IDENTITIES FROM THE HOLOMORPHIC PROJECTION OF MODULAR FORMS[1]

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1. Introduction.

Set $\mathfrak{H} = \{z = x + iy \in \mathbb{C} \ : \ y > 0\}$ and let Γ be the modular group

$$PSL_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\} / \pm 1.$$

The complex vector space of modular forms for Γ of weight $2k$, denoted $\mathcal{M}_{2k}(\Gamma) = \mathcal{M}_{2k}$, consists of functions $f : \mathfrak{H} \to \mathbb{C}$ that satisfy

$$f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) z) = f(\frac{az+b}{cz+d}) = (cz+d)^{2k} f(z) \tag{1.1}$$

for each $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$ and are holomorphic on $\mathfrak{H}$. We also require them to be holomorphic at infinity. In other words f has the Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$. If $a_0 = 0$ then $f(z)$ has exponential decay as $y \to \infty$. We term the space of such forms $\mathcal{S}_{2k}$, the cusp forms. The spaces $\mathcal{M}_{2k}$, $\mathcal{S}_{2k}$ with $k \in \mathbb{N}$ are finite dimensional with

$$\begin{aligned} \dim \mathcal{M}_{2k} &= \left[\frac{2k}{12}\right] + 1 \text{ (or } \left[\frac{2k}{12}\right] \text{ if } 2k \equiv 2 \mod 12), \\ \dim \mathcal{S}_{2k} &= \dim \mathcal{M}_{2k} - 1 \text{ (or 0 if } 2k \leqslant 10). \end{aligned}$$

For example an element of $\mathcal{M}_{2k}$ is the Eisenstein series

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{m=1}^{\infty} \sigma_{2k-1}(m) q^m \tag{1.2}$$

where $k \geq 2$, B_k is the kth Bernoulli number, ($B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = \frac{-1}{30}$, $\dots$), $q = e^{2\pi i z}$ and the divisor sum $\sigma_k(n) = \sum_{d|n} d^k$.

If $f \in \mathcal{M}_k$ and $g \in \mathcal{M}_l$ then $fg \in \mathcal{M}_{k+l}$ so we get the well known result $E_4(z)E_4(z) = cE_8(z)$ since $\dim \mathcal{M}_8 = 1$. Comparing Fourier expansions we obtain the identity

$$\sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i) = \sigma_7(n) \tag{1.3}$$

for all $n \geq 1$. What kinds of identities of this type are possible in more general settings?

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2. The space $C^\infty(\Gamma\backslash\mathfrak{H}, 2k)$.

This is the space of smooth functions Φ that transform as follows

$$\Phi(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)z) = \left(\frac{cz+d}{|cz+d|}\right)^{2k} \Phi(z). \tag{2.1}$$

For example if $f \in \mathcal{M}_{2k}$ then $y^k f(z) \in C^\infty(\Gamma\backslash\mathfrak{H}, 2k)$. For elements $\Phi_1, \Phi_2 \in C^\infty(\Gamma\backslash\mathfrak{H}, 2k)$ that do not grow large too quickly we have the inner product

$$\langle\, \Phi_1, \Phi_2 \,\rangle = \int_{\Gamma\backslash\mathfrak{H}} \Phi_1(z)\overline{\Phi_2(z)}\, d\mu z, \tag{2.2}$$

where $d\mu z = \frac{dxdy}{y^2}$. Also available are the Maass raising and lowering operators $R_{2k} = 2iy\frac{d}{dz} + k$ and $L_{2k} = -2iy\frac{d}{d\bar{z}} - k$ where

$$\begin{aligned} R_{2k} :& C^\infty(\Gamma\backslash\mathfrak{H}, 2k) \to C^\infty(\Gamma\backslash\mathfrak{H}, 2k+2), \\ L_{2k} :& C^\infty(\Gamma\backslash\mathfrak{H}, 2k) \to C^\infty(\Gamma\backslash\mathfrak{H}, 2k-2). \end{aligned}$$

Finally we may also define the weight $2k$ hyperbolic Laplacian $\Delta_{2k} = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + 2iky\frac{\partial}{\partial x}$. It is related to the raising and lowering operators as follows,

$$\Delta_{2k} = -L_{2k+2}R_{2k} - k(1+k), \tag{2.3}$$

$$= -R_{2k-2}L_{2k} + k(1-k). \tag{2.4}$$

See [Bu] chapter 2 for more details.

3. Holomorphic projection.

To construct identities we need to project our results into the finite dimensional space $\mathcal{M}_{2k}$. Define the Poincare series

$$P_m(z, 2k) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty\backslash\Gamma} (cz+d)^{-2k} e^{2\pi i m \frac{az+b}{cz+d}} \tag{3.1}$$

for $k \geq 2$ and $\Gamma_\infty = \{\gamma \in \Gamma : \gamma\infty = \infty\}$. It is a cusp form in $\mathcal{S}_{2k}$ for $m \geq 1$ and for any other $f \in \mathcal{S}_{2k}$ we have

$$\langle\, y^k f(z), y^k P_m(z, 2k) \,\rangle = a_m \frac{(2k-2)!}{(4\pi m)^{2k-1}}$$

with a_m the mth Fourier coefficient of $f(z)$. We may use this feature of the Poincare series to define a projection map $\pi_{hol} : C^\infty(\Gamma\backslash\mathfrak{H}, 2k) \to \mathcal{M}_{2k}$.

Lemma 3.1. *For $\Phi \in C^\infty(\Gamma\backslash\mathfrak{H}, 2k)$ satisfying $\frac{1}{y^k}\Phi(z) = c_0 + O(y^{-\varepsilon})$ as $y \to \infty$ with $k \geq 2$ and $\varepsilon > 0$ set*

$$\pi_{hol}(\Phi(z)) = c_0 + \sum_{m=1}^{\infty} \langle\, \Phi(z), y^k P_m(z, 2k) \,\rangle \frac{(4\pi m)^{2k-1}}{(2k-2)!} q^m.$$

Then $\pi_{hol}(\Phi(z)) \in \mathcal{M}_{2k}$ and $\langle\, y^k f(z), \Phi \,\rangle = \langle\, y^k f(z), y^k \pi_{hol}\Phi \,\rangle$ for every $f \in \mathcal{S}_{2k}$.

Note that if $g(z)$ is already an element of $\mathcal{M}_{2k}$ then $\pi_{hol}(y^k g(z)) = g(z)$ and in that sense it is a projection. This idea originated in [St]. See [Za1] for a proof of the above lemma.

So we are led to the following question. What kinds of identities are possible using

(i) Multiplication : $M_{2k}(\Gamma) \times M_{2l}(\Gamma) \to M_{2k+2l}(\Gamma)$,
(ii) $R_{2k} : C^\infty(\Gamma\backslash\mathfrak{H}, 2k) \to C^\infty(\Gamma\backslash\mathfrak{H}, 2k+2)$,
(iii) $L_{2k} : C^\infty(\Gamma\backslash\mathfrak{H}, 2k) \to C^\infty(\Gamma\backslash\mathfrak{H}, 2k-2)$ and
(iv) $\pi_{hol} : C^\infty(\Gamma\backslash\mathfrak{H}, 2k) \to M_{2k}(\Gamma)$?

4. Repeatedly raising and lowering holomorphic modular forms.

To see what happens when we repeatedly apply the Maass raising operator it is useful to re-express things in terms of the simpler operator $D_R = 2iy^2 \frac{d}{dz}$. We obtain

$$R_{2(k+n-1)}R_{2(k+n-2)} \dots R_{2k} = \sum_{j=0}^{n} \binom{n}{j} \frac{(j+k-1)!}{(k-1)!} y^{j-n} D_R^{n-j} \tag{4.1}$$

for $k > 0$ and $R_{2(n-1)} \dots R_0 = y^{-n} D_R^n$.

Similarly for the lowering operator we have

$$L_{2(k-n+1)}L_{2(k-n+2)} \dots L_{2k} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} \frac{k!}{(k-j)!} y^{j-n} D_L^{n-j} \tag{4.2}$$

for $k \geq n+1$ and $D_L = 2iy^2 \frac{d}{d\bar{z}}$. Formulas (4.1) and (4.2) may be verified by induction. For convenience, when it is clear that we are dealing with an element of $C^\infty(\Gamma \backslash \mathfrak{H}, 2k)$, we shall write R^n instead of $R_{2(k+n-1)}R_{2(k+n-2)} \dots R_{2k}$ and L^n for $L_{2(k-n+1)}L_{2(k-n+2)} \dots L_{2k}$.

4.1. Lowering modular forms.

For $f(z) = \sum_{m=0}^\infty a_m q^m \in \mathcal{M}_{2k}$ we compute $L^n y^k f(z)$. Actually the answer is rather easy because

$$\begin{aligned} L_{2k} y^k f(z) &= (-2iy\frac{d}{d\bar{z}} - k) y^k f(z) \\ &= -2iy(\frac{d}{d\bar{z}} y^k) f(z) - 2iy^{k+1} \frac{d}{d\bar{z}} f(z) - k y^k f(z) = 0. \end{aligned}$$

Therefore $L^n y^k f(z) = 0$ for any $n > 0$. On the level of the Fourier coefficients we have the following.

$$\begin{aligned} L^n y^k f(z) = & \sum_{m=0}^\infty a_m \left((-1)^n \sum_{j=0}^{n} \binom{n}{j} \frac{k!}{(k-j)!} y^{j-n} D_L^{n-j} (y^k q^m) \right) \\ = & \sum_{m=0}^\infty a_m q^m \left((-1)^n \sum_{j=0}^{n} \binom{n}{j} \frac{k!}{(k-j)!} y^{j-n} y^{k+n-j} (-1)^{n-j} \frac{(k+n-j-1)!}{(k-1)!} \right) \end{aligned}$$

since $D_L^n(y^k q^m) = y^{k+n}(-1)^n \frac{(k+n-1)!}{(k-1)!} q^m$. On simplifying we see that we must have the identity

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} (k-j+1)_{n-1} = 0, \tag{4.3}$$

for $n \geq 1$ where we define

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1 & \text{if } m = 0, \\ (a)(a+1)\dots(a+m-1) & \text{if } m > 0. \end{cases} \tag{4.4}$$

We can prove this more directly. One method is to consider $\frac{d^m}{dx^m}(1+x)^n x^r$ at $x = -1$. This may be evaluated in two ways. First use the binomial expansion of $(1+x)^n$ to get

$$\sum_{i=0}^{n} \binom{n}{i} (r-m+i+1)_m (-1)^{i+r-m}.$$

Secondly use Leibnitz' formula to show that

$$\frac{d^m}{dx^m}(1+x)^n x^r = \sum_{i=0}^{m} \binom{m}{i} (n-i+1)_i (1+x)^{n-i} (r-m+i+1)_{m-i} x^{r-m+i}.$$

At $x = -1$ the only non-zero term is when $i = n$ (provided $m \geq n$) and so we obtain

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (r-m+i+1)_m = \begin{cases} (-1)^n \frac{m!}{(m-n)!} (r-m+n+1)_{m-n} & m \geq n, \\ 0 & m < n. \end{cases} \tag{4.5}$$

Replace i by $n-i$ and set $m = n-1, k = r+1$ to see (4.3).

4.2. Raising modular forms.

As in the previous example apply (4.1) and the formula

$$D_R^n(y^k q^m) = y^{k+n} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{(k+n-1)!}{(k+l-1)!} y^l (4\pi m)^l q^m, \tag{4.6}$$

to get

$$R^n y^k f(z) = \sum_{m=1}^{\infty} a_m q^m \sum_{l=0}^{n} M_l(n,k)(4\pi m)^l y^{k+l},$$

for $f(z) = \sum_{m=1}^{\infty} a_m q^m$ with

$$M_l(n,k) = (-1)^l \frac{n!}{l!} \sum_{j=0}^{n-l} \binom{j+k-1}{j} \binom{k+n-j-1}{k+l-1}.$$

Now for $p, l \geq 0$ we have the identity

$$\sum_{j=0}^{m} \binom{p+j}{p} \binom{p+l+m-j}{p+l} = \binom{2p+l+m+1}{2p+l+1}. \tag{4.7}$$

To see this note that $\sum_{j=0}^{\infty} \binom{p+j}{p} x^j = \frac{1}{(1-x)^{p+1}}$ for $|x| < 1$. Consequently

$$\sum_{i=0}^{\infty} \binom{p+i}{p} x^i \sum_{j=0}^{\infty} \binom{p+l+j}{p+l} x^j = \frac{1}{(1-x)^{p+1+p+l+1}} = \sum_{j=0}^{\infty} \binom{2p+l+1+j}{2p+l+1} x^j$$

and comparing the coefficients of x^m on the above line yields (4.7). Thus $M_l(n,k) = (-1)^l \frac{n!}{l!} \binom{2k-1+n}{2k-1+l}$ and

$$R^n y^k f(z) = \sum_{m=1}^{\infty} a_m q^m \sum_{l=0}^{n} \frac{n!}{l!} \binom{2k-1+n}{2k-1+l} (-4\pi m)^l y^{k+l}. \tag{4.8}$$

If $f(z) = \sum_{m=0}^{\infty} a_m q^m \in \mathcal{M}_{2k}$ then the above formula remains true with a small alteration. Define

$$E(m,l) = \begin{cases} m^l & \text{if } m > 0 \text{ or } l > 0 \\ 1 & \text{if } m = 0 \text{ and } l = 0. \end{cases} \tag{4.9}$$

Then

$$R^n y^k f(z) = \sum_{m=0}^{\infty} a_m q^m \sum_{l=0}^{n} (-4\pi)^l \frac{n!}{l!} \binom{2k-1+n}{2k-1+l} E(m,l) y^{k+l}. \tag{4.10}$$

To work out the holomorphic projection of $R^n y^k f(z)$ and similar functions we use the next result.

Lemma 4.1. *For $\Phi \in C^\infty(\Gamma\backslash\mathfrak{H}, 2k)$ with*

$$\Phi(z) = \sum_{m=0}^{\infty} u_m(y) q^m, \quad u_m(y) = \sum_{i=0}^{r_m} v_m(i) y^{l_m(i)}$$

and $l_0(i) \leq k$ then $\pi_{hol}\Phi(z) = \sum_{m=0}^{\infty} c_m q^m$ where c_0 is the constant part of $\frac{1}{y^k} u_0(y)$ and for $m \geq 1$

$$c_m = \sum_{i=0}^{r_m} v_m(i) \frac{\Gamma(l_m(i)+k-1)}{(2k-2)!} (4\pi m)^{k-l_m(i)}.$$

Proof. To compute the c_m we evaluate $\langle \Phi, y^k P_m \rangle$ by unfolding it.

$$\begin{aligned}
\langle \Phi(z), y^k P_m(z,2k) \rangle &= \int_{\Gamma\backslash\mathfrak{H}} \Phi(z) y^k \overline{P_m(z,2k)}\, d\mu z \\
&= \int_{\Gamma\backslash\mathfrak{H}} \Phi(z) y^k \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} \overline{(cz+d)^{-2k}} e^{\overline{2\pi i m\gamma z}}\, d\mu z \\
&= \int_{\Gamma\backslash\mathfrak{H}} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} \Phi(\gamma z) \mathrm{Im}(\gamma z)^k \overline{(cz+d)^{-2k}} e^{\overline{2\pi i m\gamma z}}\, d\mu z \\
&= \int_0^\infty \int_0^1 \Phi(z) y^k e^{-2\pi i m\bar{z}} \frac{dxdy}{y^2} \\
&= \int_0^\infty u_m(y) y^{k-2} e^{-4\pi m y}\, dy \\
&= \sum_{i=0}^{r_m} v_m(i)\Gamma(l_m(i)+k-1)(-4\pi m)^{-l_m(i)-k+1}.
\end{aligned}$$

The result follows by applying lemma 3.1. ♦

Set $\Phi(z) = R^n y^k f(z)$ for $f(z) = \sum_{m=1}^\infty a_m q^m \in \mathcal{S}_{2k}$. Then $\Phi \in C^\infty(\Gamma, 2(n+k))$ and we have by (4.8) that $\Phi(z) = \sum_{m=1}^\infty u_m(y) q^m$ with $u_m(y) = \sum_{i=0}^{r_m} v_m(i) y^{l_m(i)}$ where $r_m = n$, $v_m(i) = a_m \frac{n!}{i!} \binom{2k-1+n}{2k-1+i} (-4\pi m)^i$ and $l_m(i) = k+i$. Thus $\pi_{hol}(R^n y^k f(z)) = \pi_{hol}(\Phi(z)) = \sum_{m=1}^\infty c_m q^m$ with

$$c_m = a_m (4\pi m)^n \sum_{i=0}^n (-1)^i \frac{n!}{i!} \binom{2k-1+n}{2k-1+i} \frac{(2k+n+i-2)!}{(2k+2n-2)!}. \tag{4.11}$$

As with the previous example we must have $\pi_{hol}(R^n y^k f(z)) = 0$. This time the reason is that

$$\begin{aligned}
\langle R_{2k}\Phi(z), y^k P_m(z,2k+2) \rangle &= \langle \Phi(z), -L_{2k+2} y^k P_m(z,2k+2) \rangle \\
&= \langle \Phi(z), 0 \rangle = 0.
\end{aligned} \tag{4.12}$$

This shows that π_{hol} composed with the raising operator is identically zero. (See [Bu] Prop. 2.1.3 for the above relation between R_{2k} and $-L_{2k+2}$.) Therefore each $c_m = 0$ and we obtain the identity

$$\sum_{i=0}^n (-1)^i \binom{n}{i} (2k+i)_{n-1} = 0,$$

which may be verified for $n \geq 1$ with (4.5).

5. Raising Maass forms.

A Maass form is an element $\eta(z,s)$ of $C^\infty(\Gamma\backslash\mathfrak{H}, 2k)$ that is an eigenfunction of the Laplacian Δ_{2k} so that

$$\Delta_{2k}\eta = \lambda\eta = s(1-s)\eta. \tag{5.1}$$

Maass cusp forms have zero constant terms in their Fourier expansions and are non holomorphic analogs of the elements of $\mathcal{S}_{2k}$. From (5.1) it may be shown that a Maass cusp form of weight $2k = 0$ has Fourier expansion

$$\eta(z,s) = \sum_{m\neq 0} b_m \sqrt{|m|y} K_{s-1/2}(2\pi|m|y) e^{2\pi i m x} \tag{5.2}$$

where K is the K Bessel function and we are summing over all non zero integers. See [Bu], [Iw] for example. To find $R^n\eta(z,s)$ we first need to compute $D_R^n(y^{1/2}K_v(2\pi|m|y)e(mx))$ for $v = s-1/2$. Use the fact that $\frac{d}{dy}K_v(y) = -1/2(K_{v-1}(y)+K_{v+1}(y))$ to show that

$$D_R^n(y^{1/2}K_v(2\pi|m|y)e(mx)) = \sum_{i=0}^{n}\sum_{j=-i}^{i}\alpha_j^n(i)y^{1/2+n+i}K_{v+j}(2\pi|m|y)e(mx).$$

The numbers $\alpha_j^n(i)$ depend on m and may be defined recursively. (Note that the superscript n is an index not an exponent.) For $n=0$ we have $\alpha_j^0(i)=0$ unless $i=j=0$ in which case $\alpha_0^0(0)=1$. For $n \geq 0$ and $\delta = |m|m^{-1}$ we have

$$\alpha_j^{n+1}(i) = -\pi m\left(\delta\alpha_{j-1}^n(i-1)+2\alpha_j^n(i-1)+\delta\alpha_{j+1}^n(i-1)\right)+(i+n+1/2)\alpha_j^n(i).$$

Setting $\beta_j^n(i) = \frac{(-2)^n\delta^j}{(2\pi m)^i}\alpha_j^n(i)$ removes the m dependance and

$$\beta_j^{n+1}(i) = \beta_{j-1}^n(i-1)+2\beta_j^n(i-1)+\beta_{j+1}^n(i-1)-(2i+2n+1)\beta_j^n(i).$$

To isolate the dependance on j set $\beta_j^n(i) = (-1)^{n+i}\gamma_i^n\binom{2i}{j+i}$ to obtain the relation

$$\gamma_i^{n+1} = \gamma_{i-1}^n + (2i+2n+1)\gamma_i^n. \tag{5.3}$$

We may solve the recurrence (5.3) with the initial conditions $\gamma_0^0 = 1$ and $\gamma_i^0 = 0$ for $i \neq 0$ to get

$$\gamma_i^n = \frac{(2n)!}{(n+i)!(2i)!2^{n-i}} \quad \text{for} \ \ n \geqslant 1, \ 0 \leqslant i \leqslant n.$$

Therefore

$$\alpha_j^n(i) = (-4\pi m)^i\frac{(2n)!\delta^j}{(i+j)!(i-j)!(n-i)!} \tag{5.4}$$

and

$$R^n\eta(z,s) = y^{-n}D_R^n\eta(z,s) = \sum_{m\neq 0} b_m\sqrt{|m|}y^{-n}D_R^n(K_v(2\pi|m|y)e(mx))$$

$$= \sum_{m\neq 0} b_m\sqrt{|m|}\sum_{i=0}^{n}\frac{(-4\pi m)^i}{(n-i)!}y^{i+\frac{1}{2}}\sum_{j=-i}^{i}\frac{(2n)!(|m|m^{-1})^j}{(i+j)!(i-j)!}K_{v+j}(2\pi|m|y)e(mx). \tag{5.5}$$

Incidentally, with (2.3) and (2.4) we can show that $R^n\eta$ is an eigenfunction of Δ_{2n} if and only if η is an eigenfunction of Δ_0. This means that every Maass cusp form of weight $2n$ and eigenvalue $1/4-v^2$ has the Fourier expansion (5.5).

6. Holomorphic projection of raised Maass forms.

As we have seen (4.12) forces $\pi_{hol}(R^n\eta(z,s))$ to be zero. To see how the Fourier coefficients vanish we calculate $\langle R^n\eta(z,s), y^nP_m(z,2n)\rangle$. Use the fact that

$$\int_0^\infty x^{r-1}K_v(x)e^{-x}\,dx = 2^{-r}\pi^{1/2}\Gamma(r+1/2)^{-1}\Gamma(r+v)\Gamma(r-v), \tag{6.1}$$

for $\mathrm{Re}(r) > |\mathrm{Re}(v)|$ to get $\langle R^n\eta(z,s), y^nP_m(z,2n)\rangle$ equaling

$$\frac{b_m}{2}(4\pi m)^{1-n}\sum_{i=0}^{n}\frac{(-1)^i(2n)!}{(n-i)!}\sum_{j=-i}^{i}\frac{\Gamma(n+v+i+j-\frac{1}{2})\Gamma(n-v+i-j-\frac{1}{2})}{(i+j)!(i-j)!\Gamma(n+i)}.$$

Consequently $\pi_{hol}(R^n\eta(z,s)) = \sum_{m=1}^\infty c_mq^m$ with c_m given by

$$\frac{b_m}{2}(4\pi m)^{n}\sum_{i=0}^{n}\frac{(-1)^i2n(2n-1)}{(n+i-1)(n-i)!}\sum_{j=-i}^{i}\frac{\Gamma(n+v+i+j-\frac{1}{2})\Gamma(n-v+i-j-\frac{1}{2})}{(i+j)!(i-j)!}.$$

To check that this is indeed zero we'll prove it by hand.

Lemma 6.1. *For $n \geq 1, v \in \mathbb{C}$*

$$\sum_{i=0}^{n} \frac{(-1)^i}{(n+i-1)!(n-i)!} \sum_{j=-i}^{i} \frac{\Gamma(n+v+i+j-\frac{1}{2})\Gamma(n-v+i-j-\frac{1}{2})}{(i+j)!(i-j)!} = 0.$$

Proof. Note that $(1+x)^s = \sum_{l=0}^{\infty} \binom{s}{l} x^l$ is valid for all $-1 < x < 1$ and $s \in \mathbb{C}$ if we set $\binom{s}{l} = \frac{\Gamma(s+1)}{\Gamma(s-l+1)\Gamma(l+1)}$ and use the usual conventions for defining x^s. With the formula $\frac{\Gamma(l-s)}{l!} = \Gamma(-s)(-1)^l \binom{s}{l}$ for $l \in \mathbb{N}$ we see that

$$\sum_{l=0}^{2i} \frac{\Gamma(n+s+l-1)\Gamma(n+2i-l-s)}{l!(2i-l)!} = \Gamma(n+s-1)\Gamma(n-s) \sum_{l=0} 2i \binom{-n-s+1}{l} \binom{-n+s}{2i-l}.$$

Also, since $\sum_{l=0}^{\infty} \binom{-n-s+1}{l} x^l = (1+x)^{-n-s+1}$ and $\sum_{l=0}^{\infty} \binom{-n+s}{l} x^l = (1+x)^{-n+s}$ their product is $(1+x)^{1-2n}$ implying that the coefficient of x^{2i} in the above is

$$\sum_{l=0}^{2i} \binom{-n-s+1}{l} \binom{-n+s}{2i-l} = \binom{1-2n}{2i} = \binom{2n+2i-2}{2i}.$$

Therefore, to finish the proof, it suffices to show that

$$S = \sum_{i=0}^{n} (-1)^i \binom{2n-1}{n-i} \binom{2n+2i-2}{2i} = 0. \tag{6.2}$$

By adapting the identity following (4.7) we have that

$$\sum_{l=0}^{\infty} \binom{2n-2+2l}{2l} x^l = \frac{1}{2} \left((1-\sqrt{x})^{-2n+1} + (1+\sqrt{x})^{-2n+1} \right).$$

Also $\sum_{l=0}^{\infty} \binom{2n-1}{l} (-x)^l = (1-x)^{2n-1}$ so that S is the coefficient of x^n in the product

$$\begin{aligned} &\frac{1}{2} \left((1-\sqrt{x})^{-2n+1} + (1+\sqrt{x})^{-2n+1} \right) (1-x)^{2n-1} \\ &= \frac{1}{2} \left((1+\sqrt{x})^{2n-1} + (1-\sqrt{x})^{2n-1} \right) \\ &= \sum_{i=0}^{n-1} \binom{2n-1}{2i} x^i \end{aligned}$$

Thus $S = 0$ completing the proof. ⧫

The identity (6.2) also appears as a special case of the identity on p37 of [Ri].

7. Projecting products of raised modular forms.

In order to get a non zero projection we try the following. For $f = \sum a_m q^m \in \mathcal{M}_{2k_1}$, $g = \sum b_m q^m \in \mathcal{M}_{2k_2}$ we examine

$$\pi_{hol}(R^{n_1} y^{k_1} f(z) \cdot R^{n_2} y^{k_2} g(z)).$$

Clearly it is an element of $\mathcal{M}_{2(K+N)}$ for $K = \sum_i k_i$, $N = \sum_i n_i$. From (4.10) we derive

$$R^{n_1} y^{k_1} f(z) \cdot R^{n_2} y^{k_2} g(z) = \sum_{m=0}^{\infty} u_m(y) q^m$$

with

$$u_m(y) = \sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2}(4\pi)^L(-1)^L\frac{n_1!n_2!}{l_1!l_2!}\binom{2k_1-1+n_1}{n_1-l_1}\binom{2k_2-1+n_2}{n_2-l_2}$$
$$\times\, y^{K+L}\sum_{j=0}^{m}a_jb_{m-j}E(j,l_1)E(m-j,l_2) \tag{7.1}$$

where $L=\sum_i l_i$. (This convention for N, K and L will be in place from now on.) If we label the inner sum $T^*_m(l_1,l_2;f,g)$ then

$$T^*_m(l_1,l_2;f,g) = T_m(l_1,l_2;f,g) + a_0b_mE(0,l_1)E(m,l_2) + a_mb_0E(m,l_1)E(0,l_2) \tag{7.2}$$

where

$$T_m(l_1,l_2;f,g) = \sum_{j=1}^{m-1}a_jb_{m-j}j^{l_1}(m-j)^{l_2}. \tag{7.3}$$

Finally, applying lemma 4.1, we have that $\pi_{hol}(R^{n_1}y^{k_1}f(z)\cdot R^{n_2}y^{k_2}g(z)) = \sum_{m=0}^{\infty}c_mq^m$ with

$$c_m = (4\pi)^N\sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2}(-1)^L\frac{n_1!n_2!}{l_1!l_2!}\binom{2k_1-1+n_1}{n_1-l_1}\binom{2k_2-1+n_2}{n_2-l_2}$$
$$\times\,\frac{(2K+N+L-2)!}{(2K+2N-2)!}m^{N-L}T^*_m(l_1,l_2;f,g), \tag{7.4}$$

for $m\geq 1$ and $c_0=(-4\pi)^Na_0b_0E(0,n_1)E(0,n_2)$. Hence $c_0=0$ unless $n_1=n_2=0$ and in that case $c_0=a_0b_0$. So we see that the differential operator $\mathcal{P}_{n_1,n_2}$ defined by

$$\mathcal{P}_{n_1,n_2}(f,g) = (4\pi)^{-N}\pi_{hol}(R^{n_1}y^{k_1}f(z)\cdot R^{n_2}y^{k_2}g(z)) \tag{7.5}$$

gives a map $\mathcal{P}_{n_1,n_2}:\mathcal{M}_{2k_1}\times\mathcal{M}_{2k_2}\to\mathcal{S}_{2(N+K)}$ for $(n_1,n_2)\neq(0,0)$. For $n_1=n_2=0$ we have $\mathcal{P}_{0,0}(f,g)=fg$. Examples in section 9 show that this map is not identically zero.

8. Rankin-Cohen differential operators.

The map $\mathcal{P}$ above is similar to a construction of Cohen in [Co]. For $f\in\mathcal{M}_{2k_1}$, $g\in\mathcal{M}_{2k_2}$ he shows that

$$\mathcal{F}_N(f,g) = (2\pi i)^{-N}\sum_{i=0}^{N}(-1)^i\binom{2k_1-1+N}{N-i}\binom{2k_2-1+N}{i}\partial_z^if\partial_z^{N-i}g \tag{8.1}$$

is an element of $\mathcal{M}_{2(K+N)}$ where ∂_z^i means $\frac{d^i}{dz^i}$.

How are $\mathcal{F}$ and $\mathcal{P}$ related? In fact it's not hard to show that $\mathcal{F}_N$ is a certain average of the $\mathcal{P}_{n_1,n_2}$s.

Proposition 8.1. *For every $f\in\mathcal{M}_{2k_1}$, $g\in\mathcal{M}_{2k_2}$*

$$(-1)^N\sum_{i=0}^{N}(-1)^i\binom{2k_1-1+N}{N-i}\binom{2k_2-1+N}{i}\mathcal{P}_{i,N-i}(f,g) = \mathcal{F}_N(f,g). \tag{8.2}$$

Proof. If $N=0$ then the proposition is true. For $N\geq 1$ we may write the left hand side of (8.2) as $\sum_{m=1}^{\infty}d_mq^m$ with, after simplifying,

$$d_m = \sum_{i=0}^{N}(-1)^i\binom{2k_1-1+N}{N-i}\binom{2k_2-1+N}{i}T^*_m(i,N-i;f,g).$$

The Fourier coefficients of the right hand side are identical. ♦

As with $\mathcal{F}$, $\mathcal{P}$ may be expressed in terms of the derivatives of the modular forms.

Proposition 8.2. *For every $f \in \mathcal{M}_{2k_1}$, $g \in \mathcal{M}_{2k_2}$*

$$\mathcal{P}_{n_1,n_2}(f,g) = (2\pi i)^{-N} \sum_{l_1=0}^{n_1} \sum_{l_2=0}^{n_2} (-1)^L \frac{n_1! n_2!}{l_1! l_2!} \binom{2k_1-1+n_1}{n_1-l_1} \binom{2k_2-1+n_2}{n_2-l_2}$$
$$\times \frac{(2K+N+L-2)!}{(2K+2N-2)!} \partial_z^{N-l_1-l_2} \big(\partial_z^{l_1} f \, \partial_z^{l_2} g\big). \tag{8.3}$$

Proof. Compare Fourier coefficients. ⧫

Rankin in [Ran1],[Ran2] considers the general question of which polynomials in the derivatives of modular forms are again modular forms. His operator in [Ran3] includes $\mathcal{F}$ as a special case. It is formulated as follows. Set $r \geqslant 2$ and label r modular forms $f_i \in \mathcal{M}_{2k_i}$ for $1 \leqslant i \leqslant r$. Also define

$$V(r,N) = \{(v_1, v_2, \dots, v_r) : v_i \in \mathbb{N}, \sum_i v_i = N\},$$
$$U(r) = \{(u_1, u_2, \dots, u_r) : u_i \in \mathbb{C}, \sum_i u_i = 0\}$$

then, for a fixed $u \in U(r)$,

$$(2\pi i)^{-N} \sum_{v \in V(r,N)} \frac{\partial^{v_1} f_1}{(2k_1-1+v_1)! v_1!} \cdots \frac{\partial^{v_r} f_r}{(2k_r-1+v_r)! v_r!} u_1^{v_1} \dots u_r^{v_r} \tag{8.4}$$

which we'll denote by $\mathcal{G}_N(f_1, f_2, \dots, f_r)$ is an element of $\mathcal{M}_{2(K+N)}$. This operator may also be expressed as an average, this time of

$$\mathcal{P}_{n_1,\dots,n_r}(f_1, \dots f_r) = (4\pi)^{-N} \pi_{hol}(R^{n_1} y^{k_1} f_1(z) \cdots R^{n_r} y^{k_r} f_r(z)). \tag{8.5}$$

Proposition 8.3. *With the above notation $\mathcal{G}_N(f_1, \cdots, f_r)$ equals*

$$(-1)^N \sum_{v \in V(r,N)} \frac{u_1^{v_1}}{(2k_1-1+v_1)! v_1!} \cdots \frac{u_r^{v_r}}{(2k_r-1+v_r)! v_r!} \mathcal{P}_{v_1,\dots,v_r}(f_1, \dots, f_r).$$

Proof. Same as proposition 8.1. ⧫

For more information on Rankin-Cohen differential operators see [Za1], [Za2]. Similar operators for Siegel modular forms are constructed in [Eh-Ib].

9. Convolution sums involving the divisor function.

We give a straightforward application of this material to finding explicit formulas for the sums

$$S_m(n_1, n_2; r_1, r_2) = \sum_{j=1}^{m-1} j^{n_1} (m-j)^{n_2} \sigma_{r_1}(j) \sigma_{r_2}(m-j). \tag{9.1}$$

Glaisher [Gl], Ramanujan [Ram] (pp 136-162) and Lahiri [La] were the first to systematically evaluate S_m for small values of n_1, n_2, r_1, r_2. Ramanujan manipulated the expressions

$$P = 1 - 24 \sum_{m=1}^{\infty} \frac{m q^m}{1-q^m}, \quad Q = 1 + 240 \sum_{m=1}^{\infty} \frac{m^3 q^m}{1-q^m}, \quad R = 1 - 504 \sum_{m=1}^{\infty} \frac{m^5 q^m}{1-q^m}, \tag{9.2}$$

to obtain his identities and this work was extended in [La]. Ramanujan's series P, Q, R are none other than E_2, E_4, E_6, see [Be]. Also go to [H-O-S-W] in this volume for an interesting elementary method employing a generalization of Liouville's identity to find explicit formulas for S_m and other sums.

Set

$$G_{2k} = \frac{-B_{2k}}{4k} E_{2k} = \frac{-B_{2k}}{4k} + \sum_{m=1}^{\infty} \sigma_{2k-1}(m) q^m. \tag{9.3}$$

While G_{2k} is in $\mathcal{M}_{2k}$ for $k \geq 2$ the series G_2 is not in the zero space $\mathcal{M}_2$. If we let $G_2^*(z) = G_2(z) + (8\pi y)^{-1}$ then G_2^* does transform correctly under the action of Γ and it has weight 2. Although it is no longer holomorphic we do have $yG_2^*(z) \in C^\infty(\Gamma\backslash\mathfrak{H}, 2)$.

Set

$$\begin{aligned} S_m^*(l_1, l_2; r_1, r_2) &= T_m^*(l_1, l_2; G_{r_1+1}, G_{r_2+1}) \\ &= S_m(l_1, l_2; r_1, r_2) - \frac{B_{r_1+1}}{2(r_1+1)} \sigma_{r_2}(m) E(0, l_1) E(m, l_2) \\ &\quad - \frac{B_{r_2+1}}{2(r_2+1)} \sigma_{r_1}(m) E(0, l_2) E(m, l_1). \end{aligned} \tag{9.4}$$

Thus the sum we are interested in, (9.1), arises naturally in the Fourier coefficients of $\mathcal{P}_{l_1,l_2}(G_{r_1+1}, G_{r_2+1})$. Then it can be seen that (7.4) implies that the expressions S_m^* satisfy the relation

$$\begin{aligned} \sum_{l_1=0}^{n_1} \sum_{l_2=0}^{n_2} (-1)^L \frac{n_1! n_2!}{l_1! l_2!} \binom{r_1+n_1}{r_1+l_1} \binom{r_2+n_2}{r_2+l_2} \frac{(r_1+r_2+N+L)!}{(r_1+r_2+2N)!} m^{N-L} S_m^*(l_1, l_2; r_1, r_2) \\ = \alpha_m \quad \text{for } r_1, r_2 \geq 3 \text{ and odd} \end{aligned} \tag{9.5a}$$

where $\sum_{m=1}^{\infty} \alpha_m q^m$ is in $\mathcal{S}_{2N+2+r_1+r_2}$ if $(n_1, n_2) \neq (0, 0)$. If $(n_1, n_2) = (0, 0)$ then $\sum_{m=0}^{\infty} \alpha_m q^m$ is in $\mathcal{M}_{2N+2+r_1+r_2}$ and $\alpha_0 = B_{r_1+1} B_{r_2+1} (4(r_1+1)(r_2+1))^{-1}$.

This means that we can express $S_m^*(n_1, n_2; r_1, r_2)$ (and hence $S_m(n_1, n_2; r_1, r_2)$ by (9.4)) in terms of the sums $S_m^*(l_1, l_2; r_1, r_2)$ for $0 \leq l_1 < n_1$ and $0 \leq l_2 < n_2$ and the coefficients of a cusp form in $\mathcal{S}_{r_1+r_2+2+2N}$. This cusp form may be identified by calculating its first few terms.

If $r_1 = 1$ or $r_2 = 1$ then the recurrence relation is slightly different to take into account the extra factor in the constant term of G_2^*. We have (each for $m \geq 1$)

$$S_m^*(0, 0; 1, 1) = \frac{-m}{2} \sigma_1(m) + \frac{5}{12} \sigma_3(m), \tag{9.5b}$$

$$\begin{aligned} \sum_{l_1=0}^{n_1} \sum_{l_2=0}^{n_2} (-1)^L \frac{n_1! n_2!}{l_1! l_2!} \binom{1+n_1}{1+l_1} \binom{1+n_2}{1+l_2} \frac{(2+N+L)!}{(2+2N)!} m^{N-L} S_m^*(l_1, l_2; 1, 1) \\ + \left((-1)^{n_1} + (-1)^{n_2}\right) \frac{\sigma_1(m)}{2} m^{N+1} \frac{N!(N+1)!}{(2N+2)!} = \beta_m \quad \text{for } (n_1, n_2) \neq (0, 0), \end{aligned} \tag{9.5c}$$

$$\begin{aligned} \sum_{l_1=0}^{n_1} \sum_{l_2=0}^{n_2} (-1)^L \frac{n_1! n_2!}{l_1! l_2!} \binom{1+n_1}{1+l_1} \binom{r+n_2}{r+l_2} \frac{(1+r+N+L)!}{(1+r+2N)!} m^{N-L} S_m^*(l_1, l_2; 1, r) \\ + (-1)^{n_2} \frac{\sigma_r(m)}{2} m^{N+1} \frac{N!(N+r)!}{(2N+r+1)!} = \gamma_m \quad \text{for } r \geq 3 \text{ odd}, \end{aligned} \tag{9.5d}$$

where $\sum_{m=1}^{\infty} \beta_m q^m$ is in $\mathcal{S}_{2N+4}$. Also $\sum_{m=1}^{\infty} \gamma_m q^m$ is in $\mathcal{S}_{2N+r+3}$ if $(n_1, n_2) \neq (0, 0)$. If $(n_1, n_2) = (0, 0)$ then $\sum_{m=0}^{\infty} \gamma_m q^m$ is in $\mathcal{M}_{r+3}$ and $\gamma_0 = B_{r+1}(48(r+1))^{-1}$.

Recall that $S_m^*(n_1, n_2; r_1, r_2) = S_m(n_1, n_2; r_1, r_2)$ unless $n_1 n_2 = 0$ and in that case S_m^* has the two extra terms given by (9.4). Note that $(9.5b)$ appears in [Za1] where it is proved by finding $\pi_{hol}((yG_2^*)^2)$. The relations $(9.5a)$, $(9.5c)$ and $(9.5d)$ generalize this idea to cover all the other cases.

9.1 Examples.

To illustrate these ideas (and check the equations) we'll give some examples. For $S^*_m(0,0;3,3)$ use relation (9.5a) with $n_1 = n_2 = 0$ to see that $S^*_m(0,0;3,3) = \alpha_m$ with $\sum_{m=0}^{\infty} \alpha_m q^m$ in $\mathcal{M}_8$ and $\alpha_0 = B_4^2/64 = 1/(64 \cdot 900)$. Thus $\alpha_m = \sigma_7(m)/120$ and

$$S^*_m(0,0;3,3) = \frac{1}{120}\sigma_7(m). \tag{9.1.1}$$

When $n_1 = 1$ and $n_2 = 0$ (9.5a) implies that

$$\frac{m}{2}S^*_m(0,0;3,3) - S^*_m(1,0;3,3) = \alpha_m$$

with $\sum_{m=1}^{\infty} \alpha_m q^m$ in $\mathcal{S}_{10}$. Consequently

$$S^*_m(1,0;3,3) = S^*_m(0,1;3,3) = \frac{m}{2}S^*_m(0,0;3,3) = \frac{1}{240}m\sigma_7(m). \tag{9.1.2}$$

When $n_1 = 1$ and $n_2 = 1$ (9.5a) implies that

$$\frac{8}{45}m^2 S^*_m(0,0;3,3) - \frac{4}{5}mS^*_m(1,0;3,3) + S^*_m(1,1;3,3) = \alpha_m$$

with $\sum_{m=1}^{\infty} \alpha_m q^m$ in $\mathcal{S}_{12}$. The one dimensional space $\mathcal{S}_{12}$ contains the discriminant function $\sum_{m=1}^{\infty} \tau(m)q^m$. Since $\tau(1) = 1$ and $S^*_1(1,1;3,3) = 0$ we have $\alpha_m = -\tau(m)/540$ and

$$S^*_m(1,1;3,3) = \frac{1}{540}(\tau(m) + m^2\sigma_7(m)). \tag{9.1.3}$$

From (9.1.1), (9.1.2) and (9.1.3) we get

$$S_m(0,0;3,3) = \frac{1}{120}(\sigma_7(m) - \sigma_3(m)) \tag{9.1.4}$$

$$S_m(1,0;3,3) = S_m(0,1;3,3) = \frac{1}{240}m(\sigma_7(m) - \sigma_3(m)), \tag{9.1.5}$$

$$S_m(1,1;3,3) = \frac{1}{540}(\tau(m) + m^2\sigma_7(m)). \tag{9.1.6}$$

Continuing this procedure we obtain.

$$S_m(2,0;3,3) = \frac{1}{2160}(4\tau(m) + 5m^2\sigma_7(m) - 9m^2\sigma_3(m)), \tag{9.1.7}$$

$$S_m(2,1;3,3) = \frac{1}{1080}(m^3\sigma_7(m) - m\tau(m)), \tag{9.1.8}$$

$$S_m(2,2;3,3) = \frac{1}{30888}(13m^4\sigma_7(m) - 22m^2\tau(m) + 9\tau(m) + 2160r(m)), \tag{9.1.9}$$

where $r(m) = \sum_{j=1}^{m-1} \sigma_3(m)\tau(m-j)$ comes from the cuspform $G_4\Delta \in \mathcal{S}_{16}$.

With $r_1 = r_2 = 1$ we have $S^*_m(0,0;1,1) = -m/2\sigma_1(m) + 5/12\sigma_3(m)$ by (9.5b). By (9.5c)

$$S^*_m(1,0;1,1) = \frac{m}{2}S^*_m(0,0;1,1) = \frac{1}{24}(-6m^2\sigma_1(m) + 5m\sigma_3(m)). \tag{9.1.10}$$

Also

$$\frac{4}{30}m^2 S^*_m(0,0;1,1) - \frac{2}{3}mS^*_m(1,0;1,1) + S^*_m(1,1;1,1) - \frac{1}{60}m^3\sigma_1(m) = 0$$

so that

$$S_m^*(1,1;1,1) = \frac{1}{12}(m^2\sigma_3(m) - m^3\sigma_1(m)). \tag{9.1.11}$$

Therefore

$$S_m(0,0;1,1) = \frac{1}{12}(\sigma_1(m) - 6m\sigma_1(m) + 5\sigma_3(m)), \tag{9.1.12}$$

$$S_m(1,0;1,1) = \frac{1}{24}(m\sigma_1(m) - 6m^2\sigma_1(m) + 5m\sigma_3(m)), \tag{9.1.13}$$

$$S_m(1,1;1,1) = \frac{1}{12}(m^2\sigma_3(m) - m^3\sigma_1(m)). \tag{9.1.14}$$

Finally we consider the case $r_1 = 1, r_2 = 3$. By (9.5d) we have $S_m^*(0,0;1,3) + \frac{1}{8}\sigma_3(m)m = \gamma_m$ with $\sum \gamma_m q^m$ in $\mathcal{M}_6$ and $\gamma_0 = -1/5760$. Hence $\gamma_m = 7\sigma_5(m)/80$ for $m \geq 1$ and

$$S_m^*(0,0;1,3) = \frac{1}{80}(7\sigma_5(m) - 10m\sigma_3(m)). \tag{9.1.15}$$

Similarly

$$\frac{m}{3}S_m^*(0,0;1,3) - S_m^*(1,0;1,3) + \frac{1}{60}m^2\sigma_3(m) = \gamma_m$$

with $\sum \gamma_m q^m$ in $\mathcal{S}_8$. This implies that

$$S_m^*(1,0;1,3) = \frac{1}{240}(7m\sigma_5(m) - 6m^2\sigma_3(m)). \tag{9.1.16}$$

We also have

$$S_m^*(0,1;1,3) = \frac{1}{120}(7m\sigma_5(m) - 3m^2\sigma_3(m)). \tag{9.1.17}$$

For the last calculation (9.5d) gives

$$\frac{m^2}{7}S_m^*(0,0;1,3) - \frac{m}{4}S_m^*(0,1;1,3) - \frac{m}{2}S_m^*(1,0;1,3) + S_m^*(1,1;1,3) - \frac{1}{336}m^3\sigma_3(m) = \gamma_m$$

with $\sum \gamma_m q^m$ in $\mathcal{S}_{10}$. Thus, as before, $\gamma_m = 0$ and

$$S_m^*(1,1;1,3) = \frac{1}{60}(m^2\sigma_5(m) - m^3\sigma_3(m)). \tag{9.1.18}$$

Equations (9.1.15), (9.1.16), (9.1.17) and (9.1.18) imply that

$$S_m(0,0;1,3) = \frac{1}{240}(21\sigma_5(m) - 30m\sigma_3(m) + 10\sigma_3(m) - \sigma_1(m)), \tag{9.1.19}$$

$$S_m(1,0;1,3) = \frac{1}{240}(7m\sigma_5(m) - 6m\sigma_3(m) - m\sigma_1(m)), \tag{9.1.20}$$

$$S_m(0,1;1,3) = \frac{1}{120}(7m\sigma_5(m) - 12m^2\sigma_3(m) + 5m\sigma_3(m)), \tag{9.1.21}$$

$$S_m(1,1;1,3) = \frac{1}{60}(m^2\sigma_5(m) - m^3\sigma_3(m)). \tag{9.1.22}$$

The equations (9.1.4), (9.1.5), (9.1.12), (9.1.13), (9.1.14), (9.1.19), (9.1.20), (9.1.21) and (9.1.22) appear in [La] and [H-O-S-W].

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