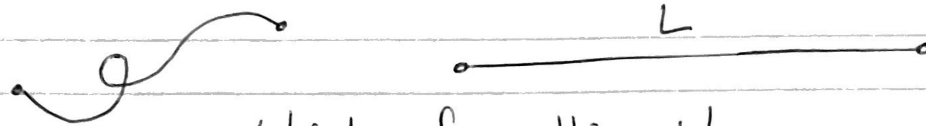
13.3 Arc length and curvature

We have already looked at arc length for curves



think of pulling the thread tight

In our vector notation we get a formula that works in 2 or 3 dimensions

Arc length $L = \int_a^b |\vec{r}\,'(t)|\, dt$

Example 1 Find the length of the helix given by $\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j} + t\vec{k}$ between the points $(1,0,0)$ and $(1,0,2\pi)$.

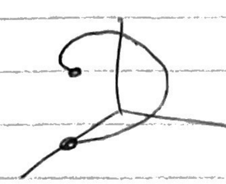
Solution. We see the starting point has $a = t = 0$ and the end point $b = t = 2\pi$

$$\text{Length} = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt$$

$$= \int_0^{2\pi} |\langle -\sin t, \cos t, 1\rangle|\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\, t \Big]_0^{2\pi} = 2\sqrt{2}\,\pi.$$

So the length is $2\sqrt{2}\,\pi$



More generally, if we want the length between $\vec{r}(0)$ and $\vec{r}(t)$, it is given by the <u>arc length function</u>

$$s(t) = \int_0^t |\vec{r}\,'(u)|\,du$$

In the last example we find

$$s(t) = \int_0^t \sqrt{2}\,du = \sqrt{2}\,u\Big]_0^t = \sqrt{2}\,t$$

and this tells us that, when t changes by 1 unit, the length moved along the curve is $\sqrt{2}$ units.

To get rid of this $\sqrt{2}$ factor we can rewrite $\vec{r}(t)$ in terms of s

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

$$= \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle \quad \text{since } t = \frac{s}{\sqrt{2}}$$

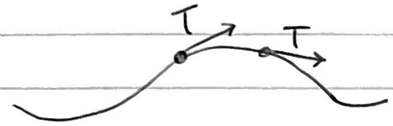
with this new parameter s corresponding exactly to how far along the curve you move.

For example, if you move 10 units along this helix from the point $(1,0,0)$ where do you finish?

Answer: At point $\left(\cos\frac{10}{\sqrt{2}}, \sin\frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}} \right)$.

Curvature

Recall the unit tangent vector $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$



As we move along the curve, using the arc length parameter s, we see that $\vec{T}$ changes.

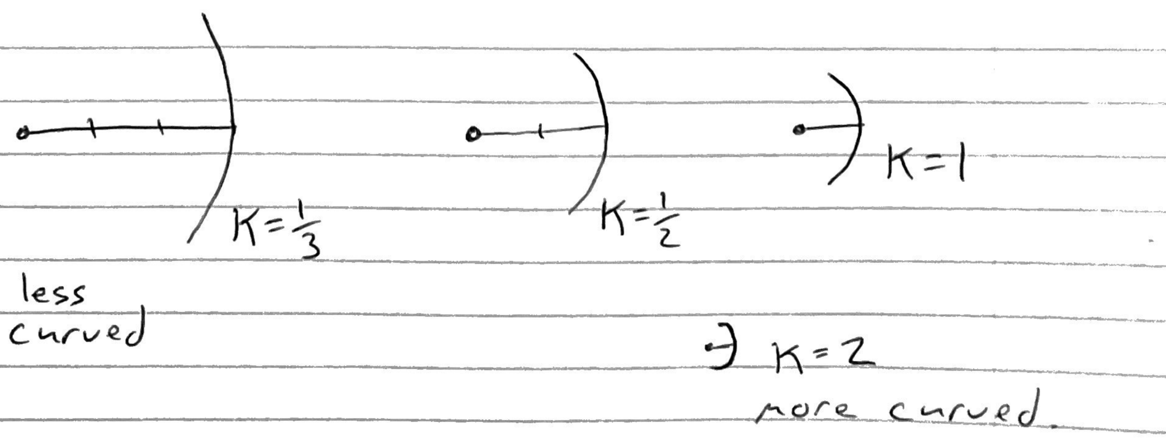
The magnitude of this rate of change is called the curvature

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| \qquad \text{(Greek letter kappa).}$$

Changing back to the parameter t we get

$$\kappa(t) = \left| \frac{\vec{T}\,'(t)}{\vec{r}\,'(t)} \right| .$$

Exercise: Use $\vec{r}(t) = \langle a\cos t, a\sin t \rangle$ to show that points on a circle of radius a all have curvature $\frac{1}{a}$.



Another formula for curvature is nice because it avoids T, though it does need a cross product:

$$K(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Using this formula for example with $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ finds

$$K(t) = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t^2+9t^4)^{3/2}}$$

in general, and $K(0) = 2$ is the curvature at $\vec{r}(0)$.

See the textbook.

The unit normal vector

The vector $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$.

Why? Because

$$0 = \frac{d}{dt} 1 = \frac{d}{dt} |\vec{T}(t)|^2 = \frac{d}{dt}(\vec{T}(t) \cdot \vec{T}(t))$$

$$= \vec{T}'(t) \cdot \vec{T}(t) + \vec{T}(t) \cdot \vec{T}'(t)$$

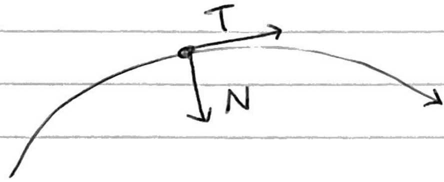
by the product rule

$$= 2\vec{T}(t) \cdot \vec{T}'(t)$$

and zero dot product means perpendicular.

So define $\vec{N}(t) = \dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$ to get

a unit vector that is orthogonal to $\vec{T}(t)$



called the unit normal.

Going back to the helix in example 1
we find

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

$$\vec{T}(t) = \left\langle -\frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

and $\vec{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$