \cap \mathcal{L}$ in Y is contained in a ∇-special subvariety.*

Here, by a ∇-special subvariety $X \subset Y$, we mean a subvariety such that $\mathrm{Gal}(\nabla|_X)$ is a strict subgroup of $G = Gal(\nabla)$. We use Theorem A to prove several functional transcendence results for uniformizers of a $(G, G/B)$-structure on an algebraic variety Y. The classical and most studied examples of such structures come from Shimura varieties. Take G to be a connected semi-simple algebraic $\mathbb{Q}$-group. Then for K, a maximal compact subgroup of $G(\mathbb{R})$, we have that $\Omega = G(\mathbb{R})/K$ is a bounded symmetric domain. It is known that the compact dual $\check{\Omega}$ of Ω is given as the quotient $\check{\Omega} = G(\mathbb{C})/B$ for a Borel subgroup B of G. This quotient $\check{\Omega}$ is a homogeneous projective variety and Ω is a semi-algebraic subset (if we assume $K \subset B$). Given an arithmetic lattice $\Gamma \subset G(\mathbb{Q})$, the analytic quotient $Y := \Gamma\backslash\Omega = \Gamma\backslash G(\mathbb{R})/K$ has the structure of an algebraic variety and is called a (pure) Shimura variety. As detailed in Subsection 3.2, a $(G, G/B)$-structure on Y can taken to be the system of partial differential equations satisfied by a uniformization function $q : \Omega \to Y := \Gamma\backslash\Omega$.

We will work, in this paper, with more general $(G, G/B)$-structures. For example, our theory allows for quotients of Ω (as above) by an arbitrary lattice $\Gamma \subset G(\mathbb{R})$. Furthermore, it

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also applies to the differential equations satisfied by conformal mappings of circular polygons [16, Chapter 4]. In any case, we will show that attached to any $(G, G/B)$-structure is a G-principal connection ∇ and so are able to apply Theorem A and obtain the following result.

Corollary B. *Let υ be a uniformization of an irreducible $(G, G/B)$-structure on an algebraic variety Y. Assume $W \subset G/B \times X$ is an irreducible algebraic subvariety intersecting the graph of υ. Let U be an irreducible component of this intersection such that*

$$\dim W < \dim U + \dim X, \quad \dim U > 0.$$

Then the projection of U to X is contained in a ∇-special subvariety of X.

As we shall later see in the paper, many more results naturally follow from Theorem A. Moreover, if we assume in addition that the $(G, G/B)$-structure is *simple* (see Definition 5.3), a natural assumption, then we are also able to obtain an Ax-Schanuel Theorem (including with derivatives) for products of Y. This gives a result which has a slightly weaker conclusion than the general case of the main theorems of [27] and [10] but which applies to more general situations.

Theorem C. *Let $(Y, \mathscr{Y})$ be a simple $(G, G/B)$-structure on Y, $\hat{t}_1 \dots \hat{t}_n$ be n formal parameterizations of (formal) neighborhoods of points $p_1, \dots p_n$ in G/B and $\upsilon_1, \dots, \upsilon_n$ be solutions of $\mathscr{Y}$ defined in a neighborhood of $p_1, \dots p_n$ respectively. If*

$$\text{tr.deg.}_{\mathbb{C}}\mathbb{C}\left(\hat{t}_i, (\partial^\alpha \upsilon_i)(\hat{t}_i) : 1 \leq i \leq n,\ \alpha \in \mathbb{N}^{\dim Y}\right) < \dim Y + n \dim G$$

then there exist $i < j$ such that

$$\text{tr.deg.}_{\mathbb{C}}\mathbb{C}(\upsilon_i(\hat{t}_i), \upsilon_j(\hat{t}_j)) = \text{tr.deg.}_{\mathbb{C}}\mathbb{C}(\upsilon_i(\hat{t}_i)) = \text{tr.deg.}_{\mathbb{C}}\mathbb{C}(\upsilon_j(\hat{t}_j)) = \dim G.$$

Theorem C does not (in full generality) give any details about the kinds of special subvarieties (or correspondences) that can occur - a problem we will tackle in general in sequel to this paper. Nevertheless, it will be crucial in giving a model theoretic analysis of the relevant partial differential equations. For example, as a consequence we are able to show that the sets $\mathscr{Y}$ defined by the equations (along with some natural inequalities), in a differentially closed field, are strongly minimal and geometrically trivial. Building on this model theoretic analysis, we are able to give a complete analysis in the case of hyperbolic curves. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind and let j_Γ be a uniformizing function for Γ. Notice here that $G = \mathrm{PSL}_2$, $\Omega = \mathbb{H}$ and $\check{\Omega} = \mathbb{CP}_1$. Let $\hat{t}_1, \dots, \hat{t}_n$ be formal parameterizations of neighborhoods of points $p_1, \dots p_n$ in $\mathbb{H}$. We write δ_i for the derivations induced by differentiation with respect to $\hat{t}_i$. We prove the Ax-Schanuel Theorem with derivatives for j_Γ:

Theorem D. *Assume that $\hat{t}_1, \dots, \hat{t}_n$ are geodesically independent, namely $\hat{t}_i$ is nonconstant for $i = 1, \dots, n$ and there are no relations of the form $\hat{t}_i = \gamma \hat{t}_j$ for $i \neq j$, $i, j \in \{1, \dots, n\}$ and γ is an element of $Comm(\Gamma)$, the commensurator of Γ. Then*

$$\textit{tr.deg.}_{\mathbb{C}}\mathbb{C}(\hat{t}_1, j_\Gamma(\hat{t}_1), j'_\Gamma(\hat{t}_1), j''_\Gamma(\hat{t}_1), \dots, \hat{t}_n, j_\Gamma(\hat{t}_n), j'_\Gamma(\hat{t}_n), j''_\Gamma(\hat{t}_n)) \geq 3n + \textit{rank}(\delta_i \hat{t}_j).$$

Notice that the statement given above, in the spirit of Ax's original paper, is slightly stronger than the one found for example in [31] whereby $t_1, \dots, t_n$ are assumed to be the coordinate functions of some complex *analytic* subvariety of some open subset $D \subset \mathbb{H}^n$. Besides being a stronger result than the Ax-Lindemann-Weierstrass theorem from [14], it also generalizes the setup by dropping the assumption that the quotient is genus zero.

1.1. **Applications of Ax-Schanuel theorems.** Over the last decade, functional transcendence results, often in the form of the Ax-Lindemann-Weierstrass type results for certain analytic functions have played a key role in the class of diophantine problems known as *special points* problems or problems of *unlikely intersections*. See for instance, [30, 32, 36, 20, 14]. The Ax-Schanuel theorem generalizes the Ax-Lindemann-Weierstrass. In the setting of pure Shimura varieties, the Ax-Schanuel theorem of [27] has recently been applied to certain diophantine problems [15]. Various Ax-Schanuel results have also been applied to various cases of the Zilber-Pink conjecture [3, 11].

Over the past several years, in a series of works Aslanyan, and later Aslanyan, Kirby and Eterović [1, 2, 5, 6, 4] develop the connection between Ax-Schanuel type transcendence statements and the existential closedness of certain reducts of differentially closed fields related to equations satisfied by the j-function. This series of work builds on the earlier program of Kirby, Zilber and others mainly around the exponential function, see e.g. [22]. We expect our results to contribute nontrivially to this line of work. The earlier work on the exponential function utilizes the transcendence results of Ax [7], where a differential algebraic proof of the functional version of Schanuel's conjecture is given. Intermediate differential algebraic results from Ax's work are utilized in the program studying existential closedness results around the exponential function. A different approach was required in [1, 2, 5, 6, 4] for studying the j-function, in part since the intermediate results of the functional transcendence results of [31] take place in the o-minimal rather than abstract differential setting. For instance, the motivation for a differential algebraic proof of Ax-Schanuel results and this issue is pointed out specifically following Theorem 1.3 of [5] and in Section 4.4 of [4]. Our results open up the possibility of adapting a similar approach to existential closedness around more general automorphic functions since the general technique of our proof is differential algebraic along lines similar to Ax's work. Besides this issue, it is expected that our generalizations of the Ax-Schanuel results of [31] can be used to establish existential closedness results for more general automorphic functions beyond the modular j-function.

1.2. **Organization of the paper.** The paper is organized as follows. In Section 2 we give the necessary background in the Cartan approach to the study of linear differential equations. In particular, we recall the definition (and basic properties) of a G-principal connection and its associated Cartan connection form. In Section 3 we prove Theorem A and derive some of its corollaries. We also introduce the idea of geometric structures (or $(G, G/B)$ structures) and apply Theorem A in this setting to obtain Corollary B . In Section 4 we recall Scanlon's work on covering maps, show that they are part of the formalism of geometric structures and detail the intersection of our work with other similar work in the literature. In section 5 we use Theorem A to study products of geometric structures (Theorem C) and use those to give a model theoretic study of the $(G, G/B)$ structures. Section 6 and 7 are devoted to proving Theorem D, i.e., the full Ax-Schanuel Theorem with derivatives in the case of curves.

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2. The Cartan approach to linear differential equations

In this section, we set up some notations and conventions about principal connections. A complete reference is Sharpe's book [35] or the third part of Epstein-Elzanowski's book [19]. Throughout, we will be working over the field of complex numbers $\mathbb{C}$. Analytic functions or manifolds mean holomorphic.

2.1. **Principal connection.** Let G be an algebraic group, Y a smooth algebraic variety and $\pi\colon P \to Y$ a principal bundle modeled over G, *i.e.* endowed with an action of G, denoted by R or by $\cdot$, that induces an isomorphism

$$P \times G \xrightarrow{\sim} P \underset{Y}{\times} P \quad (p, g) \mapsto (p, p \cdot g) = (p, R_g(p)).$$

The fibers of π are principal homogeneous G-spaces. The election of a point p in a fiber $P_y = \pi^{-1}(y)$ induces an isomorphism of G-spaces,

$$G \xrightarrow{\sim} P_y, \quad g \mapsto p \cdot g,$$

and an isomorphism of groups,

$$G \xrightarrow{\sim} \mathrm{Aut}_G(P_y), \quad g \mapsto \sigma \text{ with } \sigma(p \cdot h) = p \cdot gh$$

between G and the group $\mathrm{Aut}_G(P_y)$ of G-equivariant automorphisms of P_y. Note that this pair of isomorphisms conjugate the left action of G on itself with the action of $\mathrm{Aut}_G(P_y)$ on P_y. A *gauge transformation* of P is a G-equivariant map $F\colon P \to P$ such that $\pi \circ F = \pi$. That means that for each fiber $F|_{P_y} \in \mathrm{Aut}_G(P_y)$.

We define the *vertical bundle* $T(P/Y)$ as the kernel of $d\pi$, it is a subbundle of TP. A *connection* is a section ∇ of the exact sequence,

$$0 \to T(P/Y) \to TP \to TY \underset{Y}{\times} P \to 0.$$

Thus, it is a map

$$\nabla\colon TY \underset{Y}{\times} P \to TP, \quad (v, p) \mapsto \nabla_{v,p},$$

satisfying $d\pi(\nabla_{v,p}) = v$.

The image $\nabla(TY \times_Y P) \subset TP$ is a *distribution of vector fields*[1] *of rank* $\dim Y$ *on* P, the so-called ∇-horizontal distribution $\mathcal{H}_\nabla$. We have a canonical decomposition of the tangent bundle $TP = T(P/Y) \oplus \mathcal{H}_\nabla$ as the direct sum of its vertical and ∇-horizontal subbundles.

[1]Let $D \subset TY$ be a subset of the tangent bundle of a smooth manifold Y such that for each $y \in Y$, the fiber above y, $D_y \subseteq T_yY$ is a d-dimensional subspace. Further, suppose that for any point y there is an Zariski open neighborhood U of y such that for any $z \in U$, we have independent vector fields $X_1(z), \dots, X_d(z)$ regular vector fields whose span is D_z.

A connection induces a ∇-horizontal lift operator of vector fields on Y to P,

$$\nabla\colon \mathfrak{X}_Y \to \pi_*\mathfrak{X}_P; \quad v \mapsto \nabla_v \text{ with } (\nabla_v)(p) = \nabla_{v(\pi(p)),p}$$

that lifts vector fields in open subset $U \subset Y$ up to ∇-horizontal vector fields in $\pi^{-1}(U) \subset P$.

This operator is $\mathcal{O}_X$-linear. By abuse of notation, this lift operator is represented by the same symbol ∇. It completely determines the connection. Note that usually, the symbol ∇ is used to denote the associated covariant derivative. The horizontal lift of rational vector fields on Y defines a map $\mathrm{D} : \mathbb{C}(P) \to \mathbb{C}(P) \otimes_{\mathbb{C}(Y)} \Omega^1(Y)$ where $\Omega^1(Y)$ is the $\mathbb{C}(Y)$-vector space or rational 1-forms. Such a D extends the differential structure of $\mathbb{C}(Y)$ given by the exterior derivative, $\mathrm{d} : \mathbb{C}(Y) \to \Omega^1(Y)$, and satisfies Leibniz rule : $\mathrm{D}(ab) = a\mathrm{D}(b) + b\mathrm{D}(a)$.

We say that the connection ∇ is principal if it is G-equivariant: for all $(v,p) \in TY \times_Y P$ and $g \in G$ we have $\nabla_{v,R_g(p)} = dR_g(\nabla_{v,p})$. This is equivalent to requiring that the ∇-horizontal distribution is G-invariant, or that the image of the ∇-horizontal lift operator consists of G-invariant vector fields. We say that the connection ∇ is *flat* if the lift operator is a Lie algebra morphism, that is:

$$\nabla_{[v,w]} = [\nabla_v, \nabla_w].$$

In what follows, a connection means a G-principal flat connection.

We are mostly concerned with *rational connections.* The definition above is the definition of a regular connection. A rational connection on a bundle P over Y is a regular connection on the restriction $P|_{Y^\circ}$ of the bundle above a Zariski open subset $Y^\circ \subset Y$. Given a G-invariant rational connection ∇, we may replace the base space Y by a suitable Zariski open subset Y° such that $\nabla|_{Y^\circ}$ is regular.

Example 2.1. The most important example of a rational G-invariant connection is a linear differential equation in fundamental form. Let us fix an algebraic irreducible curve Y and a non constant rational function $y \in \mathbb{C}(Y)$. As $\mathbb{C}(Y)$ is an algebraic extension of $\mathbb{C}(y)$, the derivation $\frac{d}{dy}$ can be uniquely extended to $\mathbb{C}(Y)$. Our equation is

$$\frac{dU}{dy} = A(y)U \text{ with } U \text{ an invertible } n \times n \text{ matrix of unknowns and } A \in \mathfrak{gl}_n(\mathbb{C}(Y)).$$

In this situation G is the linear group $\mathrm{GL}_n(\mathbb{C})$ with coordinate ring $\mathbb{C}\left[U_i^j, \frac{1}{det(U)}\right]$, P is $Y \times G$, and the action of G on P is given by right translations: $(y,U)\cdot g = (y, Ug)$. We see $\frac{d}{dy}$ as a rational vector field on Y and the linear differential system gives us its ∇-horizontal lift

$$\nabla_{\frac{d}{dy}} = \frac{\partial}{\partial y} + \sum_{i,j,k} A_i^j U_j^k \frac{\partial}{\partial U_i^k}.$$

Then ∇-horizontal lift operator is determined by the above formula and $\mathcal{O}_Y$-linearity. The field $\mathbb{C}(P)$ with the derivation $\nabla_{\frac{d}{dy}}$ is called the field of the universal solution of the linear equation. As the vector fields $\sum_k U_j^k \frac{\partial}{\partial U_i^k}$ are right invariant on G, ∇ is a principal connection. The compatibility with Lie bracket is straightforward as for any couple of vector fields v and w on Y, $[v, w]$ is colinear to v; thus ∇ is flat. The equation defines a regular connection outside of the set of poles of the rational functions A_i^j and zeroes of $\frac{d}{dy}$ as a vector field on Y.

2.2. **Basic facts about singular foliations.** We will use freely vocabulary and results from holomorphic foliation theory while recalling the relevant objects in this subsection.

A *singular foliation* $\mathcal{F}$ of rank m on an algebraic variety P is a m-dimensional $\mathbb{C}(P)$-vector subspace of rational vector fields in P, $\mathfrak{X}_P$, stable by Lie bracket. Such vector fields are said to be tangent to $\mathcal{F}$. We say that the foliation $\mathcal{F}$ is regular at $p \in M$ if there is a basis $\{v_1, \ldots, v_m\}$ of $\mathcal{F}$ such that $v_i(p)$ is defined for $i = 1, \ldots, m$ and $v_1(p), \ldots, v_r(p)$ are $\mathbb{C}$-linearly independent, otherwise we say that $\mathcal{F}$ is singular at p. The set of singular points of $\mathcal{F}$ form a Zariski closed subset $\operatorname{sing}(\mathcal{F})$ of codimension ≥ 2. We say that the foliation $\mathcal{F}$ is regular if $\operatorname{sing}(\mathcal{F}) = \emptyset$.

An *integral submanifold* of $\mathcal{F}$ is an m-dimensional analytic submanifold $\mathcal{S} \subset P$ (not necessarily embbeded in P) whose tangent space at each point is generated by the values of vector fields in $\mathcal{F}$. Maximal connected integral submanifolds are called *leaves*. Through any regular point passes a unique leaf. Any connected integral submanifold of $\mathcal{F}$ determines completely a leaf by analytic continuation. For general results about Zariski closures of leaves of singular foliations we refer to [13], in particular the Zariski closure of a leaf is irreducible.

A subvariety $V \subset P$ is $\mathcal{F}$-invariant if the vector fields tangents to $\mathcal{F}$ whose domain is dense in V restrict to rational vector fields on V. In such case $\mathcal{F}|_V$ is a singular foliation in V of the same rank. Leaves of $\mathcal{F}|_V$ are leaves of $\mathcal{F}$ contained in V.

We say that $\mathcal{F}$ is *irreducible* if and only if it does not admit any rational first integrals, that is, if $f \in \mathbb{C}(P)$ such that $vf = 0$ for all $v \in \mathcal{F}$ then $f \in \mathbb{C}$. From Theorem 1.4 in [13] we have that $\mathcal{F}$ is irreducible if and only if it has a dense leaf. Let $\mathcal{L}$ be leaf of $\mathcal{F}$ and $\overline{\mathcal{L}}$ its Zariski closure. Then $\overline{\mathcal{L}}$ is an irreducible $\mathcal{F}$-invariant variety and the restricted foliation $\mathcal{F}|_{\overline{\mathcal{L}}}$ is irreducible.

If ∇ is a rational connection then the space

$$\Gamma^{\mathrm{rat}}(\mathcal{H}_\nabla) = \mathbb{C}(P) \otimes_{\mathbb{C}(Y)} \nabla(\mathfrak{X}_Y)$$

of rational ∇-horizontal vector field is a singular foliation in P if and only if ∇ is flat. Moreover, if ∇ is a regular flat connection then $\Gamma^{\mathrm{rat}}(\mathcal{H}_\nabla)$ is a regular foliation. By abuse of notation $\Gamma^{\mathrm{rat}}(\mathcal{H}_\nabla)$-invariant varieties are called ∇-invariant varieties and leaves of $\Gamma^{\mathrm{rat}}(\mathcal{H}_\nabla)$ are called ∇-horizontal leaves. If ∇ is regular and $\mathcal{L}$ is a ∇-horizontal leaf then the projection $\mathcal{L} \to Y$ is a topological cover in the usual topology.

Note that the foliation $\Gamma^{\mathrm{rat}}(\mathcal{H}_\nabla)$ has non horizontal leaves (called *vertical leaves*) included in the fibers of P at non regular points of ∇.

2.3. **The Galois group.** Let us consider ∇ a regular connection. We say that a variety $Z \subset P$ is ∇-invariant if $\mathcal{H}_\nabla|_Z \subset TZ$ and the projection on Y is dominant. The intersection of ∇-invariant varieties is ∇-invariant and therefore for each point $p \in P$ there is a minimal ∇-invariant variety Z such that $p \in Z$.

Frobenius theorem ensures that through any $p \in P$, there exists a unique ∇-horizontal leaf denoted by $\mathcal{L}_p$. It is clear that $\pi|_{\mathcal{L}_p}\colon \mathcal{L}_p \to Y$ is surjective. Moreover if $Z \subset P$ is a ∇-invariant variety and $p \in Z$ then the Zariski closure of the leaf, $\overline{\mathcal{L}_p} \subset Z$. Note that, by the G-invariance of the distribution $\mathcal{H}_\nabla$, we have $\mathcal{L}_p \cdot g = \mathcal{L}_{p\cdot g}$.

Lemma 2.2. *The following are equivalent:*

(a) *Z is a minimal ∇-invariant variety.*
(b) *For any $p \in Z$, Z is the Zariski closure of $\mathcal{L}_p$.*
(c) *Z is the Zariski closure of a ∇-horizontal leaf.*

Proof. $(a) \Rightarrow (b)$ Assume that Z is minimal, and let $p \in Z$. As Z is ∇-invariant it implies $\mathcal{L}_p \subset Z$ and therefore $\overline{\mathcal{L}_p} \subset Z$. We have that $\overline{\mathcal{L}_p}$ is a ∇-invariant variety, and then by minimality of Z we have $Z = \overline{\mathcal{L}_p}$.

$(b) \Rightarrow (c)$ Trivial.

$(c) \Rightarrow (a)$ Let us consider $p \in Z$ such that $Z = \overline{\mathcal{L}_p}$. Note that Z is irreducible. Let us see that Z is minimal. Let us consider $W \subset Z$ a ∇-invariant subvariety and $q \in W$. We have,

$$\overline{\mathcal{L}_q} \subseteq W \subseteq \overline{\mathcal{L}_p} = Z.$$

Let us consider $p' \in \mathcal{L}_p$ in the same fiber than q. Note that there is $g \in G$ such that $q \cdot g = p'$. Then we have $\overline{\mathcal{L}_q} \cdot g = Z$. If follows that Z and W have the same dimension. They are irreducible and therefore they are equal. □

Lemma 2.3. *Let Z be a minimal ∇-invariant variety for a regular connection. Then,*

$$\mathrm{Gal}(Z) = \{g \in G\colon Z \cdot g = Z\}$$

is an algebraic subgroup of G, and $\pi|_Z\colon Z \to Y$ is a $\mathrm{Gal}(Z)$-principal bundle.

Proof. We just need to note that the isomorphism,

$$P \times G \xrightarrow{\sim} P \underset{Y}{\times} P, \quad (p, g) \mapsto (p, p \cdot g)$$

maps $Z \times \mathrm{Gal}(Z)$ onto $Z \underset{Y}{\times} Z$. □

Note that if Z is a minimal ∇-invariant variety then any other is of the form $Z \cdot g$ for some $g \in G$ and $\mathrm{Gal}(Z \cdot g) = g^{-1}\mathrm{Gal}(Z)g$. It follows that P is the disjoint union of minimal ∇-invariant varieties, each one of them a principal bundle, all of them modeled over conjugated subgroups of G.

Definition 2.4. The *Galois group* of ∇, $\mathrm{Gal}(\nabla)$ is the algebraic group $\mathrm{Gal}(Z)$ for any minimal ∇-invariant variety Z. It is a well defined abstract algebraic group, but its immersion as a subgroup of G depends on the choice of Z.

Example 2.5. Let us consider example 2.1. Then, $\mathcal{L}_p$ is analytic subvariety obtained by the analytic continuations of the germ of a solution of the linear equation with initial condition $p \in P$ along any path in Y starting at $\pi(p)$. The differential field $(\mathbf{C}(\overline{\mathcal{L}_p}), \nabla_{\frac{d}{dy}})$ is a Picard-Vessiot extension of $(\mathbf{C}(Y), \frac{d}{dy})$. We usually chose a point of the form $p = (y_0, \mathrm{id}) \in Y \times G$ so that Galois group is called Picard-Vessiot group at y_0.

2.4. **Cartan connection form.** An alternative way to encode a principal connection ∇ is through its connection form Ω. First, let us define the structure form ω, which is canonically attached to the principal bundle. We differentiate the action R of G on Y, with respect the second factor along $P \times \{e\}$ so that we obtain a trivialization

$$d_2R\colon P \times \mathfrak{g} \xrightarrow{\sim} T(P/Y).$$

Such trivialization defines the structure form $\omega = \mathrm{pr}_2 \circ d_2R^{-1}$ of the bundle,

$$\omega\colon T(P/Y) \to \mathfrak{g}.$$

Note that if $g \in G$ and F is a gauge transformation then dR_g and dF map the vertical bundle $T(P/Y)$ onto itself. Therefore $R_g^*(\omega)$ and $F^*(\omega)$ are well defined as $\mathfrak{g}$-valued forms on $T(P/Y)$. The structure form have the following properties:

(1) Right G-covariance: for $g \in G$, $R_g^*\omega = \mathrm{Adj}_{g^{-1}} \circ \omega$;
(2) Left gauge-invariance: for any gauge transformation $F^*(\omega) = \omega$.
(3) For each fiber P_y the form $\omega_y = \omega|_{P_y}$ satisfy the Maurer-Cartan structure equation:

$$d\omega_y = -\frac{1}{2}[\omega_y, \omega_y].$$

Given a principal connection ∇ there is a unique way to extend the structure form ω to a $\mathfrak{g}$-valued 1-form Ω on P that vanish along the horizontal distribution, the so-called Cartan connection form:

$$\Omega\colon TP \to \mathfrak{g}, \quad v_p \mapsto \omega(v_p - \nabla_{d\pi(v_p),p}).$$

It is clear that Ω and ∇ determine each other as $\mathcal{H}_\nabla = \ker(\omega_\nabla)$. The gauge-invariance property of ω extends partially to Ω.

Definition 2.6. We say that a gauge transformation $F\colon P \to P$ is a *gauge symmetry* of ∇ if for any $p \in P$ and $v \in T_{\pi(p)}Y$, we have that $dF(\nabla_{v,p}) = \nabla_{v,F(p)}$.

Summarizing, the connection form Ω, attached to a flat principal connection has the following properties:

(1) $\Omega|_{T(P/Y)} = \omega$.
(2) $\Omega = 0$ on the horizontal distribution
(3) Ω is G-covariant ; For $g \in G$, $g^*\Omega = \mathrm{Adj}_{g^{-1}} \circ \Omega$ (right covariant);
(4) Ω is gauge-invariant; for any gauge symmetry F of ∇, $F^*\Omega = \Omega$ (left invariant)
(5) Ω satisfies Cartan structure equation

$$d\Omega = -\frac{1}{2}[\Omega, \Omega].$$

Example 2.7. Going on with example 2.1, the structure form ω is,

$$\omega = (U^{-1}dU)|_{T(P/Y)}$$

and the connection form is:

$$\Omega = U^{-1}dU - U^{-1}AUdt.$$

Proposition 2.8. *Let $\mathcal{L}$ be a leaf of $\mathcal{H}_\nabla$. The restriction of Ω to the Zariski closure of this leaf, $\overline{\mathcal{L}}$, takes values in* $\mathrm{Lie}(\mathrm{Gal}(\overline{\mathcal{L}}))$.

Proof. Note that $\overline{\mathcal{L}} \to Y$ is a principal $\mathrm{Gal}(\overline{\mathcal{L}})$ bundle. Therefore, $\omega|_{T(\overline{\mathcal{L}}/Y)}$ takes values in $\mathrm{Lie}(\mathrm{Gal}(\overline{\mathcal{L}}))$. If follows that $\Omega|_{\overline{\mathcal{L}}}$ takes values in $\mathrm{Lie}(\mathrm{Gal}(\overline{\mathcal{L}}))$. □

3. ∇-Special subvarieties and Ax-Schanuel

Definition 3.1. Let ∇ be a flat G-principal connection over Y with Galois group G. A subvariety $X \subset Y$ is ∇-*special* if the group $\mathrm{Gal}(\nabla|_X)$ is a strict subgroup of G.

Let $H \subset G$ be an algebraic subgroup. A subvariety $X \subset Y$ is H-*special* if $\mathrm{Gal}(\nabla|_X) \subset H$.

Example 3.2. The connection on the trivial $(\mathbb{C}^*)^2$- bundle over $\mathbb{C}^2 = Y$ given by

$$dU - \begin{bmatrix} d(y_2 y_1) & 0 \\ 0 & d(y_1) \end{bmatrix} U = 0$$

with solution $U = \begin{bmatrix} e^{y_2 y_1} & 0 \\ 0 & e^{y_1} \end{bmatrix}$ has special subvarieties. Its Galois group is $\mathbb{C}^* \times \mathbb{C}^*$ but its restriction to lines $y_2 = q \in \mathbb{Q}$ has Galois group $\mathbb{C}^*$

The relation between the ∇-special subvarieties and special Shimura subvarieties (Hodge type locus) is not clear from the definitions. Both have a monodromy group and in both cases the Zariski closure of that monodromy group is not G. Throughout, unless otherwise stated, we use *special* subvarieties as short for ∇-special subvarieties.

The first version of our Ax-Schanuel theorem is the following.

Theorem 3.3. *Let ∇ be a G-principal connection on $P \to Y$ with Galois group G. Let V be an algebraic subvariety of P, $\mathcal{L}$ is an horizontal leaf and $\mathcal{V}$ an irreducible component of $V \cap \mathcal{L}$. If $\dim V < \dim(\mathcal{V}) + \dim G$ and $\dim(\mathcal{V}) > 0$ then the projection of $\mathcal{V}$ in Y is contained in a special subvariety.*

Corollary 3.4. *Let $\hat{\mathcal{V}} : \mathrm{spf}\mathbb{C}[[s_1, \ldots, s_n]] \to \mathcal{L} \subset P$ be a non constant formally parameterized space in a horizontal leaf of ∇ and V its Zariski closure. If $\dim V < \mathrm{rk}(\hat{\mathcal{V}}) + \dim G$ then the projection of V in Y is a special subvariety.*

Proof. As $\mathcal{V} = V \cap \mathcal{L}$ is a germ of analytic subvariety containing $\hat{\mathcal{V}}$ then $\dim \mathcal{V} \geq \mathrm{rk}(\hat{\mathcal{V}})$. The inequality in the hypothesis of the corollary implies the one of the theorem. □

3.1. **Proof of Theorem 3.3.** The proof of the theorem follows from the next lemmas about the connection form Ω. We can assume V is the Zariski closure of the component $\mathcal{V}$ of $V \cap \mathcal{L}$ and assume V is irreducible.

Lemma 3.5. *The restriction of Ω to V has kernel of positive dimension at the generic point.*

Proof. As $\dim \mathcal{V} > 0$ we have that the set of point $p \in V$ such that the dimension of the kernel of $\Omega|_V$ at p is greater than or equal to $\dim \mathcal{V}$ is a Zariski closed subset. This set contains the points of $\mathcal{V}$ for which the tangent vectors are in the kernels. This analytic space is Zariski dense in V. It hence follows that the rank of $\Omega|_V$ is smaller than $\dim V - \dim \mathcal{V} < \dim G$. □

Lemma 3.6. *If $\dim \ker(\Omega|_V) = \dim Y$ then $V = P$.*

Proof. As $\ker(\Omega|_V) \subset \ker(\Omega)|_V$ and $\dim\ker(\Omega) = \dim Y$, the horizontal leaf through $p \in V$ is included in V. By hypothesis this leaf is Zariski dense in P. □

Lemma 3.7. *There is a Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ such that $\forall p \in V$, $\Omega_p(T_pV) = \mathfrak{k}$.*

Proof. We show that $\Omega_p(T_pV)$ does not depend on $p \in V$. Let $e_1, \ldots, e_q$ be a basis of $\mathfrak{g}$ and decompose,

$$\Omega|_V = \sum_{i=1}^{q} \Omega_i e_i.$$

We may assume that first $\Omega_1, \ldots, \Omega_k$ form a maximal set of linearly independent 1-forms over $\mathbf{C}(V)$ among the Ω_i's. We have then:

$$\Omega_{k+i} = \sum_{j=1}^{k} b_{ij}\Omega_j$$

with b_{ij} rational on V. We consider a vector field D on V in the kernel of Ω. By taking Lie derivatives we obtain:

$$\mathrm{Lie}_D(\Omega_{k+i}) = \sum_{j=1}^{k}(D \cdot b_{ij})\Omega_j + \sum_{j=1}^{k} b_{ij}\mathrm{Lie}_D\Omega_j.$$

From Cartan formula ($\mathrm{Lie}_D = i_D \circ d + d \circ i_D$) and Cartan structural equation ($d\Omega_j$ is a combination of the 2-forms $\Omega_k \wedge \Omega_\ell$ with constant coefficients) we have that $\mathrm{Lie}_D\Omega_j = 0$ for any $j = 1, \ldots, q$, and therefore for $i = 1, \ldots, q-k$ we have

$$\sum_{j=1}^{k}(D \cdot b_{ij})\Omega_j = 0.$$

By the linear independence we obtain that $D \cdot b_{ij} = 0$. It hence follows that the b_{ij}'s are first integrals of the foliation of V defined by $\ker(\Omega|_V)$ and thus are constant on $\mathcal{V}$. Since V is the Zariski closure of $\mathcal{V}$, the functions b_{ij} are constant. We have thus proved that the image of Ω_p is a fixed linear subspace $\mathfrak{k} \subset \mathfrak{g}$.

We claim that $\mathfrak{k}$ is a Lie subalgebra. Indeed, if we let e_1, e_2 be two elements of $\mathfrak{k}$ and let v_1 and v_2 be two vector fields on V such that $\Omega|_V(v_i) = e_i$ for $i = 1, 2$. Then by Cartan's structural equation,

$$[e_1, e_2] = d\Omega(v_1, v_2)$$

On the other hand, as Ω is a 1-form with values in $\mathfrak{k}$, we have that $d\Omega$ is a 2-form with values in $\mathfrak{k}$.

□

Lemma 3.8. *The Lie algebra $\mathfrak{k}$ is the Lie algebra of an algebraic subgroup $K \subset G$.*

Proof. There exists a connected Lie subgroup $K \subset G$ such that $\mathrm{Lie}(K) = \mathfrak{k}$. Choosing a point $p_0 \in V \subset P$ makes an identification of the fiber P_0 through p_0 with G such that p_0 is the identity and the action of G on P_0 is the right translation on G.

The image of $V_0 = V \cap P_0$ under this identification gives an algebraic variety through the identity. As $\Omega(TV_0) = \mathfrak{k}$, its tangent is spanned by the vector fields generating right translation by K. So K is the connected component of the identity of V and thus is algebraic. $\square$

Consider the quotient of P by the action of K, namely $\rho : P \to P/K$ and $\ell : \mathfrak{g} \to \mathfrak{g}/\mathfrak{k}$.

Lemma 3.9. *The foliation $\ell \circ \Omega = 0$ is ρ-projectable on a foliation $\mathcal{F}$ on P/K of dimension equal to the dimension of Y.*

A foliation given by a distibution $\mathcal{H}$ on P is ρ-projectable if there is a foliation $\mathcal{G}$ on P/K such that $\mathcal{H} \oplus \ker d\rho = d\rho^{-1}\mathcal{G}$. This means that the image by ρ of a leaf of $\mathcal{H}$ is a leaf of $\mathcal{G}$.

Proof. The distribution given by the kernel of $\ell \circ \Omega$ is a foliation as the sub Lie algebra structure of $\mathfrak{k}$ implies the Frobenius condition. Vector fields v on P with $\Omega(v) \in \mathfrak{k}$ are in the kernel of $\ell \circ \Omega$ and also in the space $\ker(\Omega)$. So the foliation has dimension $\dim Y + \dim K$ and its leaves contain the orbits of K thus it is ρ-projectable. $\square$

Lemma 3.10. *The dimension of $\rho(V)$ is between* 1 *and* $\dim Y - 1$.

Proof. As $\ell \circ \Omega|_V = 0$, we have that V is included in a leaf of $\ell \circ \Omega$. Thus $\rho(V)$ is included in a leaf of $\mathcal{F}$. Such a leaf is a $(\dim Y)$-dimensional immersed analytic subset so that $\dim \rho(V) \leq \dim Y$. If $\rho(V)$ is a point then it projects on a point in Y, but this projection must contain a non constant formal curve thus $0 < \dim \rho(V)$.

If $\rho(V)$ has dimension $\dim Y$, then $\rho(V)$ is an algebraic leaf of $\mathcal{F}$. The existence of such a leaf implies that the Lie algebra of the Galois group of ∇ is $\mathfrak{k}$. This is not possible by hypothesis and then $\dim \rho(V) < \dim Y$. $\square$

We can now conclude the proof of Theorem 3.3.

Proof. As $\dim \rho(V) < \dim Y$ its projection in Y is contained in a strict algebraic subvariety X of Y. It suffices to build a strict $\nabla|_X$-invariant subvariety of $P|_X$. But we have that $\rho(V)$ is an algebraic subvariety of P/K such that $\ker d\pi|_V = 0$. So the map $\rho(V) \to X$ is a finite map. It is an algebraic leaf of $\mathcal{F}|_X$ this implies that $\rho^{-1}(\rho(V))$ is a $\nabla|_X$-invariant K-principal subbundle: $\mathrm{Gal}(\nabla|_X)$ has Lie algebra included in $\mathfrak{k}$ thus X is special. $\square$

3.2. **Uniformizing equation and Ax-Schanuel.** Let G be an algebraic group and B an algebraic subgroup. A $(G, G/B)$-structure on an algebraic variety Y is usually defined using charts on Y with values in G/B and change of charts in G. Here is a algebraic version of this notion.

The jet space $J^*(Y, G/B)$ of invertible jets of map is endowed with an action of G by postcomposition. As it is a jet space, its ring has a $\mathcal{D}_Y$-differential structure.

Definition 3.11. A *rational* $(G, G/B)$*-structure* on Y is a $\mathcal{D}_Y$-subvariety $\mathscr{C}$ of $J^*(Y, G/B)$ with a Zariski open $Y^o \subset Y$ such that $\mathscr{C}|_{Y^o}$ a G-principal sub-bundle of $J^*(Y^o, G/B)$.

As $\mathcal{D}_Y$-varieties have no $\mathcal{O}_X$ torsion, $\mathscr{C}$ is well defined if we know $\mathscr{C}|_{Y^o}$. Since $\dim Y = \dim G/B$, one has an isomorphism from $J^*(Y, G/B)$ to $J^*(G/B, Y)$. If $\mathscr{C}$ is a $(G, G/B)$-structure, then its image under this isomorphism is denoted by $\mathscr{U}$ and is a finite dimensional (over $\mathbb{C}$) $\mathcal{D}_{G/B}$-subvariety of $J^*(G/B, Y)$. A local analytic solution v of $\mathscr{Y}$ is a holomorphic map defined on a open subset $v : U \to Y$, whose jet $j^\infty v : U \to J^*(G/B, Y)$ takes values in $\mathscr{Y}$. These solutions are called uniformizations of the $(G/B, G)$-structure.

We call $\mathscr{Y}$ the space of uniformizations of the $(G, G/B)$-structure C. As the datas $\mathscr{C}$ and $\mathscr{Y}$ are equivalent, we will use any of them to denote the $(G, G/B)$-structure.

Lemma 3.12. *A geometric structure $\mathscr{C}$ on an algebraic variety Y is a principal bundle with a principal connection on some Zariski open subset Y^o.*

Proof. By definition we have that $\mathscr{C}$ is a G principal bundle over Y^o. The $\mathcal{D}_Y$-structure of $\mathscr{C}$ gives a lift of vector fields on Y° to $\mathscr{C}$, it is a connection on $\mathscr{C}|_{Y^\circ}$. The group G acts on $\mathscr{C}$ by post-composition. The action of $\mathcal{D}_Y$ is the infinitesimal part of the action by pre-composition. These two commute and hence the connection is G-invariant. □

The following statement is another (more usual) version of the Ax-Schanuel theorem. It follows by applying Corollary 3.4 to $\mathscr{C}$.

Corollary 3.13. *Let υ be a uniformization of a $(G, G/B)$-structure on an m-dimensional algebraic variety Y. Assume that $\hat{\gamma}$ is a non constant formal curve on G/B such that*

$$\text{tr.deg.}_{\mathbf{C}}\mathbf{C}\,(\hat{\gamma}, (\partial^\alpha \upsilon)(\hat{\gamma}) : \alpha \in \mathbf{N}^m) < 1 + \dim G$$

then the Zariski closure $\overline{\upsilon(\hat{\gamma})}$ is a special subvariety of Y

From Corollary 3.13 we get the Ax-Schanuel theorem without derivatives as stated for instance in [27].

Corollary 3.14. *Let υ be a uniformization of an irreducible $(G, G/B)$-structure on an m-dimensional algebraic variety Y. Assume $W \subset G/B \times Y$ is an irreducible algebraic subvariety intersecting the graph of υ. Let U be an irreducible component of this intersection such that*

$$\dim W < \dim U + \dim Y.$$

Then the projection of U to Y is contained in a special subvariety of Y.

Proof. Let $j^0 : \mathscr{C} \to G/B \times Y$ be the 0-jet projection. Then $(j^0)^{-1}(W)$ intersect the graph of the jet of υ on $\tilde{U}$ such that $j^0(\tilde{U}) = U$.

Now $\dim({j^0}^{-1}(W)) = \dim W + \dim B < \dim U + \dim X + \dim B = \dim \tilde{U} + \dim G$. Using Corollary 3.13 the result follows. □

We also obtain the following.

Corollary 3.15. *Let υ be a uniformization of a $(G, G/B)$-structure on a m dimensional algebraic variety Y. Let $A \subset G/B$ be an irreducible algebraic subvariety. If $\upsilon(A)$ has a proper Zariski closure in Y, then $\upsilon(A)$ is contained in a special subvariety of Y.*

Proof. Let $U \subset G/B \times Y$ be the graph of $\upsilon|_A$ and V be its Zariski closure. Then

$$\dim V \leq \dim A + \dim \overline{\upsilon(A)} < \dim U + \dim X.$$

□

4. Differential equations and covering maps

Our main technical results have applications to algebraicity problems for certain analytic functions, where our applications overlap with some existing results. In this section, we outline some settings for existing results which overlap our applications. We work in the context of covering maps following [34], but we will explain the setup.

Let G be an algebraic group over $\mathbb{C}$ with a regular action on a complex variety X such that $X = G/B$ for some algebraic subgroup B of G. Let U be a complex submanifold of $X(\mathbb{C})$ and Γ be a Zariski dense subgroup of $G(\mathbb{C})$ with the property that the induced action of Γ on X preserves U. Assume that we have a complex analytic map $\upsilon : U \to Y$ which is a covering map of the complex algebraic variety Y expressing $Y(\mathbb{C})$ as $\Gamma \setminus U$. So, υ^{-1} is a multivalued function with branches corresponding to elements of Γ.

Under those assumptions Scanlon shows that there is a differential algebraically constructible function $\tilde{\chi} : X \to Z$, for some algebraic variety Z, called the *generalized Schwarzian derivative* associated to υ such that for any differential field F having field of constants $\mathbb{C}$ and points $a, b \in X(F)$ one has that $\tilde{\chi}(a) = \tilde{\chi}(b)$ if and only if $a = gb$ for some $g \in G(\mathbb{C})$. For the situation we have in mind, the restriction to differential fields F with $m := \dim X$ commuting derivatives is enough. From Scanlon's construction, it follows that the generalized Schwarzian derivative is defined on an order k jet space $\tilde{\chi} : J^*_{k,m}(X) \to Z$. Thus the map $\chi := \tilde{\chi} \circ \upsilon^{-1} : J^*_{k,m}(Y) \to Z$ is a well-defined analytic map and induces a differential analytic map $Y(M) \to Y(M)$ for M any field of meromorphic functions in m variables.

Now, also assume that the restriction of υ to some set containing a fundamental domain is definable in an o-minimal expansion of the reals as an ordered field. One of the main results of [34], Theorem 3.12, is that under this assumption the function $\chi := \tilde{\chi} \circ \upsilon^{-1}$, for any choice of a branch of υ^{-1}, is also differential algebraically constructible. The function χ is called the *generalized logarithmic derivative* associated to υ.

Example 4.1. When $G = \mathrm{PSL}_2$, it is classically known that $\tilde{\chi} : J^*_3(X) \to \mathbb{C}$ can be taken to be the Schwarzian derivative $S(x) = \left(\frac{x''}{x'}\right)' - \frac{1}{2}\left(\frac{x''}{x'}\right)^2$ and $\chi : J^*_3(Y) \to \mathbb{C}$ is given by $S(y) + R(y)y'^2$ with $R(y) = S(\upsilon^{-1})$.

The definability property in Scanlon's theorem is analogous to the hypothesis that can be made on the principal connection $S(x) = R(y)$ to have all singulatities regular. Both imply that R is a rational function.

Using χ and $\tilde{\chi}$, we can define the differential equation satisfied by υ and υ^{-1}. From the composition of jets one gets two maps: $c_1 : J^*_{k,m}(X) \times J^*_{k,m}(Y) \to J^*_k(X, Y)$ and $c_2 : J^*_{k,m}(Y) \times J^*_{k,m}(X) \to J^*_k(Y, X)$. The algebraic subvariety given by $\chi - \bar{\chi} = 0$ projects by c_1 on a algebraic subvariety of $J^*_k(X, Y)$ and by c_2 on a algebraic subvariety of $J^*_k(Y, X)$

Let us assume that $\bar{t} = (t_1, \dots t_m)$ are the coordinates on $U \subset X = G/B$ for some algebraic subgroup B of G and $\bar{y} = (y_1, \dots y_m)$ be coordinates on Y. By construction, we have that $\upsilon(\bar{t})$ satisfies the algebraic differential equation

$$\mathscr{Y} = c_1\left(\{\chi(\bar{y}) - \tilde{\chi}(\bar{t}) = 0\}\right) \subset J^*_k(G/B, Y)$$

and the inverse branches $\upsilon^{-1}(\bar{y})$ satisfies

$$\mathscr{C} = c_2\left(\{\chi(\bar{y}) - \tilde{\chi}(\bar{t}) = 0\}\right) \subset J^*_k(Y, G/B).$$

Remark 4.2. The $*$ appearing in the jet space J_k^* means that we are adding to the explicit equations above inequations ensuring that the rank of the jacobian matrix of solutions is m.

Proposition 4.3. *The equations above define a rational $(G, G/B)$-structure $\mathscr{C}$ on Y.*

Proof. Since $\bar{y}$ are coordinates on Y, using c_2 it follows that $\chi(\bar{y})$ is a rational function R on Y. Let Y° be the domain of R on which our differential equation can be written $\bar{\chi}(\bar{t}) = R(y)$ in $J_k^*(Y, G/B)$. By construction, it follows that $\mathscr{C}$ coincides with the differential subvariety of $J^*(Y, G/B)$ defined by this equation.

By Scanlon's construction, we have a map $\mathscr{C} \times G \to \mathscr{C}$ insuring that $\mathscr{C}$ is a G-principal bundle on Y°. As a smooth finite dimensional $\mathcal{D}_{Y^\circ}$-subspace of $J^*(Y^\circ, G/B)$, $\mathscr{C}$ gives rise to a connection. The properties of $\bar{\chi}$ ensure that this connection is G-invariant. □

It hence follows that our Ax-Schanuel Theorems (corollaries 3.13 and 3.14) hold in the case of covering maps given in the Scanlon theory. In the coming subsections, we will describe the settings in which existing versions of an Ax-Schanuel Theorem for covering maps exists and have some overlap with our results.

4.1. **Modular curves.** Let $j : \mathbb{H} \to \mathbb{A}^1(\mathbb{C})$ be the classical modular j-function. In the notation above we take $G = \mathrm{SL}_2$ and

$$B = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : ad = 1 \right\}$$

the subgroup of lower triangular matrices so that $X = \mathrm{SL}_2(\mathbb{C})/B \cong \mathbb{CP}_1$. As well-known, if we take $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, then the quotient $Y(1) = \Gamma \setminus \mathbb{H}$ can be identified with the affine line $\mathbb{A}^1(\mathbb{C})$. We take U to be the open subset of $\mathbb{H}$ such that $j : U \to \mathbb{A}^1(\mathbb{C}) \setminus \{0, 1728\}$ is a covering map. The restriction of j to the domain $F = \{z \in \mathbb{H} \,:\, |Re(z)| \leq \frac{1}{2} \text{ and } Im(z) \geq \frac{\sqrt{3}}{2}\}$, which contain a fundamental domain, is definable in $\mathbb{R}_{an,exp}$. Hence j is a solution to a $(G, G/B)$-structure as describe above. In this case, this structure can be taken to be the well-known Schwarzian differential equation satisfied by j.

In this setting, the Ax-Schanuel Theorem is a result of Pila and Tsimerman [31, Theorem 1.1]

Theorem 4.4. *Let $V \subset (\mathbb{P}^1)^n \times Y(1)^n$ be an algebraic subvariety, and let U be a component of $V \cap \Gamma(j^n)$, where $\Gamma(j^n)$ denotes the graph of the j-function applied to $\mathbb{H}$ inside each of the n copies of $\mathbb{P}^1$. Then $\dim U = \dim V - n$ unless the projection of U to $Y(1)^n$ is contained in a proper weakly special subvareity of $Y(1)^n$.*

Replacing Γ in the above results with $\Gamma(N)$, the kernel of the reduction mod N map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ (also replacing j with suitable j_N and $Y(1)$ with $Y(N)$), one obtains the same result; j and j_N are interalgebraic as functions over $\mathbb{C}$. Pila and Tsimerman [31, Theorems 1.2 and 1.3] prove the more general version of Ax-Schanuel which includes the first and second derivatives of j (and replaces n with $3n$). Our work gives a uniform proof of this result for $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ any Fuchsian group of the first kind.

4.2. **Pure Shimura Varieties.** We follow closely the exposition given in [9]. Let G to be a connected semi-simple algebraic $\mathbb{Q}$-group and K a maximal compact subgroup of $G(\mathbb{R})$. Then it follows that $\Omega = G(\mathbb{R})/K$ is a bounded symmetric domain. It is known (cf. [21, Proposition 7.14]) that the compact dual $\check{\Omega}$ of Ω is given as the quotient $\check{\Omega} = G(\mathbb{C})/B$ for a Borel subgroup B and is a homogeneous projective variety. One can always assume that $K \subset B$, so that Ω is a semi-algebraic subset of $\check{\Omega}$.

Given an arithmetic lattice $\Gamma \subset G(\mathbb{Q})$, the analytic quotient $Y := \Gamma\backslash\Omega = \Gamma\backslash G(\mathbb{R})/K$ has the structure of an algebraic variety and is called a pure (connected) Shimura variety. The quotient map $q : \Omega \to Y := \Gamma\backslash\Omega$ is a covering map[2] and the result [23, Theorem 1.9] shows that it is definable in $\mathbb{R}_{an,exp}$ on some fundamental domain. Hence q is a solution to a $(G, G/B)$-structure on Y as defined above.

We fix $Y = \Gamma\backslash\Omega$ a connected pure Shimura variety and $q : \Omega \to Y$ the quotient map.

Definition 4.5. A *weakly special subvariety* of Y is a Shimura variety Y' given as

$$Y' = \Gamma'\backslash G'(\mathbb{R})/K'$$

where G' is an algebraic $\mathbb{Q}$-subgroup of G, the group $\Gamma' = \Gamma \cap G'(\mathbb{Q})$ is an arithmetic lattice, and $K' = K \cap G'(\mathbb{R})$.

It follows that if Y' is a weakly special subvariety of Y, then Y' is algebraic, which by definition means that there is an algebraic subvariety V of $\check{\Omega}$ such that $Y' = V \cap \Omega$.

Theorem 4.6. [37, Theorem 1.2] *An irreducible subvariety $Z \subset Y$ is weakly special if and only if some (all) components of $q^{-1}(Z)$ are algebraic.*

An essential part of Pila's strategy for attacking various diophantine problems associated with the geometry of certain analytic covering maps (see e.g. [29]) is to identify the (weakly) special subvarieties of Y. In this setting, the most general transcendence result to which our applications are related comes from [27].

Theorem 4.7. [27, Theorem 1.1] *Let $W \subset \Omega \times Y$ be an algebraic subvariety. Let W_0 be a component of $W \cap D$ of positive dimension, where D is a the graph of the map $q : \Omega \to Y$. Suppose that*

$$\dim W < \dim W_0 + \dim Y.$$

Then the projection of W_0 to Y is contained in a proper weakly special subvariety of Y.

4.3. **Ball quotients.** In this subsection we outline the setting of the recent manuscript [10]. Let $G := \mathrm{PU}(n, 1)$, the group of holomorphic automorphisms of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$. Concretely, let $\mathrm{U}(n, 1)$ denote the group of linear transformations of $\mathbb{C}^{n+1}$ leaving invariant the form:

$$z_1\bar{z}_1 + z_2\bar{z}_2 + \ldots + z_n\bar{z}_n - z_0\bar{z}_0,$$

namely,

$$\mathrm{U}(n, 1) = \left\{ g \in \mathrm{GL}_{n+1}(\mathbb{C}) \,|\, g^T \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \bar{g} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

[2]To obatain a covering map, one might need to restrict q to an open subset U of Ω avoiding ramification points of the original map.

For $g \in \mathrm{U}(n,1)$, define the map $\phi_g : \mathbb{B}^n \to \mathbb{B}^n$ as follows: if

$$g = \begin{pmatrix} A & a_1 \\ a_2 & a_0 \end{pmatrix}$$

where A is a $n \times n$ matrix, a_1 is a column vector, and a_2 is a row vector, then for $z \in \mathbb{B}^n$,

$$\phi_g(z) = \frac{Az + a_1}{a_2 z + a_0}.$$

It is not hard to show that the map ϕ_g is the identity if and only if $g \in \{e^{it} I_{n+1} \ : \ t \in \mathbb{R}\} \cong \mathrm{S}^1$. Then we define $\mathrm{PU}(n,1) = \mathrm{U}(n,1)/\mathrm{S}^1$ and note that $\mathrm{PU}(n,1)$ is the group of holomorphic automorphisms of $\mathbb{B}^n$.

Using [21, Proposition 7.14] it follows that the compact dual, $X = \mathbb{CP}_n$, of $\mathbb{B}^n$ can be written as a quotient G/B for some algebraic subgroup B of G. If we let $\Gamma \subset G$ be a lattice and let Y be the quotient $\Gamma \backslash \mathbb{B}^n$, then the quotient map $\upsilon : \mathbb{B}^n \to Y$ is a covering map. From [10, Theorem 3.4.5], we have that υ is definable in $\mathbb{R}_{an,exp}$ on some fundamental domain. Hence, once again, the uniformizer υ is a solution to a $(G, G/B)$-structure on Y as defined above. Note that by results of Mok [25], the quotient Y has the structure of a quasi-projective algebraic variety. In this setting, Baldi and Ullmo [10, Theorem 1.22] established the Ax-Schanuel conjecture:

Theorem 4.8. *Let $W \subset \mathbb{B}^n \times Y$ and Π be the graph of the quotient map. Let U be an irreducible component of $W \cap \Pi$ such that $codim\, U < codim\, W + codim\, \Pi$ or equivalently $\dim W < \dim U + \dim Y$. If the projection from U to Y is positive dimensional, then it is contained in a strict totally geodesic subvariety of Y.*

Theorem 4.8 generalizes the earlier non-arithmetic Ax-Lindemann-Weierstrass theorem of Mok [26]. In the setting of this subsection, by Corollary 5.6.2 of [10], the totally geodesic subvarieties are precisely the bi-algebraic subvarieties for the map υ. Keeping in mind this connection will be essential later for observing applications of our results which generalize the Ax-Schanuel result of [10]. There are also similar results of [8] (stated in terms of geodesic subvarieties) for $\Gamma \leq SO(n,1)$, another setting to which our results would likely apply[3].

5. The product case and applications to Model theory

In this section we use Theorem 3.3 to study products of $(G, G/B)$-structures. In particular we show that an Ax-Schanuel type theorem holds in this setting. We then apply this result to give a model theoretic study of set defined, in a differentially closed field, by the $(G, G/B)$ structures. We show that the definable sets are strongly minimal, geometrically trivial and in the case of covering maps, satisfy a weak form of the Ax-Lindemann-Weierstrass Theorem with derivatives.

[3]One would simply have to establish that the uniformizers are o-minimally definable.

5.1. **Product of $(G, G/B)$-structures.** In this subsection we apply some of the results of Section 3 to products of $(G, G/B)$-structures. We first observe that the following Lemma holds.

Lemma 5.1. *Let G be an algebraic group, B a subgroup and $\mathscr{Y} \subset J^*(G/B, Y)$ a rational $(G, G/B)$-structure on Y. If the Galois group of the associated charts set $\mathscr{C}$ is G then*

1. *$\mathscr{Y}$ is irreducible,*
2. *there is no proper $\mathcal{D}_{G/B}$- subvariety of $\mathscr{Y}$.*

Proof. Using the isomorphism between $J^*(G/B, Y)$ and $J^*(Y, G/B)$, we need to prove that $\mathscr{C}$ is irreducible. The hypothesis on the Galois group means that $\mathscr{C}$ is the Zariski closure of an horizontal leaf. Irreducibility of the Zariski closure of a leaf of an holomorphic foliation is proved in [13].

The isomorphism between $J^*(G/B, Y)$ and $J^*(Y, G/B)$ exchanges the two differential structures thus a proper $\mathcal{D}_{G/B}$-subvariety of $\mathscr{Y}$ gives a proper $\mathcal{D}_Y$-subvariety of $\mathscr{C}$. By our Galois assumption there is no $\mathcal{D}_Y$-subvariety of $\mathscr{C}$. □

For $i = 1 \dots n$ let G_i be simple algebraic groups with trivial center, $B_i \subset G_i$ a subgroup and $\mathscr{C}_i \subset J_*(Y_i, G_i/B_i)$ a rational $(G_i, G_i/B_i)$-structure on Y_i whose G_i-invariant connection is denoted by ∇_i. As

$$\prod_{i=1}^{n} J_*(Y_i, G_i/B_i) \subset J_*\left(\prod_{i=1}^{n} Y_i, \prod_{i=1}^{n} G_i/B_i\right) = \tilde{J}$$

the subset $\mathscr{C} = \prod_i \mathscr{C}_i$ is a $(\prod_i G_i, \prod_i G_i/B_i)$-structure on $Y = \prod_i Y_i$ whose connection will be ∇. A uniformization of this structure is a map $\bar{\upsilon} = (\upsilon_1, \dots, \upsilon_n)$, where the i^{th} factor is a uniformization of Y_i and depends only of variables G_i/B_i.

Theorem 5.2. *Let V be an algebraic subvariety of $\tilde{J}$ and $\bar{\upsilon}$ a unifomization with graph $\mathcal{L}$ in $\tilde{J}$. Assume*

1. *The Galois group of the i^{th} factor is G_i,*
2. *V is the Zariski closure of $V \cap \mathcal{L}$ and $\dim(V \cap \mathcal{L}) > 0$,*
3. *$\dim V < \dim(V \cap \mathcal{L}) + \dim G_1 + \ldots + \dim G_n$,*
4. *the projection of V on each G_i/B_i is dominant.*

Then

(a) *there exist two indices $i < j$ such that the projection X_{ij} of V in $Y_i \times Y_j$ is proper special subvariety whose projections on both factors is onto.*

(b) *Denote by π_i and π_j the projections of X_{ij} on Y_i and Y_j. There is an isomorphism*

$$\varphi \colon \pi_i^* \mathscr{C}_i \to \pi_j^* \mathscr{C}_j$$

defined over V_{ij} such that $\varphi_* \pi_i^* \nabla_i = \pi_j^* \nabla_j$.

(c) *Assume the factors have finitely many maximal irreducible $\{e\}$-special subvarieties then X_{ij} is a correspondence.*

Proof. (a) Let X be the projection of V in $Y_1 \times \ldots \times Y_n$. By assumption (3) and Theorem 3.3, X is a special subvariety of $Y_1 \times \ldots \times Y_n$. The projection $\mathscr{C}$ to $\mathscr{C}_i$ is compatible with the connections. As the projection of V on G_i/B_i is dominant so is the projection of V on Y_i and then of X on Y_i. This implies that the projection from $\mathscr{C}|_X$ in $\mathscr{C}_i$ has a ∇_i-invariant image. By lemma 5.1, this projection is dominant. If follows that $\mathrm{Gal}(\nabla|_X)$ is a subgroup of $G_1 \times \ldots \times G_n$ that projects onto each component G_i. Then, by Goursat-Kolchin lemma there are indices $i < j$ and an algebraic group isomorphism,

$$\sigma\colon G_i \to G_j$$

such that

$$\mathrm{Gal}(\nabla|_X) \subset \left\{ (g_1, \ldots, g_n) \subset \prod_i G_i \colon \sigma(g_i) = g_j \right\}$$

Now, let us consider $\nabla_{ij} = \nabla_i \times \nabla_j$ as a connection on

$$\mathscr{C}_i \times \mathscr{C}_j \to Y_i \times Y_j.$$

It follows, $\mathrm{Gal}(\nabla_{ij}|_{V_{ij}}) \subset G_\sigma = \{(g_1, g_j) \in G_i \times G_j \colon \sigma(\bar{g}_i) = \bar{g}_j\}$. Therefore $X_{ij} \subset Y_i \times Y_j$ is a proper special subvariety.

(b) Let T_{ij} be the Zariski closure of an horizontal leaf of $\nabla_{ij}|_{X_{ij}}$ with Galois group G_σ then $T_{ij} \subset \mathscr{C}_i \times \mathscr{C}_j$ is the is the graph of the connection preserving isomorphism φ of the statement.

(c) We may consider $Y_i^\circ \subset Y_i$ the complement of $\{e\}$-special subvarieties. One can restrict bundles and connections above the products of Y_i°. Thus we may assume that factors have no $\{e\}$-special subvarieties.

As $X_{ij} \subset V_i \times V_j$ is a G_σ-special subvariety and for $y \in Y_i$, $\{y\} \times Y_j$ is a $(\{e\} \times G)$-special subvariety, then the intersection $X_j = X_{ij} \cap \{y\} \times Y_j$ is empty, has dimension 0 or is a special subvariety with group $G_\sigma \cap (\{e\} \times G)$. As G_σ is the graph of an isomorphism the latter is $\{e\} \times \{e\}$. As Y_j has no $\{e\}$-special subvarieties, X_j is empty of has dimension 0.

The projections of X_{ij} are onto thus X_j is generically (on y) not empty. Then $\dim X_{ij} = \dim Y_i$ and $\dim X_{ij} = \dim Y_j$ and the two projections are dominant. This proves the assertion (c), that is X_{ij} is a correspondence. □

In the following definition, we extract some of the key properties of geometric structures needed to apply the above theorem and to formulate a number of applications

Definition 5.3. A $(G, G/B)$ structure $\mathscr{Y}$ (or $\mathscr{C}$) on an algebraic variety Y is said to be *simple* if

(1) G is a centerless simple group,
(2) The Galois group of $\mathscr{C}$ is G
(3) Y as finitely many maximal irreductible $\{e\}$-special subvarieties.

Remark 5.4. If $\dim Y = 1$ the condition (3) trivially holds since any curve with a geometric structure has no special subvarieties.

Example 5.5. Simple ball quotients are described by Appell's bivariate hypergeometric systems F_1. It is a rank 3 linear connection on a vector bundle on $Y = \mathbb{CP}_1 \times \mathbb{CP}_1$ with singularities along 7 lines: $\{0, 1, \infty\} \times \mathbb{CP}_1$, $\mathbb{CP}_1 \times \{0, 1, \infty\}$ and the diagonal. This connection gives a rational $(\mathrm{PSL}_3(\mathbb{C}), \mathbb{CP}_2)$-structure on Y called $\mathscr{C}_{hyp}$ or $\mathscr{Y}_{hyp}$ (see [38]).

For clever choices of the exponents in F_1 system, see for instance [17], solutions of $\mathscr{Y}_{hyp}$ are built from the quotient of the ball $\mathbb{B} \subset \mathbb{CP}_2$ by a lattice $\Gamma \subset \mathrm{PSU}(2,1) \subset \mathrm{PSL}_3(\mathbb{C})$. As this lattice is included in $\mathrm{Gal}(\mathscr{C}_{hyp})$, the Galois group of this geometric structure is $\mathrm{PSL}_3(\mathbb{C})$ and it satisfies (1) and (2) in definition 5.3.

To see that the third condition is satisfied, let us consider X a $\{e\}$-special curve in Y and denote $\upsilon : \mathbb{B} \to Y$ the quotient map. The condition $\mathrm{Gal}(\mathscr{C}_{\mathrm{hyp}}|_{\mathrm{X}}) = \{e\}$ implies that the restriction of an inverse branch $\upsilon^{-1}|_X$ is a rational map on X with values in $\mathbb{B}$. As X is complete, by Liouville theorem $\upsilon^{-1}|_X$ must be constant which is in contradiction with the definition of $\mathscr{C}_{hyp}$.

Corollary 5.6. *Let $(Y, \mathscr{Y})$ be a simple $(G, G/B)$-structure on Y, $\hat{t}_1 \dots \hat{t}_n$ be n formal parametrizations of (formal) neighborhoods of points $p_1, \dots p_n$ in G/B and $\upsilon_1, \dots, \upsilon_n$ be solutions of $\mathscr{Y}$ defined in a neighborhood of $p_1, \dots p_n$ respectively. If*

$$\mathrm{tr.deg.}_{\mathbb{C}}\mathbb{C}\left(\hat{t}_i, (\partial^\alpha \upsilon_i)(\hat{t}_i) : 1 \le i \le n,\ \alpha \in \mathbb{N}^{\dim Y}\right) < \dim Y + n \dim G$$

then there exist $i < j$ such that

$$\mathrm{tr.deg.}_{\mathbb{C}}\mathbb{C}(\upsilon_i(\hat{t}_i), \upsilon_j(\hat{t}_j)) = \mathrm{tr.deg.}_{\mathbb{C}}\mathbb{C}(\upsilon_i(\hat{t}_i)) = \mathrm{tr.deg.}_{\mathbb{C}}\mathbb{C}(\upsilon_j(\hat{t}_j)) = \dim G.$$

5.2. **Strong minimality of the differential equations for uniformizers.** We begin by recalling some of the relevant notions from the model theoretic approach to the study of differential equations. We let $\mathcal{L}_m = \{0, 1, +, \cdot\} \cup \Delta$ denote the language of differential rings, where $\Delta = \{\partial_1, \dots, \partial_m\}$ is a set of unary function symbols. From a model theoretic perspective, differential fields are regarded as $\mathcal{L}_m$-structures where the symbols ∂_i are interpreted as derivations, while the other symbols interpreted as the usual field operations.

A differential field (K, Δ) is differentially closed if it is existentially closed in the sense of model theory, namely if every finite system of Δ-polynomial equations with a solution in a Δ-field extension already has a solution in K. We use m-DCF_0 to denote the common first order[4] theory of differentially closed fields in $\mathcal{L}_m$. It follows that m-DCF_0 has quantifier elimination, meaning that every definable subset of a differentially closed field (K, Δ) is a boolean combination of Kolchin closed sets.

Remark 5.7. We will use the following model theoretic conventions

(1) The notation υ will be used both for a tuple and an element.
(2) We say that a tuple υ is algebraic over a differential field K, and write $\upsilon \in K^{alg}$, if each coordinate of υ is algebraic over K.

We fix a saturated model $(\mathbb{U}, \Delta)$ of m-DCF_0 and assume that $\mathbb{C}$, the field of complex numbers, is its field of constants of, i.e., $\mathbb{C} = \{\upsilon \in \mathbb{U} : \partial(\upsilon) = 0 \text{ for all } \partial \in \Delta\}$. Given a differential field subfield K of $\mathbb{U}$ and υ a tuple of elements from $\mathbb{U}$, the complete type of υ over K, denoted $tp(\upsilon/K)$, is the set of all $\mathcal{L}_m$-formulas with parameters from K that υ satisfies. It is not hard to see that the set

$$I_{p,K} = \{f \in K\{X\} : f(X) = 0 \in p\} = \{f \in K\{X\} : f(\upsilon) = 0\}$$

[4]The description given here is not a first order axiomatization. We refer the reader to [24] for the basic model theory of m-DCF_0.

is a differential prime ideal in the differential polynomial ring $K\{X\}$, where $p = tp(\upsilon/K)$. Using quantifier elimination, it is not hard to see that the map $p \mapsto I_{p,K}$ is a bijection between the set of complete types over K and differential prime ideals in $K\{X\}$. Furthermore, it follows that a tuple υ_1 is a realization of $tp(\upsilon/K)$ if and only if $I_{p,K}$ is the vanishing ideal of υ_1 over K. Therefore in what follows there is no harm to think of $p = tp(\upsilon/K)$ as the ideal $I_{p,K}$

Definition 5.8. Let $K \subset \mathbb{U}$ be a differential field and υ is a tuple from $\mathbb{U}$.

(1) Let $F \subset \mathbb{U}$ be a differential field extension of K. We say that $tp(\upsilon/F)$ *is a non-forking extension* of $tp(\upsilon/K)$ if $K\langle \upsilon \rangle$ is algebraically disjoint from F over K, i.e, if $y_1, \dots, y_k \in K\langle \upsilon \rangle$ are algebraically independent over K then they are algebraically independent over F.
(2) We say that $tp(\upsilon/K)$ has *U-rank 1* (or is minimal) if and only if $\upsilon \notin K^{alg}$ but every forking extension of $tp(\upsilon/K)$ is algebraic, that is has only finitely many realizations.

Remark 5.9. Let K and υ be as above and let $p = tp(\upsilon/K)$. Let F be a differential field extension. Assume further that $tr.deg._K K\langle \upsilon \rangle = r$.

(1) If K is algebraically closed then p has a unique non-forking extension to F, namely $tp(\hat{\upsilon}/F)$ for any $\hat{\upsilon}$ realizing p such that $\mathrm{tr.deg.}_F F\langle \hat{\upsilon} \rangle = r$.
(2) We have that $tp(\upsilon/F)$ is algebraic if and only if $\upsilon \in F^{alg}$.
(3) Also $tp(\upsilon/F)$ is a nonforking extension of $tp(\upsilon/K)$ if and only if $\mathrm{tr.deg.}_K K\langle \upsilon \rangle = \mathrm{tr.deg.}_F F\langle \upsilon \rangle$.
(4) In particular, the assumptions p has U-rank 1 and $\mathrm{tr.deg.}_F F\langle \upsilon \rangle < r$ (so that $tp(\upsilon/F)$ is a forking extension of p) implies that $\upsilon \in F^{alg}$.

Let $\mathscr{Y} \subset \mathbb{U}^\ell$ be a definable set and K any differential field over which $\mathscr{Y}$ is defined. Assume that the order of $\mathscr{Y}$, $\mathrm{ord}(\mathscr{Y}) = \sup\{\mathrm{tr.deg.}_K K\langle \upsilon \rangle : \upsilon \in \mathscr{Y}\}$, is finite; say $\mathrm{ord}(\mathscr{Y}) = r$. By the (complete) *type p of $\mathscr{Y}$ over K* we mean that $p = tp(\upsilon/K)$ for any $\upsilon \in \mathscr{Y}$ such that $\mathrm{tr.deg.}_K K\langle \upsilon \rangle = r$. Recall that we say that $\mathscr{Y}$ is *strongly minimal* if it cannot be written as the disjoint union of definable sets of order r, and for *any* differential field extension F of K and element $\upsilon \in \mathscr{Y}$, we have that $tr.deg(F\langle \upsilon \rangle / F) = 0$ or r. We will make use of the following fact.

Fact 5.10. *The definable set $\mathscr{Y}$ is strongly minimal if and only if its type over K has U-rank 1, if $\mathscr{Y}$ cannot be written as the disjoint union of K-definable sets of order r, and for* any *element $\upsilon \in \mathscr{Y}$, we have that $tr.deg(K\langle \upsilon \rangle / K) = 0$ or r.*[5]

Definition 5.11. Let $\mathscr{Y} \subset \mathbb{U}^m$ be a strongly minimal set and K any differential field over which $\mathscr{Y}$ is defined. We say that $\mathscr{Y}$ *geometrically trivial* if for any distinct $\upsilon_1, \dots, \upsilon_\ell \in \mathscr{Y}$, if the collection consisting of $\upsilon_1, \dots, \upsilon_\ell$ together with all their derivatives $\partial^\alpha \upsilon_i$ is algebraically dependent over K then for some $i < j$, the pair υ_i, υ_j together with their derivatives are algebraically dependent over K.

[5]This formulation is precisely suited for the argument we give later in the paper, and we have phrased it in this way so that it might be used more easily as a black box for non-experts. Taken together, the conditions might equivalently be written as - $\mathscr{Y}$ is the zero set of a prime differential ideal P, such that any differential ideals containing P (even after base change to a larger differential field) have the property that their zero sets are finite.

We now aim to show that the set defined by the partial differential equations for any uniformizer is strongly minimal and geometrically trivial. We will need a basic tool from model theory (more precisely stability theory) sometimes called the *Shelah reflection principle*. We restrict our exposition to types of finite order. Let $F = F^{alg}$ be any algebraically closed differential field and let $p = tp(\upsilon/F)$ for some tuple υ. Assume that $\text{tr.deg.}_F F\langle\upsilon\rangle = \ell$. We say that a sequence $(\upsilon_i)_{i=1}^{\infty}$ is a *Morley sequence in* p if υ_{i+1} realizes the (unique) *non forking extension* of p over $F_i = F\langle\upsilon_1, \ldots, \upsilon_i\rangle^{alg}$, i.e., in particular $\text{tr.deg.}_{F_i} F_i\langle\bar{\upsilon}_{i+1}\rangle = \ell$. It follows that one can take $\upsilon_1 = \upsilon$.

In general, when given a differential variety, $\mathscr{Y}$ defined over a differential field K, p the type of a generic solution of $\mathscr{Y}$ over K, υ a realization of p, and F a differential field extension of K, we say that $tp(\upsilon/F)$ is a *forking extension* of p if $\text{tr.deg.}_F F\langle\upsilon\rangle < \text{tr.deg.}_K K\langle\upsilon\rangle$. Otherwise, $tp(\upsilon/F)$ is a non forking extension of p (if K is algebraically closed, then this extension is unique). The next result gives a characterization of the kinds of fields one needs to consider while characterizing forking extensions of a type.

The Shelah reflection principle. [33, Lemma 2.28] *Let K be any differential field and let $p = tp(\upsilon/K)$ for some tuple υ with* $\text{tr.deg.}_K K\langle\upsilon\rangle = r$. *Let $F = F^{alg}$ be an algebraically closed differential field extension such that $tp(\upsilon/F)$ is a forking extension of p. Then there is a finite initial segment $(\upsilon_1, \ldots, \upsilon_k)$ of a Morley sequence $(\upsilon_i)_{i=1}^{\infty}$ in $tp(\upsilon/F)$ such that* $\text{tr.deg.}_K K\langle\upsilon_1, \ldots, \upsilon_k\rangle < k \cdot r$. [6]

The intuition here is that we start with a Morley sequence in $tp(\upsilon/F)$ and forking is captured in the fact that at some point this appropriately chosen sequence ceases to be a Morley sequence in $p = tp(\upsilon/K)$ (recall that we can start with $\upsilon_1 = \upsilon$). We can now apply Theorem 5.2 to study the set defined by the differential equations for uniformizers. Let us first explain the translation from the geometrical setting to m-DCF_0.

We have fixed a $(G, G/B)$-structure $\mathscr{C}$ on an algebraic variety Y with the group G simple and centerless and with $\dim(G) = k$. We assume that the Galois group of the associated charts connection on $\mathscr{C}$ is G. We consider an open subset of U of G/B and assume that $\bar{t} = (t_1, \ldots t_m)$ are the coordinates on U realizing a transcendence basis of $\mathbb{C}(G/B)$. We assume $Y \subset \mathbb{C}^{\ell}$. Now a uniformization of the $(G, G/B)$-structure on Y, say $\upsilon : U \to Y \subset \mathbb{C}^{\ell}$, will be described by a system of partial differential equations in variables $t_1, \ldots t_m$ and unknowns the ℓ coordinates of υ.

We assume throughout that our universal field $\mathbb{U}$ contains elements $t_1, \ldots t_m$ such that $\partial_i t_i = 1$ and $\partial_j t_i = 0$. We denote by $\mathscr{Y}$ the $(\partial_1, \ldots, \partial_m)$-differential equations satisfied by υ together with the inequations ensuring that the rank of the jacobian matrix of solutions is m. By abuse of notation, $\mathscr{Y} \subset \mathbb{U}^{\ell}$ also denotes the solution set it defines. Next let K, with $\mathbb{C} \subseteq K \subseteq \mathbb{C}(\bar{t})^{alg}$, for some (any) differential field over which $\mathscr{Y}$ is defined. In general $K \neq \mathbb{C}$.

Remark 5.12. We have the following observations

[6] Note that this implies (assuming one selects a sequence with the given properties of minimal length) that we have $\text{tr.deg.}_F F\langle\upsilon_k\rangle \leq \text{tr.deg.}_{K\langle\upsilon_1,\ldots,\upsilon_{k-1}\rangle} K\langle\upsilon_k\rangle < r$, while $(\upsilon_1, \ldots, \upsilon_{k-1})$ is a Morley sequence over K. In fact, one can actually arrange that $\text{tr.deg.}_F F\langle\upsilon_k\rangle = \text{tr.deg.}_{K\langle\upsilon_1,\ldots,\upsilon_{k-1}\rangle} K\langle\upsilon_k\rangle$ or even more specifically that a *canonical base* for the forking extension is contained in the algebraic closure of the initial segment of the Morley sequence.

(1) The assumption on the rank of the jacobian matrix is implicitly part of the formalism of Subsection 3.2. When $dim(Y) = 1$, this corresponds to the assumption that one only considers non-constant solutions.
(2) Using Lemma 5.1 (2) for any $\upsilon \in \mathscr{Y}$, we have that $tr.deg._{\mathbb{C}(\bar{t})}\mathbb{C}(\bar{t})\langle \upsilon \rangle = k$. Hence it also follows that $tr.deg._K K\langle \upsilon \rangle = k$.

We need the following abstract reformulation of Corollary 5.6.

Corollary 5.13. *Let $\upsilon_1, \ldots, \upsilon_n \in \mathscr{Y}$ be distinct solutions. If*

$$tr.deg._{\mathbb{C}(\bar{t})}\mathbb{C}(\bar{t})\langle \upsilon_1, \ldots, \upsilon_n \rangle < kn,$$

then for some $i < j$, we have that

$$tr.deg._{\mathbb{C}(\upsilon_i)}\mathbb{C}(\upsilon_i, \upsilon_j) \leq m - 1.$$

If we further assume that $(Y, \mathscr{Y})$ has finitely many maximal irreducible $\{e\}$-special subvarieties then $\upsilon_i \in \mathbb{C}(\upsilon_j)^{alg}$, that is each component of υ_i is algebraic over $\mathbb{C}(\upsilon_j)$.

Proof. Assume that $\mathrm{tr.deg.}_{\mathbb{C}(\bar{t})}\mathbb{C}(\bar{t})\langle \upsilon_1, \ldots, \upsilon_n \rangle = r < kn$ and let $\mathcal{C}$ be a finitely generated (over $\mathbb{Q}$) algebraic closed subfield of $\mathbb{C}$ such that

$$\mathrm{tr.deg.}_{\mathcal{C}(t)}\mathcal{C}(\bar{t})\langle \upsilon_1, \ldots, \upsilon_n \rangle = r.$$

For example we can take $\mathcal{C}$ to be generated by the coefficients of the polynomial defining the algebraic relations over $\mathbb{C}(\bar{t})$ between $\upsilon_1, \ldots, \upsilon_n$ and derivatives.

Applying Seidenberg's embedding theorem to the field $\mathcal{C}(\bar{t})\langle \upsilon_1, \ldots, \upsilon_n \rangle$, we may assume that $\upsilon_1, \ldots, \upsilon_n$ are elements of $\mathcal{M}(U)$, the field of meromorphic functions on an open connected domain $U \subset \mathbb{C}^m$. Using Theorem 5.2, since

$$\mathrm{tr.deg.}_{\mathcal{C}}\mathcal{C}(\bar{t})\langle \upsilon_1, \ldots, \upsilon_{l+1} \rangle < kn + m,$$

for some $1 \leq i \leq n$, we have that $tr.deg._{\mathbb{C}(\upsilon_i)}\mathbb{C}(\upsilon_i, \upsilon_j) \leq m - 1$.

If we further assume that $(Y, \mathscr{Y})$ has finitely many maximal irreducible $\{e\}$-special subvarieties then from Theorem 5.2 (c) we get that $\upsilon_i \in \mathbb{C}(\upsilon_j)^{alg}$. $\square$

Theorem 5.14. *Assume that $(Y, \mathscr{Y})$ is simple. Then $\mathscr{Y}$ is strongly minimal and geometrically trivial. Furthermore, if we let $(F, \partial_1, \ldots, \partial_m)$ be a differential extension of K and let υ_1, $\upsilon_2 \in \mathscr{Y}$ with υ_1, $\upsilon_2 \notin \mathscr{Y}(F^{alg})$ then if $\upsilon_2 \in F\langle \upsilon_1 \rangle^{alg}$, we have that $\upsilon_2 \in \mathbb{C}(\upsilon_1)^{alg}$.*

Proof. Using Lemma 5.1 and Fact 5.10 (see also remark 5.12) to show that $\mathscr{Y}$ is strongly minimal, all we have to show is that its type over K has U-rank 1. Let $\upsilon \in \mathscr{Y}$ be such that $p = tp(\upsilon/K)$ is the type of $\mathscr{Y}$ over K. As pointed out in Remark 5.12 we have that $\mathrm{tr.deg.}_K K\langle \upsilon \rangle = k$. We need to show that every forking extension of p is algebraic. Suppose that $F = F^{alg}$ is a differential field extension of K such that $q = tp(\upsilon/F)$ is a forking extension of p. Using the Shelah reflection principle we can hence find distinct $\upsilon_1, \ldots, \upsilon_{r+1} \in \mathscr{Y}$, an initial segment in a Morley sequence in q, such that

- $\mathrm{tr.deg.}_K K\langle \upsilon_1, \ldots, \upsilon_r \rangle = k \cdot r$; but
- $\mathrm{tr.deg.}_K K\langle \upsilon_1, \ldots, \upsilon_{r+1} \rangle < k \cdot (r + 1)$.

Using Corollary 5.13, since $\operatorname{tr.deg.}_{\mathbb{C}(\bar{t})}\mathbb{C}(\bar{t})\langle \upsilon_1, \ldots, \upsilon_{r+1}\rangle < k \cdot (r+1)$, for some $1 \leq i \leq r$, we have that $\upsilon_{r+1} \in \mathbb{C}(\upsilon_i)^{alg}$. Note that we use here that $\upsilon_1, \ldots, \upsilon_r$, and derivatives are algebraically independent over K. Hence $\operatorname{tr.deg.}_K K\langle \upsilon_1, \ldots, \upsilon_{r+1}\rangle = k \cdot r$, that is $\upsilon_{r+1} \in K\langle \upsilon_1, \ldots, \upsilon_r\rangle^{alg}$. It hence follows that the only way forking can occur is if q is algebraic which is what we aimed to show.

Finally, given that $\mathscr{Y}$ is strongly minimal, the statement of Corollary 5.13 is precisely geometric triviality. The last implication is a direct consequence of Corollary 5.13 and geometric triviality. □

5.3. **The case of covering maps.** Assume now that $\upsilon : U \to Y$ is a covering map. So we have Γ a Zariski dense subgroup of $G(\mathbb{C})$ with the property that the induced action of Γ on G/B preserves U and υ is a covering map of the complex algebraic variety Y expressing $Y(\mathbb{C})$ as $\Gamma\backslash U$. We also assume that the restriction of υ to some set containing a fundamental domain is definable in an o-minimal expansion of the reals as an ordered field.

Let $\operatorname{Comm}_G(\Gamma)$ be the commensurator of Γ. Recall that by a $\operatorname{Comm}_G(\Gamma)$-correspondence (also known as Hecke correspondence) on $Y(\mathbb{C}) \times Y(\mathbb{C})$ we mean a subset of the form

$$X_g = \{\upsilon(\bar{\tau}) \times \upsilon(g \cdot \bar{\tau}) : \bar{\tau} \in U\}$$

where $g \in \operatorname{Comm}_G(\Gamma)$. It follows that X_g is given by equations $\Phi_g(X, Y) = 0$ for some set Φ_g of polynomials with complex coefficients. So $\Phi_g(\upsilon(\bar{t}), \upsilon(g\bar{t})) = 0$. With this notation, for $g_1, g_2 \in G(\mathbb{C})$ we more generally say that $\upsilon(g_1\bar{t})$ and $\upsilon(g_2\bar{t})$ are in $\operatorname{Comm}_G(\Gamma)$-correspondence if $\Phi_g(\upsilon(g_1\bar{t}), \upsilon(g_2\bar{t})) = 0$ for some $g \in \operatorname{Comm}_G(\Gamma)$. In other words if g_1 and g_2 are in the same coset of $\operatorname{Comm}_G(\Gamma)$. By analogy with the case of curves, the set of polynomials Φ_g is called Γ-special.

We keep the assumptions and notations from the previous subsection: we assume that our universal differential field $\mathbb{U}$ contains elements $t_1, \ldots t_m$ such that $\partial_i t_i = 1$ and $\partial_j t_i = 0$. By abuse of notation, we denote by $\mathscr{Y}$ the set of solution of the $(G, G/B)$ structure for υ and we write K, with $\mathbb{C} \subseteq K \subseteq \mathbb{C}(\bar{t})^{alg}$, for some (any) finitely generated differential field extension of $\mathbb{C}$ over which $\mathscr{Y}$ is defined. We also write $k = \dim G$.

Proposition 5.15. *Let $\upsilon_1, \upsilon_2 \in \mathscr{Y}$ be two distinct solutions. There is an embedding of $K\langle \upsilon_1, \upsilon_2\rangle$ into the field of meromorphic functions on some open connected domain V contained in the fundamental domain of Γ such that $\upsilon_i = \upsilon(g_i\bar{t})$ for some $g_i \in G(\mathbb{C})$. Consequently, if there is a set P of polynomials in $\mathbb{C}[X, Y]$ such that $P(\upsilon_1, \upsilon_2) = 0$ (ie $\upsilon_1 \in \mathbb{C}(\upsilon_2)^{alg}$), then P is Γ-special.*

Proof. We first show that we can write $\upsilon_i = \upsilon(g_i\bar{t})$ for some $g_i \in G(\mathbb{C})$, where $\upsilon : U \to Y$ is the covering map. Let $V \subset U$ be a open connected domain which is properly contained in a fundamental domain of action of Γ. Applying Seidenberg's embedding theorem, we may assume that υ_1, υ_2 have coordinates in $\mathcal{M}(V)$, the field of meromorphic functions on V. It follows that for some functions $\phi_i : V \to U$, we can write $\upsilon_i = \upsilon(\phi_i(\bar{t}))$. Now since υ_i is a solution to $\chi(y) = \tilde{\chi}(\bar{t})$, we have that

$$\tilde{\chi}(\phi_i(\bar{t})) = \chi(\upsilon(\phi_i(\bar{t}))) = \tilde{\chi}(\bar{t}).$$

From $\tilde{\chi}(\phi_i(\bar{t})) = \tilde{\chi}(\bar{t})$ it follows that $\phi_i(\bar{t}) = g_i\bar{t}$ for some $g_i \in G(\mathbb{C})$.

Finally if we assume there is the set P of polynomials in $\mathbb{C}[X,Y]$ such that $P(\upsilon_1,\upsilon_2)=0$, then we have that $P(\upsilon(g_1\bar{t}),\upsilon(g_2\bar{t}))=0$. Standard arguments using double cosets and the commensurator $\mathrm{Comm}_G(\Gamma)$ of Γ (cf. proof of [14, Lemmas 5.15]) shows that g_1 and g_2 are in the same coset of $\mathrm{Comm}_G(\Gamma)$. Hence P is Γ-special. □

Combining the above with Theorem 5.14 we obtain what can be considered a weak form of the Ax-Lindemann-Weierstrass Theorem with derivatives:

Corollary 5.16. *Assume that $(Y,\mathscr{Y})$ is simple. Let $\upsilon_1,\dots,\upsilon_n\in\mathscr{Y}$ are distinct solutions that are not in any $\mathrm{Comm}_G(\Gamma)$-correspondence. Then*

$$tr.deg._K K\langle \upsilon_1,\dots,\upsilon_n\rangle = nk,$$

that is the solutions and their derivatives are algebraically independent over K.

We can use Corollary 5.16 and arithmeticity to give a characterization of ω-categoricity of the pregeometry associated with the solution set $\mathscr{Y}$ generalizing that given in [14]. First we recall the following deep result of Margulis:

Fact 5.17. *Let Γ an irreducible lattice. Then Γ is arithmetic if and only if Γ has infinite index in $\mathrm{Comm}_G(\Gamma)$.*

Corollary 5.18. *Assume that Γ an irreducible lattice and $(Y,\mathscr{Y})$ simple. Then $\mathscr{Y}$ is non-ω-categorical if and only if Γ is arithmetic.*

6. Products of curves

In this section, we pay particular attention to the case $\dim Y = 1$. We aim to show how all the concepts define in Section 5 and Subsection 3.2 can be explicitly derived in this situation. We then apply the relevant results to study the fibers of the Schwarzian equation for Fuchsian groups.

6.1. **Projective structure on curves.** Consider the group $G=\mathrm{PSL}_2(\mathbb{C})$ and its subgroup B of lower triangular matrices so that $G/B=\mathbb{CP}_1$. A $(G,G/B)$-structure on a curve Y is usually called a projective structure (cf.[18]). Let us describe it in the fomalism of Subsection 3.2. We consider Y a complex affine algebraic curve defined by a polynomial equation,

$$P(y,w)=0.$$

Without loss of generality we assume that y is a local coordinate at every point of y, that is, the differential form dy has no zeroes on Y. The algebraic structure of the jet space $J(Y,\mathbb{CP}_1)$ is given by its ring of regular functions and the latter is the $\mathcal{D}_Y$-algebra generated by $\mathcal{O}_Y\otimes\mathcal{O}_{\mathbb{CP}_1}$. Let us take t to be the affine coordinate in $\mathbb{CP}_1$ and write the ring of regular functions on $J(Y,\mathbb{C})\subset J(Y,\mathbb{CP}_1)$ as

$$\mathbb{C}[Y][t,\dot{t},\ddot{t},\dots]$$

The $\mathcal{D}_Y$ differential structure of this ring is given by the action of

$$\frac{d}{dy}=\frac{\partial}{\partial y}+\dot{t}\frac{\partial}{\partial t}+\ddot{t}\frac{\partial}{\partial \dot{t}}+\dddot{t}\frac{\partial}{\partial \ddot{t}}+\cdots.$$

The open subset $J^*(Y,\mathbb{CP}_1)$ is the set of jets of submersive maps. It is defined by the inequation $\dot{t}\neq 0$. Its ring of regular functions is denoted by $\mathcal{O}_{J^*}$. To describe the set

$\mathscr{C} \subset J^*(Y, \mathbb{CP}_1)$ of jets of charts of the projective structure, we need to introduce the Schwarzian derivative with respect to the coordinate y, namely $S_y(t) = \frac{\dddot{t}}{\dot{t}} - \frac{3}{2}\left(\frac{\ddot{t}}{\dot{t}}\right)^2$.

Let R be a rational function on Y and consider in $\mathcal{O}_{J^*}$ the $\frac{d}{dy}$-ideal generated by

$$S_y(t) - 2R.$$

Since $S_y(t_1) = S_y(t_2)$ if and only if $t_1 = \frac{at_2+b}{ct_2+d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$, the local analytic solutions of this equation are charts of a projective structure. The zero set of this differential ideal is $\mathscr{C}$.

Let us give a down to earth description of $\mathscr{C}$ and the induced $\mathscr{Y}$. As the equation has order 3 and is degree 1 in $\dddot{t}$, $\mathscr{C}$ is isomorphic as an algebraic variety to $J_2^*(Y, \mathbb{CP}_1)$ and the $\mathcal{D}_Y$-structure induced on its ring of regular maps $\mathbb{C}[Y][t, \dot{t}, \frac{1}{\dot{t}}, \ddot{t}]$ is given by the above equation

$$\frac{d}{dy} = \frac{\partial}{\partial y} + \dot{t}\frac{\partial}{\partial t} + \ddot{t}\frac{\partial}{\partial \dot{t}} + \left(\frac{3}{2}\frac{\ddot{t}^2}{\dot{t}} + 2R\dot{t}\right)\frac{\partial}{\partial \ddot{t}}.$$

The set of uniformizations $\mathscr{Y} \subset J^*(\mathbb{CP}_1, Y)$ is isomorphic to $J_2^*(\mathbb{CP}_1, Y)$ as an algebraic variety. We also have that the open subsets $J^*(Y, \mathbb{C})$ and $J^*(\mathbb{C}, Y)$ are isomorphic as algebraic varieties. The ring of regular functions on $J^*(\mathbb{C}, Y)$ is

$$\mathbb{C}[t][Y][y', \frac{1}{y'}, y'', \ldots]$$

and the isomorphism $J^*(Y, \mathbb{C}) \simeq J^*(\mathbb{C}, Y)$ is given by usual formula to express the derivation of a reciprocal function, namely $y' = \frac{1}{\dot{t}}$, $y'' = \frac{-\ddot{t}}{\dot{t}^3}$, ...

But now notice that as pro-algebraic varieties $J^*(Y, \mathbb{CP}_1) \simeq J^*(\mathbb{CP}_1, Y)$ and under this isomorphism $\mathscr{C}$ and $\mathscr{Y}$ coincide. Moreover, the differential structures are not the same as on $J^*(\mathbb{CP}_1, Y)$ the differential structure of the srtuctural ring is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y'} + \cdots.$$

The subset $\mathscr{Y}$ is the zero set of the differential ideal generated by $S_t(y) + 2Ry'^2 = 0$. As already mentioned $\mathscr{Y}$ is isomorphic to $J_2^*(Y, \mathbb{CP}_1)$. The $\frac{d}{dt}$ -differential structure induced on its ring of regular maps $\mathbb{C}[t][Y][y', \frac{1}{y'}, y'']$ is given by the above equation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \left(\frac{3}{2}\frac{y''^2}{y'} - 2Ry'^3\right)\frac{\partial}{\partial y''}.$$

To describe the connection form observe that the choice of our coordinates on open subset $J_2^*(Y, \mathbb{C})$ induces a trivialization,

$$Y \times \mathrm{PSL}_2(\mathbb{C}) \to J_*^2(Y, \mathbb{CP}_1), \quad \left(y, w, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \mapsto (y, w, t, \dot{t}, \ddot{t})$$

where $t = -b/a$, $\dot{t} = 1/a^2$, $\ddot{t} = -2c/a^3$, or equivalently $y' = a^2$, $y'' = 2ca^3$. We see, by direct substitution that the linear matrix differential equation in $Y \times \mathrm{PSL}_2(\mathbb{C})$,

$$\frac{dU}{dy} = A(y,w)U \quad \text{where} \quad A(y,w) = \begin{bmatrix} 0 & 1 \\ R(y,w) & 0 \end{bmatrix} \tag{6.1}$$

and $R(y,w)$ is a rational function in Y is equivalent to,

$$S_y(t) - 2R(y,w) = 0, \quad S_t(y) + 2R(y,w)y'^2 = 0$$

in the corresponding systems of coordinates on $J_2^*(Y,\mathbb{C})$. Let us define the matrix-valued rational 1-form on $J_2^*(Y,\mathbb{CP}_1)$:

$$\Omega = U^{-1}dU - U^{-1}A(y,w)U dy$$

This 1-form Ω is the *connection form* of the differential equation. It has the following properties (some of which were already pointed out in Subsection 2.4):

(a) The kernel of Ω is the $\mathrm{PSL}_2(\mathbb{C})$-connection $\mathcal{F}$ tangent to the graphs of solutions of the Schwarzian differential equation.
(b) Ω takes values in $\mathfrak{sl}_2(\mathbb{C})$.
(c) If X is the infinitesimal generator of a monoparametric group of right translations $\{R_{\exp(\varepsilon B)} : \varepsilon \in \mathbb{C}\}$ for certain $B \in \mathfrak{sl}_2(\mathbb{C})$ then $\Omega(X) = B$.
(d) Ω is adj-equivariant $R_g^*(\Omega) = \mathrm{Adj}_g^{-1} \circ \Omega$.
(e) $d\Omega + \frac{1}{2}[\Omega,\Omega] = 0$.

6.2. **Algebraic relations between solutions of n Schwarzian equations.** Let us consider $Y_1,\ldots,Y_n$ affine algebraic curves, and for each Y_i the bundle $J_i = J_2^*(\mathbb{CP}_1, Y_i)$. The product $\tilde{J} = J_1 \times \ldots \times J_n$ is a $(\mathrm{PSL}_2(\mathbb{C}))^n$-bundle over the product $\tilde{Y} = Y_1 \times \ldots \times Y_n$. Let us consider n Schwarzian equations,

$$S_{t_i} y_i + 2R_i(y_i,w_i)y_i'^2 = 0.$$

Each one is seen as a $\mathrm{PSL}_2(\mathbb{C})$-invariant connection ∇_i in J_i over Y_i with connection form Ω_i. We consider $\tilde{\nabla}$ the product $\tilde{\nabla} = \nabla_1 \times \ldots \times \nabla_n$ which is a $(\mathrm{PSL}_2(\mathbb{C}))^n$ invariant connection on $\tilde{J}$ over $\tilde{Y}$.

We are now ready to state the relevant version of Theorem 5.2. Notice that, since we are working on curves, the factors Y_i have no $\{e\}$-special subvarieties.

Theorem 6.1. *Let (Y_i,∇_i) be algebraic curves with simple $(\mathrm{PSL}_2(\mathbb{C}),\mathbb{CP}_1)$-structures. Assume that*

$$\hat{\mathcal{V}} \colon \mathrm{Spf}\mathbb{C}[[s_1,\ldots,s_k]] \to \tilde{J}$$

is a non componentwise constant formal parameterized space in a horizontal leaf of $\tilde{\nabla}$, and let V be the Zariski closure of $\hat{\mathcal{V}}$. The following are equivalent:

(a) $\dim V < 3n + \mathrm{rank}(\ker \Omega|_V)$.
(b) *There are two different indices $1 \le i < j \le n$, a curve $X_{ij} \subset Y_i \times Y_j$ with both projections dominant, and a $\mathrm{PSL}_2(\mathbb{C})$-bundle isomorphism between $f_i^*(J_i)$ and $f_j^*(J_j)$ such that $f_i^*\Omega_i = f_j^*\Omega_j$.*

In particular, under the hypothesis of Theorem 6.1, the coordinates y_i and y_j are algebraically dependent on X_{ij}. In other words it follows that if υ_i and υ_j be solutions of the corresponding equations then $\upsilon_i(t_i(s))$ and $\upsilon_j(t_j(s))$ are algebraically dependent over $\mathbb{C}$.

Note that the rank of $\ker \Omega|_V$ is at least the dimension of the smallest analytic subvariety containing $\hat{\mathcal{V}}$ and thus is greater or equal to the rank of the jacobian of our n formal power series in k variables. A more precise description of the possible dimension of a non-trivial intersection of V with a leaf is as follows

Corollary 6.2. *Let* (Y_1, ∇_1), (Y_2, ∇_2) *be algebraic curves with simple* $(\mathrm{PSL}_2(\mathbb{C}), \mathbb{CP}_1)$*-structures. If there exist an horizontal leaf* $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \subset J_1 \times J_2$ *and an algebraic subvariety* $V \subset J_1 \times J_2$ *such that* $\dim V \cap \mathcal{L} > 0$ *and such that* V *is the Zariski closure of a positive dimensional irreducible component of* $V \cap \mathcal{L}$*, then* $\dim V$ *is* 4*,* 7 *or* 8*.*

Proof. By lemma 3.5, the rank of $\ker \Omega|_V$ can be 1 or 2. From the above theorem, if $\dim V < 7$, then V is the graph of a gauge correspondence between ∇_1 and ∇_2 thus $\dim V = \dim J_1 = \dim J_2 = 4$. □

6.3. **Orthogonality of fibers.** We can now use Theorem 5.14 and Theorem 6.1 to prove several key results that were obtained in [14] in the special case of hyperbolic curves of genus 0. Our focus remains on the Schwarzian equation

$$S_t(y) + 2R(y, w)y'^2 = 0 \tag{$\star$}$$

attached to a complex affine algebraic curve Y. So we now work in the context of 1-DCF_0, i.e., $\mathbb{U}$ is an ordinary differentially closed field. As in Section 4, we assume that $\mathbb{U}$ contains an element t such that $t' = 1$ (so $\mathbb{U}$ contains the differential field $\mathbb{C}(t)$).

We assume for the remainder of this section that Y is a *hyperbolic curve.* More precisely, let $\hat{Y}$ be a smooth projective completion of the affine curve Y. We assume that we have a Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ of the first kind, such that if C_Γ denotes the set of cusps of Γ and $\mathbb{H}_\Gamma := \mathbb{H} \cup C_\Gamma$, then there is a meromorphic mapping $j_\Gamma : \mathbb{H}_\Gamma \to \hat{Y}(\mathbb{C})$. The map j_Γ is called a *uniformizer* and is an automorphic function for Γ

$$j_\Gamma(g\tau) = j_\Gamma(\tau) \quad \text{for all } \; g \in \Gamma \text{ and } \tau \in \mathbb{H}$$

and so factorizes in a bi-rational isomorphism of $\Gamma \setminus \mathbb{H}_\Gamma$ into $\hat{Y}(\mathbb{C})$. We have that j_Γ is a solution of the Schwarzian equation $(\star)$ for some rational function R.

Using [28, Corollary B.1], we have Γ is Zariski dense in $\mathrm{Gal}(\nabla)$, for the corresponding connection ∇. So $\mathrm{Gal}(\nabla) = G = \mathrm{PSL}_2(\mathbb{C})$. We can hence apply Theorem 5.14, to conclude that the set defined by equation $(\star)$ is strongly minimal and satisfies the refined version of geometric triviality. We also have that equation $(\star)$ satisfies the weak for of the Ax-Lindemann-Weierstrass Theorem given in Corollary 5.16. We now aim to recover all the remaining main theorems from the paper [14].

For $a \in \mathbb{U}$, by a fiber of the Schwarzian equation $(\star)$ for a uniformizer j_Γ of a Fuchsian group Γ of the first kind, we mean an equation of the form

$$\chi_{\Gamma, \frac{d}{dt}}(y) = a,$$

where

$$\chi_{\Gamma, \frac{d}{dt}}(y) := S_t(y) + (y')^2 R_{j_\Gamma}(y, w). \tag{6.2}$$

The proof of the following result is identical to that of Theorems 6.2 in [14] (see also [20, Proposition 5.2]).

Proposition 6.3. *The set defined by $\chi_{\Gamma,\frac{d}{dt}}(y) = a$, with $a \in \mathbb{U}$, is strongly minimal and geometrically trivial. If $a_1, \dots, a_n$ satisfy $\chi_{\Gamma,\frac{d}{dt}}(a_i) = a$ and are dependent, then there exist $i, j \leq n$ and a Γ-special polynomial P such that $P(a_i, a_j) = 0$.*

For the next results, we will need some more notions from model theory.

Definition 6.4. Let $\mathscr{X}$ and $\mathscr{Y}$ be two strongly minimal sets both defined over some differential field $K \subset \mathcal{U}$.

1. $\mathscr{X}$ and $\mathscr{Y}$ are *nonorthogonal* if there is some definable (possibly with additional parameters) relation $\mathscr{R} \subset \mathscr{X} \times \mathscr{Y}$ such that the images of the projections of $\mathscr{R}$ to $\mathscr{X}$ and $\mathscr{Y}$ respectively are infinite and these projections are finite-to-one.
2. $\mathscr{X}$ and $\mathscr{Y}$ are *non weakly orthogonal* if they are nonorthogonal, that is there is an infinite finite-to-finite relation $\mathscr{R} \subseteq \mathscr{X} \times \mathscr{Y}$, and the formula defining $\mathscr{R}$ can be chosen to be over K^{alg}.

Remark 6.5. Suppose $\mathscr{X}$ and $\mathscr{Y}$ (as above) are nonorthogonal and that the relation $\mathscr{R} \subset \mathscr{X} \times \mathscr{Y}$ witnessing nonorthogonality is defined over some differential field F extending K. Then by definition for any $v \in \mathscr{X} \setminus F^{alg}$ there exist $v_1 \in \mathscr{Y} \setminus F^{alg}$ such that $(v, v_1) \in \mathcal{R}$. In that case $F\langle v\rangle^{alg} = F\langle v_1\rangle^{alg}$, that is v, v_1 and derivatives are algebraically dependent over F.

We will need the following important fact. We restrict ourselves to geometrically trivial strongly minimal sets as this is all we need for the Schwarzian equations. We direct the reader to [33, Corollary 2.5.5] for the more general context.

Fact 6.6. *Let $\mathscr{X}$ and $\mathscr{Y}$ be strongly minimal sets both defined over some differential field K. Assume further that they are both geometrically trivial. If $\mathscr{X}$ and $\mathscr{Y}$ are nonorthogonal, then they are non weakly orthogonal.*

Let Γ_1 and Γ_2 be two Fuchsian groups. We say that Γ_1 is commensurable with Γ_2 in *wide sense* if Γ_1 is commensurable to some conjugate of Γ_2. We will now answer a question that was left open in [14], where only the case $a = b = 0$ (and genera 0) was established.

Theorem 6.7. *Let Γ_1 and Γ_2 be two Fuchsian groups of the first kind. Assume further that Γ_1 is not commensurable with Γ_2 in the wide sense. For $a, b \in \mathbb{U}$, the strongly minimal sets defined by equations $\chi_{\Gamma_1,\frac{d}{dt}}(y) = a$ and by $\chi_{\Gamma_2,\frac{d}{dt}}(y) = b$ are orthogonal.*

We will need the following lemma, a weaker form of which is Theorem 6.3 of [14]. We use the notation $\chi^{-1}_{\Gamma,\frac{d}{dt}}(a)$ for the set defined by $\chi_{\Gamma,\frac{d}{dt}}(y) = a$.

Lemma 6.8. *Let Γ_1 and Γ_2 be two Fuchsian groups of the first kind. For $a \neq b$, the strongly minimal sets $\chi^{-1}_{\Gamma_1,\frac{d}{dt}}(a)$ and $\chi^{-1}_{\Gamma_2,\frac{d}{dt}}(b)$ are orthogonal.*

Proof. Throughout, we respectively use $\mathcal{M}(U)$ and $\mathbb{D}(p, r)$ for the field of meromorphic functions on a domain $U \subset \mathbb{C}$, and the open complex disk centered at $p \in \mathbb{C}$ with radius r. As both $\chi^{-1}_{\Gamma_1,\frac{d}{dt}}(a)$ and $\chi^{-1}_{\Gamma_2,\frac{d}{dt}}(b)$ are strongly minimal and geometrically trivial, if

$\chi^{-1}_{\Gamma_1,\frac{d}{dt}}(a) \not\perp \chi^{-1}_{\Gamma_2,\frac{d}{dt}}(b)$, then there is a finite-to-finite correspondence between the sets, defined over $\mathbb{Q}\langle a,b\rangle$. Using Seidenberg's embedding theorem, we regard a, b as meromorphic functions on a domain $U \subset \mathbb{C}$. In what follows $\tilde{a}$ denotes a meromorphic function such that $S_{\frac{d}{dt}}(\tilde{a}) = a$. The function $\tilde{b}$ is defined similarly.

Using the holomorphic inverse function theorem, we claim that without loss of generality, it is enough to prove the result for the case $a = 0$. Indeed, since $j_{\Gamma_1}(\tilde{a}(t))$ is interalgebraic with $j_{\Gamma_2}(g\tilde{b}(t))$ for some $g \in \mathrm{GL}_2(\mathbb{C})$, we have that $j_{\Gamma_1}(t)$ is interalgebraic with $j_{\Gamma_2}(g\tilde{b}(\tilde{a}^{-1}(t))$ (since $\tilde{b}$ is defined up to composition with linear fractional transformations, we can assume that there is a common regular point for $\tilde{a}$ and $\tilde{b}$ and work locally around this point). Letting $\tilde{c} = \tilde{b} \circ \tilde{a}^{-1}$ and $c = S_{\frac{d}{dt}}(\tilde{c})$, we see that $\chi^{-1}_{\Gamma_1,\frac{d}{dt}}(0) \not\perp \chi^{-1}_{\Gamma_2,\frac{d}{dt}}(c)$ and by geometric triviality this occurs over $\mathbb{Q}\langle c\rangle$.

So we assume that $a = 0$. Let p be a regular point for $\tilde{b}(t)$ and let $\mathbb{D}_1 = \mathbb{D}(p, \epsilon)$ be a disc of regular points of $\tilde{b}(t)$. Also let γ be a linear fractional transformation sending $\mathbb{D}_2 = \mathbb{D}(p, \frac{1}{2}\epsilon)$ to $\mathbb{H}$.

Since $\chi^{-1}_{\Gamma_1,\frac{d}{dt}}(0) \not\perp \chi^{-1}_{\Gamma_2,\frac{d}{dt}}(b)$, we have that for some $g \in \mathrm{GL}_2(\mathbb{C})$, the solution $j_{\Gamma_2}(g\tilde{b}(t))$ is algebraic over $\mathbb{Q}\langle b, j_{\Gamma_1}(\gamma t)\rangle \subset \mathcal{M}(\mathbb{D}_1)(j_{\Gamma_1} \circ \gamma, j'_{\Gamma_1} \circ \gamma, j''_{\Gamma_1} \circ \gamma) \subset \mathcal{M}(\mathbb{D}_2)$. But notice that for any domain U such that $\mathbb{D}_2 \subseteq U \subseteq \mathbb{D}_1$, if $j_{\Gamma_2}(g\tilde{b}(t))$ is algebraic over $\mathcal{M}(U)$, then $j_{\Gamma_1}(\gamma t)$ will also be algebraic over $\mathcal{M}(U)$. This follows from the fact that $\mathcal{M}(\mathbb{D}_1) \subseteq \mathcal{M}(U)$, and $j_{\Gamma_2}(g\tilde{b}(t))$ is interalgebraic with $j_{\Gamma_1}(\gamma t)$ over $\mathbb{Q}\langle b\rangle \subset \mathcal{M}(\mathbb{D}_1)$. But $j_{\Gamma_1}(t)$ cannot be extended algebraically on a neighborhood of $\mathbb{H}$, hence $U = \mathbb{D}_2$.

The disc $\mathbb{D}_2$ is thus the maximal among domains U such that $j_{\Gamma_2}(g\tilde{b}(t))$ is algebraic over $\mathcal{M}(U)$. But such a domain satisfies $g\tilde{b}(\mathbb{D}_2) = \mathbb{H}$, that is the image of $\mathbb{D}_2$ by the regular holomorphic map $\tilde{b}$ is the disc $g^{-1}\mathbb{H}$. A corollary of Schwarz's lemma gives that biholomorphisms from a disc to a disc are restrictions of homographies. Hence $\tilde{b}$ is an homography $h \in \mathrm{PSL}_2(\mathbb{C})$ and so $b = 0$. $\square$

Proof of Theorem 6.3. By Lemma 6.8, we may assume that $a = b$ and so we need to classify non-orthogonalities between the strongly minimal sets defined by $\chi_{\Gamma_1,\frac{d}{dt}}(y) = a$ and by $\chi_{\Gamma_2,\frac{d}{dt}}(y) = a$. As easily verified (cf. [14, Lemma 6.1]), if we let $K = \mathbb{Q}\langle a\rangle$ and let

$$\partial = \frac{1}{\tilde{a}'}\frac{d}{dt},$$

then we have that K is a ∂-differential field and for each $i = 1, 2$, the sets $\chi_{\Gamma_i,\frac{d}{dt}}(y) = a$ and $\chi_{\Gamma_i,\partial}(y) = 0$ coincide. Hence $\chi_{\Gamma_1,\frac{d}{dt}}(y) = a$ is non-orthogonal to by $\chi_{\Gamma_2,\frac{d}{dt}}(y) = a$ if and only if $\chi_{\Gamma_1,\partial}(y) = 0$ is non-orthogonal to $\chi_{\Gamma_2,\partial}(y) = 0$. We can apply [14, Theorem 6.5], which state that if Γ_1 is not commensurable with Γ_2 in the wide sense, then $\chi_{\Gamma_1,\partial}(y) = 0$ and $\chi_{\Gamma_2,\partial}(y) = 0$ are orthogonal. $\square$

Finally, without giving further details, let us mention that one can also establish the Ax-Lindemann-Weierstrass Theorem with derivatives for j_Γ using the same arguments from [14]. However, this also follows from the Ax-Schanuel theorem below. One can also follow the strategy given in [14] to obtain a special case of the André-Pink conjecture for j_Γ.

7. Proof of the Ax-Schanuel Theorems for products of curves.

In this section, we prove two instances of the Ax-Schanuel Theorem with derivatives. First we tackle the case of products of hyperbolic curves, a direct generalization of the work [31] and [27] in the case of curves. We then prove the Ax-Schanuel Theorem with derivatives for non-hyperbolic curves given from "generic triangle groups".

Let us state the general problem for a Schwarzian equation. Let υ denote a solution of a Schwarzian equation

$$(\star) \qquad S_t(y) + 2R(y,w)y'^2 = 0$$

attached to a complex affine algebraic curve Y and such that $\mathrm{Gal}(\nabla) = \mathrm{PSL}_2(\mathbb{C})$. Notice that we take $\upsilon : U \to Y$ to be a uniformization function as defined in Subsection 3.2. Let $\hat{t}_1, \dots, \hat{t}_n$ be formal parameterizations of neighborhoods of points $p_1, \dots p_n$ in U. We write δ_i for the derivations induced by differentiation with respect to $\hat{t}_i$. The Ax-Schanuel Theorem with derivatives for υ is an answer to the following problem.

Problem. *Fully characterize the conditions on $\hat{t}_1, \dots, \hat{t}_n$ for which*

$$\mathrm{tr.deg.}_{\mathbb{C}}\mathbb{C}(\hat{t}_1, \upsilon(\hat{t}_1), \upsilon'(\hat{t}_1), \upsilon''(\hat{t}_1), \dots, \hat{t}_n, \upsilon(\hat{t}_n), \upsilon'(\hat{t}_n), \upsilon''(\hat{t}_n)) < 3n + \mathrm{rank}(\delta_i \hat{t}_j).$$

Using Theorem 6.1 (which holds since $\mathrm{Gal}(\nabla) = \mathrm{PSL}_2(\mathbb{C})$) with $Y = Y_i$ and $\upsilon = \upsilon_i$ for $i = 1, \dots, n$, to prove the Ax-Schanuel Theorem with derivatives we only need to characterize the conditions on pairs $\hat{t}_1, \hat{t}_2$. Moreover, from Corollary 6.2, if $\hat{t}_1, \hat{t}_2$ are nonconstant and

$$\mathrm{tr.deg.}_{\mathbb{C}}\left(\mathbb{C}(\hat{t}_1, \hat{t}_2, \upsilon(\hat{t}_1), \upsilon'(\hat{t}_1), \upsilon''(\hat{t}_1), \upsilon(\hat{t}_2), \upsilon'(\hat{t}_2), \upsilon''(\hat{t}_2)\right) = \ell \neq 7 \text{ or } 8,$$

then it must be that $\ell = 4$ and there is a polynomial $P \in \mathbb{C}[x,y]$ such that $P(\upsilon(\hat{t}_1), \upsilon(\hat{t}_2)) = 0$.

7.1. The case of hyperbolic curves.

We prove the Ax-Schanuel Theorem with derivatives for all hyperbolic curves. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind and let j_Γ be a uniformizing function for Γ.

Theorem 7.1. *Let $\hat{t}_1, \dots, \hat{t}_n$ be formal parameterizations of neighborhoods of points $p_1, \dots p_n$ in $\mathbb{H}$. Assume that $\hat{t}_1, \dots, \hat{t}_n$ are geodesically independent, namely $\hat{t}_i$ is nonconstant for $i = 1, \dots, n$ and there are no relations of the form $\hat{t}_i = \gamma \hat{t}_j$ for $i \neq j$, $i,j \in \{1, \dots, n\}$ and γ is an element of Comm(Γ), the commensurator of Γ. Then*

$$\mathrm{tr.deg.}_{\mathbb{C}}\mathbb{C}(\hat{t}_1, j_\Gamma(\hat{t}_1), j'_\Gamma(\hat{t}_1), j''_\Gamma(\hat{t}_1), \dots, \hat{t}_n, j_\Gamma(\hat{t}_n), j'_\Gamma(\hat{t}_n), j''_\Gamma(\hat{t}_n)) \geq 3n + \mathrm{rank}(\delta_i \hat{t}_j).$$

Theorem 7.1 follows from the following lemma.

Lemma 7.2. *Assume that*

$$\mathrm{tr.deg.}_{\mathbb{C}}\left(\mathbb{C}(\hat{t}_1, \hat{t}_2, j_\Gamma(\hat{t}_1), j'_\Gamma(\hat{t}_1), j''_\Gamma(\hat{t}_1), j_\Gamma(\hat{t}_2), j'_\Gamma(\hat{t}_2), j''_\Gamma(\hat{t}_2)\right) = 4,$$

then the polynomial P such that $P(j(\hat{t}_1), j(\hat{t}_2)) = 0$ is a Γ-special polynomial (and so $\hat{t}_1$ and $\hat{t}_2$ are geodesically dependent).

Proof. To simplify notation we write $j = j_\Gamma$ and t_i instead of $\hat{t}_i$. Consider the derivation δ_2 and let us write $\chi_{\Gamma,\delta_2}(x) = a$, where $a \in \mathbb{C}\langle t_1 \rangle$ for the Schwarzian equation of $j(t_1)$ (so $\chi_{\Gamma,\delta_2}(x) = 0$ is the equation of $j(t_2)$).

Claim: $j(t_1), j(t_2) \notin \mathbb{C}\langle t_1\rangle^{alg}$.

Proof of Claim. Assume for some i, $j(t_i) \in \mathbb{C}\langle t_1\rangle^{alg}$. Then for both $i = 1, 2$ we have that $j(t_i) \in \mathbb{C}\langle t_1\rangle^{alg}$; this follows because $j(t_1)$ and $j(t_2)$ are interalgebraic over $\mathbb{C}$. Now, note that we have that $\text{tr.deg.}_{\mathbb{C}}(\mathbb{C}\langle t_2, j(t_2)\rangle) = 4$. Using the assumption $\ell = 4$ we get that $t_1 \in \mathbb{C}\langle t_2, j(t_2)\rangle^{alg}$ (of course so is $j(t_1)$). Hence, it must be the case that $\mathbb{C}\langle t_1, t_2\rangle^{alg} = \mathbb{C}\langle t_2, j(t_2)\rangle^{alg} = \mathbb{C}\langle t_2, j(t_1)\rangle^{alg}$. The last equality is obtained from the interalgebraicity of $j(t_1)$ and $j(t_2)$ over $\mathbb{C}$ and from the fact that $j(t_2) \notin \mathbb{C}(t_2)^{alg}$. Hence we have that $j(t_2)$ and t_1 are interalgebraic over the field $\mathbb{C}(t_2)$ and that $\text{tr.deg.}_{\mathbb{C}}(\mathbb{C}\langle t_1, t_2\rangle) = 4$.
But notice that t_1, $j(t_1)$, $j'(t_1)$, $j''(t_1)$ are algebraically independent over $\mathbb{C}$. So, the assumption $j(t_1) \in \mathbb{C}\langle t_1\rangle^{alg}$ and $tr.deg._{\mathbb{C}}(\mathbb{C}\langle t_1, t_2\rangle) = 4$ gives that $\text{tr.deg.}_{\mathbb{C}}(\mathbb{C}\langle t_1\rangle) = 4$ and that $t_2 \in \mathbb{C}\langle t_1\rangle^{alg}$. But as observed above, we have that t_1 is interalgebraic with $j(t_2)$ over $\mathbb{C}(t_2)$. It hence follows that t_1 is interalgebraic with $j(t_2)$ over $\mathbb{C}$, since $t_2 \in \mathbb{C}\langle t_1\rangle^{alg}$. This contradicts the fact that $\text{tr.deg.}_{\mathbb{C}}(\mathbb{C}\langle j(t_2)\rangle) = 3$. □

So from the claim and strong minimality of the two equations, we have that

$$\text{tr.deg.}_{\mathbb{C}\langle t_1\rangle}(\mathbb{C}\langle j(t_i)\rangle) = 3$$

for both $i = 1$ and $i = 2$. That is $j(t_1)$ and $j(t_2)$ are generic (over $\mathbb{C}\langle t_1\rangle$) solutions of their respective equations. Since $P(j(t_1), j(t_2)) = 0$, we get that generic solutions of $\chi_{\Gamma,\delta_2}(x) = a$ and $\chi_{\Gamma,\delta_2}(x) = 0$ are interalgebraic over $\mathbb{C}\langle t_1\rangle$. So the two equations are non-orthogonal. Using Lemma 6.8 with $\Gamma_1 = \Gamma_2 = \Gamma$ we get that $a = 0$ and from Corollary 5.16 we have that P is a $Comm(\Gamma)$-correspondence. □

We hence also answer positively a question of Aslanyan [2, Section 3.4] about the existence of differential equations satisfying the Ax-Schanuel Theorem with derivatives and such that the polynomial $(X - Y)$ is the only Γ-special polynomial. Indeed, this is true of any Γ satisfying $\Gamma = \text{Comm}(\Gamma)$. Many such examples exist (cf. [12, Fact 4.9]).

7.2. **The case of generic triangle groups.** We will now exploit the fact that the proof of Lemma 7.2 only depends on the following:

(1) Theorem 6.1 and Corollary 6.2 hold for $\chi_{\Gamma,\frac{d}{dt}}(y) = 0$, i.e., $\text{Gal}(\nabla) = \text{PSL}_2(\mathbb{C})$;
(2) There is a full characterization of the structure of the set defined by any fiber of the Schwarzian equation $\chi_{\Gamma,\frac{d}{dt}}(y) = a$; and
(3) If $a \neq b$, then the set defined by $\chi_{\Gamma,\frac{d}{dt}}(y) = a$ and $\chi_{\Gamma,\frac{d}{dt}}(y) = b$ are orthogonal.

We can hence apply this technique to any context where the above three properties have been established. As it turns out, in [12], property 1 and 2 was established for generic triangle groups. We now recall this setting and show that property 3 also holds and hence obtain the Ax-Schanuel with derivatives.

Let $\triangle \subset \mathbb{H}$ be an open circular triangle with vertices v_1, v_2, v_3 and with respective internal angles $\frac{\pi}{\alpha}$, $\frac{\pi}{\beta}$ and $\frac{\pi}{\gamma}$. Using the Riemann mapping theorem, we have a unique biholomorphic mapping $J : \triangle \to \mathbb{H}$ sending the vertices v_1, v_2, v_3 to $\infty, 0, 1$ respectively. We can also extend $J(t)$ to a homeomorphism from the closure of $\triangle$ onto $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbf{P}^1(\mathbb{R})$. The function

$J(t)$ is called a *Schwarz triangle function* and is a uniformization function in the sense of Subsection 3.2. It satisfies the Schwarzian equation $(\star)$

$$R(y,w)=R_{\alpha,\beta,\gamma}(y)=\frac{1}{2}\left(\frac{1-\beta^{-2}}{y^2}+\frac{1-\gamma^{-2}}{(y-1)^2}+\frac{\beta^{-2}+\gamma^{-2}-\alpha^{-2}-1}{y(y-1)}\right).$$

We call this equation a Schwarzian triangle equation and write it as $\chi_{\alpha,\beta,\gamma,\frac{d}{dt}}(y)=0$. By a generic such equation we mean the case when $\alpha,\beta,\gamma\in\mathbb{R}_{>1}$ are algebraically independent over $\mathbb{Q}$.

Remark 7.3. In [12], the parameters α,β,γ where allowed to be arbitrary complex numbers. However, to apply Theorem 6.1 (curve case), we require that $\alpha,\beta,\gamma\in\mathbb{R}_{>1}$.

Let us now state what is known about generic Schwarzian triangle equations. The proof can be found in [12, Section 4]. We denote $\chi_{\alpha,\beta,\gamma,\frac{d}{dt}}(y)=a$ for the fiber equations.

Fact 7.4. *Let $\alpha,\beta,\gamma\in\mathbb{R}$ be algebraically independent over $\mathbb{Q}$ and let $a\in\mathbb{U}$ be arbitrary. Then*

(1) *The Galois group $Gal(\nabla)$ for the corresponding connection ∇ is* $\mathrm{PSL}_2(\mathbb{C})$.
(2) *The set defined by $\chi_{\alpha,\beta,\gamma,\frac{d}{dt}}(y)=a$ is strongly minimal, geometrically trivial and strictly disintegrated. Namely, if K is any differential field extension of $\mathbb{Q}(\alpha,\beta,\gamma)\langle a\rangle$ and $y_1,\dots,y_n$ are distinct solutions that are not algebraic over K, then*
$$\mathrm{tr.deg.}_{\mathrm{K}}\mathrm{K}(y_1,y_1',y_1'',\dots,y_n,y_n',y_n'')=3n$$
.

We only need to prove orthogonalities of the distinct fiber equations.

Proposition 7.5. *Let $\alpha,\beta,\gamma\in\mathbb{R}$ be algebraically independent over $\mathbb{Q}$. Let $a,b\in\mathcal{U}$ be such that $a\neq b$. Then the strongly minimal sets defined by $\chi_{\alpha,\beta,\gamma,\frac{d}{dt}}(y)=a$ and $\chi_{\alpha,\beta,\gamma,\frac{d}{dt}}(y)=b$ are orthogonal.*

Proof. Assume that $\alpha,\beta,\gamma\in\mathbb{R}$ are algebraically independent over $\mathbb{Q}$. We denote by C the field of constants generated by α,β,γ over $\mathbb{Q}$, that is $C=\mathbb{Q}(\alpha,\beta,\gamma)$. For $u,v,w,d\in\mathcal{U}$, we denote by $\mathscr{X}(u,v,w,d)$ the set defined by a fiber equation $\chi_{u,v,w,\frac{d}{dt}}(y)=d$ and $u'=v'=w'=0$. Notice that is a uniformly defined family.

Arguing by contradiction, assume that $\mathscr{X}(\alpha,\beta,\gamma,a)$ and $\mathscr{X}(\alpha,\beta,\gamma,b)$ are non-orthogonal. Since the sets are geometrically trivial, we have that they are non-weakly orthogonal. So we have a definable relation $\mathscr{R}\subset\mathscr{X}(\alpha,\beta,\gamma,a)\times\mathscr{X}(\alpha,\beta,\gamma,b)$ defined over $C\langle a,b\rangle^{alg}$. Let $\mathscr{R}_a$ and $\mathscr{R}_b$ be the respective projections of $\mathscr{R}$ to $\mathscr{X}(\alpha,\beta,\gamma,a)$ and $\mathscr{X}(\alpha,\beta,\gamma,b)$. By definition, $\mathscr{R}_a$ and $\mathscr{R}_b$ are finite sets defined over $C\langle a,b\rangle^{alg}$. We define $\mathscr{Y}(\alpha,\beta,\gamma,a)$ to be the complement $\mathscr{X}(\alpha,\beta,\gamma,a)\setminus\mathscr{R}_a$. The set $\mathscr{Y}(\alpha,\beta,\gamma,b)$ is defined similarly.

Then it is not hard to see that we have an $\mathcal{L}_1$-formula $\theta(u_1,u_2,u_3)$ over $\mathbb{Q}$ such that $\theta(\alpha,\beta,\gamma)$ is the sentence asserting that

$$\exists a,b\;\; a\neq b\,(\forall x,y\,(x\in\mathscr{Y}(\alpha,\beta,\gamma,a)\wedge y\in\mathscr{Y}(\alpha,\beta,\gamma,b))\implies(x,y)\in\mathscr{R}).$$

It follows by construction that if $\theta(\alpha_1,\beta_1,\gamma_1)$ holds in $\mathcal{U}$, that is $\mathcal{U}\models\theta(\alpha_1,\beta_1,\gamma_1)$, then $\alpha_1,\beta_1,\gamma_1\in\mathbb{C}$ and there is $a_1,b_1\in\mathcal{U}$ such that $a_1\neq b_1$ and the definable sets $\mathscr{X}(\alpha_1,\beta_1,\gamma_1,a_1)$

and $\mathscr{X}(\alpha_1, \beta_1, \gamma_1, b_1)$ are non-orthogonal. Using the Fact [12, Fact 2.2] with $V = \mathbb{A}^3$ and $F = \mathbb{Q}$, since α, β, γ are algebraically independent over $\mathbb{Q}$, we can find $k, l, m \in \mathbb{N}$ such that $2 \leq k \leq l \leq m$ and $\mathcal{U} \models \theta(k, l, m)$. So there is $a_1, b_1 \in \mathcal{U}$ such that $a_1 \neq b_1$ and the definable sets $\mathscr{X}(k, l, m, a_1)$ and $\mathscr{X}(k, l, m, b_1)$ are non-orthogonal. But for any $d \in \mathcal{U}$, the set $\mathscr{X}(k, l, m, d)$ is the set defined by the fiber of the Schwarzian equation $(\star)$ for the uniformizer of the Fuchsian (triangle) group $\Gamma_{(k,l,m)}$ with signature $(0; k, l, m)$. Hence this contradicts Theorem 6.7 with $\Gamma_1 = \Gamma_2 = \Gamma_{(k,l,m)}$. □

We now assume that $\alpha, \beta, \gamma \in \mathbb{R}_{>1}$ are algebraically independent over $\mathbb{Q}$ and let $J : \triangle \to \mathbb{H}$ be the corresponding Schwarz triangle function as describe above.

Theorem 7.6. *Let $\hat{t}_1, \ldots, \hat{t}_n$ be distinct formal parameterizations of neighborhoods of points $p_1, \ldots p_n$ in $\triangle$. Then*

$$\text{tr.deg.}_{\mathbb{C}}\mathbb{C}(\hat{t}_1, J(\hat{t}_1), J'(\hat{t}_1), J''(\hat{t}_1), \ldots, \hat{t}_n, J(\hat{t}_n), J'(\hat{t}_n), J''(\hat{t}_n)) \geq 3n + \text{rank}(\delta_i \hat{t}_j).$$

Theorem 7.6 follows from the following lemma.

Lemma 7.7. *Assume that*

$$\text{tr.deg.}_{\mathbb{C}}\left(\mathbb{C}(\hat{t}_1, \hat{t}_2, J(\hat{t}_1), J'(\hat{t}_1), J''(\hat{t}_1), J(\hat{t}_2), J'(\hat{t}_2), J''(\hat{t}_2)\right) = 4,$$

then $J(\hat{t}_1) = J(\hat{t}_2)$ and hence $\hat{t}_1 = \hat{t}_2$.

Proof. As in the proof of Lemma 7.2, we write t_i instead of $\hat{t}_i$ and consider the derivation δ_2 and write $\chi_{\alpha,\beta,\gamma,\delta_2}(x) = a$, where $a \in \mathbb{C}\langle t_1 \rangle$ for the Schwarzian equation of $J(t_1)$ (so $\chi_{\alpha,\beta,\gamma,\delta_2}(x) = 0$ is the equation of $J(t_2)$). It then follows (using the exact same argument) that $J(t_1), J(t_2) \notin \mathbb{C}\langle t_1 \rangle^{alg}$. Hence using strong minimality of the two equations, we have that $tr.deg._{\mathbb{C}\langle t_1 \rangle}(\mathbb{C}\langle J(t_i) \rangle) = 3$ for both $i = 1$ and $i = 2$. So $J(t_1)$ and $J(t_2)$ are generic (over $\mathbb{C}\langle t_1 \rangle$) solutions of their respective equations.

Since $\text{tr.deg.}_{\mathbb{C}}\left(\mathbb{C}(t_1, t_2, J(t_1), J'(t_1), J''(t_1), J(t_2), J'(t_2), J''(t_2)\right) = 4$, we have some polynomial $P \in \mathbb{C}[X, Y]$ such that $P(J(t_1), J(t_2)) = 0$. So generic solutions of $\chi_{\alpha,\beta,\gamma,\delta_2}(x) = a$ and $\chi_{\alpha,\beta,\gamma,\delta_2}(x) = 0$ are interalgebraic over $\mathbb{C}\langle t_1 \rangle$. Hence the two equations are non-orthogonal. Using Proposition 7.5 we get that $a = 0$ and from Fact 7.4(2) we have that P is the polynomial $(X - Y)$. So the result follows. □

References

[1] V. Aslanyan, *Adequate predimension inequalities in differential fields*, arXiv preprint arXiv:1803.04753 (2018).

[2] V. Aslanyan, *Ax-Schanuel and strong minimality for the j-function*, Ann. Pure Appl. Logic 172 (2021), no. 1, 102871, 24 pp.

[3] V. Aslanyan, *Weak modular Zilber-Pink with derivatives*, arXiv preprint arXiv:1803.05895, 2018.

[4] V. Aslanyan, S. Eterović, J. Kirby. *A closure operator respecting the modular j-function*, arXiv preprint arXiv:2010.00102 (2020).

[5] V. Aslanyan, S. Eterović, J. Kirby. *Differential Existential Closedness for the j-function*, Proc. Amer. Math. Soc. (2021) https://doi.org/10.1090/proc/15333.

[6] V. Aslanyan, J. Kirby. *Blurrings of the j-function*, arXiv preprint arXiv:2005.10167 (2020).

[7] J. Ax. *On Schanuel's conjectures*, Annals of Mathematics, 93:252-268, 1971.

[8] U. Bader and D. Fisher and N. Miller and M. Stover *Arithmeticity, superrigidity and totally geodesic submanifolds of complex hyperbolic manifolds* arXiv preprint arXiv:2006.03008. 2020.

[9] B. Bakker, J. Tsimermann, *Lectures on the Ax-Schanuel Conjecture* Arithmetic Geometry of Logarithmic Pairs and Hyperbolicity of Moduli Spaces, CRM Short Courses, Springer

[10] G. Baldi, Emmanuel Ullmo, *Special subvarieties of non-arithmetic ball quotients and Hodge Theory*, arXiv preprint arXiv:2005.03524 (2020).

[11] F. Barroero and G. Dill, On the Zilber-Pink conjecture for complex abelian varieties, arXiv preprint arXiv:1909.01271, 2019.

[12] D. Blázquez Sanz, G. Casale, J. Freitag and J. Nagloo, *Some functional transcendence results around the Schwarzian differential equation* to appear in the Annales de la Faculté des Sciences de Toulouse - Volume in honor of Prof. H. Umemura.

[13] P. Bonnet, *Minimal invariant varieties and first integrals for algebraic foliations*, Bull. Braz. Math. Soc. (N.S.) 37 (2006), no. 1, p. 1-17.

[14] G. Casale, J. Freitag and J. Nagloo, *Ax-Lindemann-Weierstrass with derivatives and the genus 0 Fuchsian groups*, Ann. of Math. (2) 192 (2020), no. 3, 721-765.

[15] C. Daw and J. Ren, *Applications of the hyperbolic Ax–Schanuel conjecture*, Compositio Mathematica 154 (2018), no. 9 1843–1888.

[16] T. Driscoll and L. Trefethen, *Schwarz-christoffel mapping.* Vol. 8. Cambridge University Press, 2002.

[17] P. Deligne and G.D. Mostow, *Monodromy of hypergeometric functions and non-lattice integral monodromy* Publications Mathématiques de l'IHÉS, Tome 63 (1986): 5-89.

[18] D. Dumas, *Complex projective structures.* Handbook of Teichmüller theory. Vol. II, 455-508, IRMA Lect. Math. Theor. Phys., 13, Eur. Math. Soc., Zürich, 2009.

[19] M. Epstein, M. Elzanowski, *Material Inhomogeneities and their Evolution, A Geometric Approach* Interaction of Mechanics and Mathematics, Springer-Verlag Berlin Heidelberg 2007

[20] J. Freitag and T. Scanlon. *Strong minimality and the j-function.* Journal of the European Mathematical Society 20.1 (2017): 119-136.

[21] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces.* Graduate Studies in Mathematics, Vol. 34, Amer. Math. Soc., Providence, RI, 2001.

[22] Kirby, Jonathan, and Boris Zilber, *Exponentially closed fields and the conjecture on intersections with tori.* Annals of Pure and Applied Logic 165.11 (2014): 1680-1706.

[23] B. Klingler, E. Ullmo, and A. Yafaev, The hyperbolic Ax-Lindemann-Weierstrass conjecture, Publ. Math. Inst. Hautes Études Sci. 123 no. 1 (2016) 333-360.

[24] T. McGrail, *The model theory of differential fields with finitely many commuting derivations,* J. Symbolic Logic 65 (2000), no. 2, 885-913.

[25] N. Mok, *Projective algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume.* In Perspectives in analysis, geometry, and topology, volume 296 of Progr. Math., pages 331-354. Springer, New York 2012.

[26] N. Mok, *Zariski closures of images of algebraic subsets under the uniformization map on finite-volume quotients of the complex unit ball.* Compositio Mathematica, volume 155, no. 11, pages 2129–2149. Cambridge University Press, 2019.

[27] N. Mok, J. Pila, J. Tsimerman, Ax-Schanuel for Shimura varieties, Annals of Mathematics 189.3 (2019): 945-978.

[28] J. Morales Ruiz, *Differential Galois theory and non-integrability of Hamiltonian systems*, Progress in Mathematics, 179. Birkhäuser Verlag, Basel, 1999.

[29] J. Pila, Rational Points of Definable Sets and Results of André-Oort-Manin-Mumford type, International Mathematics Research Notices 2009.13 (2009): 2476-2507.

[30] J. Pila, o-minimality and the André-Oort conjecture for $\mathbb{C}^n$, Annals of mathematics (2011): 1779-1840.

[31] J. Pila and J. Tsimerman, Ax-Schanuel for the j-function, Duke Math. J. 165, no. 13 (2016) 2587-2605.

[32] J. Pila and J. Tsimerman, *The André-Oort conjecture for the moduli space of abelian surfaces.* Compositio Mathematica 149.2 (2013): 204-216.

[33] A. Pillay, *Geometric stability theory*, No. 32. Oxford University Press, 1996.

[34] T. Scanlon, *Algebraic differential equations from covering maps* Adv. Math. 330 (2018), 1071-1100.

[35] R. W. Sharpe, *Differential geometry: Cartan's generalization of Klein's Erlangen program*, Springer (2000)

[36] J. Tsimerman, *The André-Oort conjecture for $\mathcal{A}_g$*. Ann. of Math.(2) 187.2 (2018): 379-390.
[37] E. Ullmo and A. Yafaev, A characterization of special subvarieties, Mathematika 57.2 (2011): 263-273.
[38] M. Yoshida, *Fuchsian Differential Equations With Special Emphasis on the Gauss-Schwarz Theory* Aspects of Mathematics

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