Pegging bicycles on tori and unicycles on annuli

Abstract: For a partition $\lambda \vdash n$ we let $H_{\lambda,m}$ stands for the number of *transitive* factorizations of a permutations $\sigma \in \mathrm{S}_n$ with conjugacy type λ, into a product of m transpositions. We look at transitive factorizations of an n–cycle in to $n+1$ transpositions, and of the product a p–cycle with a disjoint q–cycle with $p+q = n$, into $n+1$ transpositions. We will give combinatorial proofs of the following formulas:

$$H_{n,n+1} = \frac{(n^2-1)n^{n+1}}{24}$$

$$H_{p,q,n} = \frac{p^{p+1}q^{q+1}}{p!q!}(n-1)!.$$

A factorization gives rise to a peg (Properly Embedded Graph), and we prove the formulas by studying the corresponding pegs. Factorizations of an n–cycle into $n+1$ transpositions give a graph with two cycles (a bicycle) pegged into a torus with one boundary component, while factorizations of the product of a p–cycle with a q–cycle give a graph with one cycle (a unicycle) pegged into an annulus.

This is joint work with Cormac O'Sullivan and still in progress.