ADDENDUM: AX-LINDEMANN-WEIERSTRASS WITH DERIVATIVES AND THE GENUS 0 FUCHSIAN GROUPS

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The purpose of this short note is to give more details about our use of the Seidenberg embedding theorem in Section 3, more precisely in the proof of Theorem 3.2. In that proof F is assumed to be a finitely generated differential field extension of $\mathbb{C}$ and $K = F(t)$. Then for a solution j_R we apply the embedding theorem to the field $K\langle j_R\rangle$ in a field of meromorphic functions $\mathcal{M}(U)$. Of course, for this to work we should take F to be a finitely generated differential field extension of $\mathbb{Q}$. This difference affects the definition of F, which in turn affects $K = F(t)$. Then K itself is a finitely generated field extension of $\mathbb{Q}$. Nothing about the proof is affected until the first line of the proof of Lemma 3.3 on page 735. In the proof we need to make sure K includes a smooth point on the graph of (j_R, j'_R, j''_R). Of course, there are many such points in $\mathbb{C}^4$, and we aim to replace K by some K_1 which is the algebraic closure of the (differential) field generated by such a tuple over K. However, some care must be taken with the choice, because we need to ensure that the hypothesis $\text{tr.deg}_{K_1} K_1\langle j_R\rangle = 1$ or 2 is preserved. So after we apply the Seidenberg embedding theorem in the proof of Theorem 3.2., we replace K by K_1 with this condition met so that the argument proceeds as before (with the field of constants $\mathbb{C}$ replaced by the constant field of K_1 everywhere). Of course, care is required as it is a priori possible to choose the complex numbers in such a way that $\text{tr.deg}_{K_1} K_1\langle j_R\rangle = 0$.

We now argue that such a point on the graph satisfying the above condition can be chosen. Note that for our purposes, it suffices to take K to be a finitely generated differential field extension of the field of definition of the equation satisfied by j_R. If $\text{tr.deg}_{K_1} K_1\langle j_R\rangle = 0$, but $\text{tr.deg}_K K\langle j_R\rangle = 1$ or 2, then we must have chosen the tuple $\bar{c} = (c_1, c_2, c_3, c_4)$ so that $\bar{c}$ is not independent over $K\langle j_R\rangle^{alg}$. The field $C_0 = K\langle j_R\rangle^{alg} \cap \mathbb{C}$ is a countable subfield of $\mathbb{C}$. Using the fact[1] that $\text{tr.deg}_{\mathbb{C}}\mathbb{C}\,(t, j_R(t), j'_R(t), j''_R(t)) = 4$, any polynomial relation over C_0 holds for $(c_1, j_R(c_1), j'_R(c_1), j''_R(c_1))$ only when c_1 belongs to a proper analytic subset of U. There are only countably many such relations (including the one giving non-smoothness) over C_0. So, choose c_1 outside of this countable set. Then $(c_1, j_R(c_1), j'_R(c_1), j''_R(c_1))$ is independent from $K\langle j_R\rangle^{alg}$.

REFERENCES

[1] D. Blázquez Sanz, G. Casale, J. Freitag and J. Nagloo, *Some functional transcendence results around the Schwarzian differential equation* Annales de la Faculté des Sciences de Toulouse: Mathématiques 29 (5), 1265-1300, 2020

[1] This is a well-known result in differential Galois theory. See for example [1, Proposition 3.2] where it is reproved.

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